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## Tut. Sess. 6

- Pb 5.9 P. 69

Given a value of $\Theta = \theta$, the random Variable $X$ has an exponential distribution with hazard rate function $h(x) = \theta$, a constant. The random Variable $\Theta$ has a uniform distribution on the interval $(1, 11)$. Determine $S_X(0.5)$ for the unconditional distribution.

Ans:

we have

$$F_X(x) = \int F_{X|\Theta}(x|\theta)\, f(\theta)\, d\theta$$

$X|\Theta \sim exp(\theta)$, $\Theta \sim$ uniform$(1,11)$

For exp. distn. $\theta$

$f(x) = \frac{e^{-x/\theta}}{\theta}$

$F(x) = 1 - e^{-x/\theta} \Rightarrow S(x) = e^{-x/\theta}$

Mean $= E(X) = \theta$

Variance $= \theta^2$

$h(x) = 1/\theta$

See p. 469

$\Downarrow$

Note that, In our pb

$h(x) = \theta$

$\Rightarrow F(x) = 1 - e^{-\theta x}$ ----

$$\therefore F_{X|\theta}(x|\theta) = 1 - e^{-\theta x}, \quad f(\theta) = \frac{1}{11-1} = \frac{1}{10}, \quad 1 \le \theta \le 11$$

$$\therefore F_X(x) = \int_1^{11} (1 - e^{-\theta x}) \frac{1}{10}\, d\theta$$

$$= 0.1 \left[\theta + \frac{e^{-\theta x}}{x}\right]_1^{11}$$

$$= 0.1 \left[10 + \frac{e^{-11x}}{x} - \frac{e^{-x}}{x}\right]$$

$$\therefore F_X(x) = 1 + \frac{1}{10x}(e^{-11x} - e^{-x})$$

$$\therefore S_X(x) = \frac{-1}{10x}(e^{-11x} - e^{-x})$$

$$\therefore S_X(0.5) = \frac{-1}{5}(e^{-5.5} - e^{-0.5})$$

$$\therefore S_X(0.5) = 0.1205$$

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For a pareto distribution, let both $\alpha$ and $\theta$ go to infinity with the ratio $\alpha/\theta$ held constant. show that the result is an exponential distribution.

Ans:

For Pareto $\alpha, \theta$ See p. 465

$$F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}$$

let $\tau = \alpha/\theta \Rightarrow \alpha = \tau\theta$

when $\alpha \to \theta$, $\theta \to \infty$ and $\tau$ is a constant

$$\lim_{\theta\to\infty} 1 - \left(\frac{\theta}{x+\theta}\right)^{\tau\theta}$$

$$= 1 - \lim_{\theta\to\infty} \left(\frac{\theta}{x+\theta}\right)^{\tau\theta} \quad (1)$$

ملاحظة: نهاية توزيع Pareto هو التوزيع الأسي لأن $\alpha = \tau\theta$ نعوض عنها ثم نأخذ النهاية عندما $\alpha, \theta$

The limit of the logarithm is

$$\lim_{\theta\to\infty} \ln \left[\frac{\theta}{x+\theta}\right]^{\tau\theta}$$

$$= \lim \tau\theta \left[\ln\theta - \ln(x+\theta)\right]$$

$$= \tau \lim_{\theta\to\infty} \frac{\ln\theta - \ln(x+\theta)}{\theta^{-1}}$$

, By using L' Hopital theorem

$$= \tau \lim_{\theta\to\infty} \frac{\theta^{-1} - (x+\theta)^{-1}}{-\theta^{-2}} \quad \frac{x(x+\theta)}{x(x+\theta)}$$

$$= \tau \lim_{\theta\to\infty} \frac{x\theta^{-1} + \cancel{1} - \cancel{1}}{-\theta^{-2}(x+\theta)} = -\tau \lim_{\theta\to\infty} \frac{x\theta}{x+\theta}$$

$$= -\tau \lim_{\theta\to\infty} \frac{x}{1+\frac{x}{\theta}} = -\tau x$$

$$\therefore \lim_{\theta\to\infty} \ln \left[\frac{\theta}{x+\theta}\right]^{\tau\theta} = -\tau x \Rightarrow \lim_{\theta\to\infty} \left[\frac{\theta}{x+\theta}\right]^{\tau\theta} = e^{-\tau x} \quad (2)$$

Substitute (2) in (1), We get the limit of Pareto distn is $1 - \exp(-\tau x)$ which is an exponential distn.

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Pb 5.23 p. 75 Textbook

Show that as $\tau \to \infty$ in the transformed beta distribution, the result is the inverse transformed gamma distribution.

Ans:

The transformed beta pdf is

$$f(x) = \frac{\Gamma(\alpha+\tau)\,\gamma\,(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)\,x\,[1+(x/\theta)^{\gamma}]^{\alpha+\tau}}$$

See p. 463

let $\alpha$ be constant and $\theta\tau^{1/\gamma} \to \xi$

$$\Rightarrow \theta = \xi\tau^{-1/\gamma}$$

$$f(x) = \frac{\Gamma(\alpha+\tau)\,\gamma\,x^{\gamma\tau}\,(1/\xi\,\tau^{1/\gamma})^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)\,x\,[1+x^{\gamma}\theta^{-\gamma}]^{\alpha+\tau}}$$

$$= \frac{\Gamma(\alpha+\tau)\,\gamma\,x^{\gamma\tau-1}}{\Gamma(\alpha)\Gamma(\tau)\,(\xi\tau^{-1/\gamma})^{\gamma\tau}\,[1+x^{\gamma}\theta^{-\gamma}]^{\alpha+\tau}}$$

$$\therefore f(x) = \frac{\Gamma(\alpha+\tau)\,\gamma\,x^{\gamma\tau-1}}{\Gamma(\alpha)\Gamma(\tau)\,\xi^{\gamma\tau}\tau^{-\tau}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}} \quad (1)$$

$$\because \lim_{\alpha\to\infty} \frac{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{1/2}}{\Gamma(\alpha)} = 1 \quad \text{Stirling's formula}$$

$$\Rightarrow \Gamma(\tau) = e^{-\tau}\tau^{\tau-1/2}(2\pi)^{1/2} \text{ as } \tau\to\infty \quad (2)$$

$$\Rightarrow \Gamma(\alpha+\tau) = e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2} \text{ as } \alpha+\tau\to\infty \quad (3)$$

Subs. (2), (3) in (1)

$$\therefore f(x) = \frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}\cancel{(2\pi)^{1/2}}\,\gamma\,x^{\gamma\tau-1}}{\Gamma(\alpha)\,e^{-\tau}\tau^{\tau-1/2}\cancel{(2\pi)^{1/2}}\,\xi^{\gamma\tau}\tau^{-\tau}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$f(x) = \frac{e^{-\alpha}\left(\frac{\alpha+\tau}{\tau}\right)^{\alpha+\tau-1/2}\gamma\,x^{\gamma\tau-1}\cdot x^{-\gamma\tau-\gamma\alpha}}{\Gamma(\alpha)\tau^{-\alpha}\tau^{-\tau}\xi^{\gamma(\tau+\alpha)}\xi^{-\gamma\alpha}x^{-\gamma(\tau+\alpha)}(1+x^{\gamma}\xi^{-\gamma}\tau)^{\alpha+\tau}}$$

$$\because \lim_{a\to\infty}\left(1+\frac{x}{a}\right)^{a+b} = e^{x} \quad \therefore \lim_{\tau\to\infty}\left(\frac{\alpha+\tau}{\tau}\right)^{\alpha+\tau-1/2} = \lim_{\tau\to\infty}\left(1+\frac{\alpha}{\tau}\right)^{\tau+\alpha-1/2} = e^{\alpha}$$

$$\text{when } \lim_{\tau\to\infty}\left(1+\frac{\alpha}{\tau}\right)^{\alpha} = 1$$

$$\therefore f(x) = \frac{\cancel{e^{-\alpha}}\cancel{e^{\alpha}}\,\gamma\,x^{-\gamma\alpha-1}}{\Gamma(\alpha)\,\xi^{-\gamma\alpha}\cdot\left(\frac{1}{\tau}\right)^{\alpha+\tau}\left(\frac{\xi}{x}\right)^{\gamma(\tau+\alpha)}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

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$$\therefore f(x) = \frac{\gamma x^{\gamma\alpha - 1}}{\Gamma(\alpha)\, \xi^{-\gamma\alpha} \left[\frac{1}{\tau}\left(\frac{\xi}{x}\right)^{\gamma}\right]^{\alpha+\tau} \left[1 + x^{\gamma} \xi^{-\gamma} \tau\right]^{\alpha+\tau}}$$

$$f(x) = \frac{\gamma (\xi/x)^{\gamma\alpha}}{\Gamma(\alpha)\, x} \cdot \frac{1}{\left[1 + \frac{(\xi/x)^{\gamma}}{\tau}\right]^{\alpha+\tau}}$$

$$\therefore \lim_{\tau\to\infty} \left[1 + \frac{(\xi/x)^{\gamma}}{\tau}\right]^{\alpha+\tau} = e^{(\xi/x)^{\gamma}}$$

$$\therefore f(x) = \frac{\gamma (\xi/x)^{\gamma\alpha}\, e^{-(\xi/x)^{\gamma}}}{\Gamma(\alpha)\, x} \quad \text{as } \tau \to \infty$$

which is the inverse transformed gamma pdf with parameters $\alpha, \xi, \gamma$.

See p. 467 Textbook

See p. 467

for Inverse Transformed Gamma $\alpha, \theta, \tau$

$$f(x) = \frac{\tau (\theta/x)^{\tau\alpha}\, e^{-(\theta/x)^{\tau}}}{x\, \Gamma(\alpha)}$$

$\alpha \to \alpha$

$\tau \to \gamma$

$\theta \to \xi$

H.W. Assignment *

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