Tut. Sess. (7)

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pb 5.19 p. 70

Losses in 1993 follow the density function $f(x) = 3x^{-4}$, $x \geq 1$, where x is the loss in millions of dollars. Inflation of 10 % impacts all claims uniformly from 1993 to 1994. Determine the cdf of losses for 1994 and use it to determine the probability that a 1994 loss exceeds 2,200,000.

Ans:

$$\therefore F(x) = \int_1^x 3t^{-4}\,dt = 3\left[\frac{t^{-3}}{-3}\right]_1^x$$

$$\therefore F(x) = 1 - x^{-3}$$

For inflation, let $Y = 1.1X$

$$\therefore F_Y(y) = 1 - \left(\frac{y}{1.1}\right)^{-3}$$

$$\therefore pr(Y > 2.2) = 1 - F_Y(2.2)$$

$$= \left(\frac{2.2}{1.1}\right)^{-3} = \frac{1}{8} = 0.125 \quad \#$$

pb 5.14 p. 69

Consider the exponential inverse Gaussian frailty model with

$$a(x) = \frac{\theta}{2\sqrt{1+\theta x}}, \quad \theta > 0$$

a) verify that the conditional hazard rate $h_{X|\Lambda}(x|\lambda)$ of $X|\Lambda$ is indeed a valid hazard rate.

b) Determine the conditional survival function $S_{X|\Lambda}(x|\lambda)$.

c) If Λ has a gamma distribution with parameters $\theta = 1$ and α replaced by 2α, determine the marginal or unconditional survival function of X.

d) Use (c) to argue that a given frailty model may arise from more than one combination of conditional distribution of $X|\Lambda$ and frailty distribution of Λ.

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Ans:

a) $\because A(x) = \int_0^x a(t)\,dt$

$$= \int_0^x \frac{\theta}{2\sqrt{1+\theta t}}\,dt$$

$$= \frac{1}{2}\int_0^x (1+\theta t)^{-1/2}\cdot\theta\,dt$$

$$= \frac{1}{\cancel{2}}\left[\frac{(1+\theta t)^{\frac{1}{2}}}{\cancel{\frac{1}{2}}}\right]_0^x$$

$$\therefore A(x) = \sqrt{1+\theta x} - 1$$

$\because h_{X|\Lambda}(x|\lambda) = \lambda\, a(x)$, $\Lambda > 0$ Frailty model

$$\therefore h_{X|\Lambda}(x|\lambda) = \lambda\left[\frac{\theta}{2\sqrt{1+\theta x}}\right]$$

Clearly, $h_{X|\Lambda}(x|\lambda) > 0 \quad \forall\ x > 0$, also,

$$h_{X|\Lambda}(x|\lambda) \to 0 \text{ as } x \to \infty$$

$\therefore h_{X|\Lambda}(x|\lambda)$ is considered as a valid hazard rate.

b)

$\because$ The conditional survival function $S_{X|\Lambda}(x|\lambda)$ is given by

$$S_{X|\Lambda}(x|\lambda) = e^{-\lambda A(x)}$$

$$\therefore S_{X|\Lambda}(x|\lambda) = e^{-\lambda(\sqrt{1+\theta x} - 1)}$$

c) $\Lambda \sim \text{gamma}(2\alpha, 1) \Rightarrow f(x;1) = \dfrac{1^{-2\alpha}\, x^{2\alpha-1}\, e^{-x/1}}{\Gamma(2\alpha)}$

The moment generating fn of the frailty random variable Λ is

$$M_\Lambda(z) = E(e^{z\Lambda})$$

$$= \left(\frac{1}{1-z}\right)^{2\alpha} = (1-z)^{-2\alpha}$$

For $X \sim \text{gamma}(\alpha, \theta)$

$$\Rightarrow f(x;\theta) = \frac{\theta^{-\alpha} x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)}$$

$$\Rightarrow M_X(z) = E(e^{zX}) = \left(\frac{1}{1-\theta z}\right)^{\alpha}, \quad z < \frac{1}{\theta}$$

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$\because$ The marginal survival fn is

$$S_X(x) = E\left[e^{-\Lambda A(x)}\right] = M_\Lambda[-A(x)]$$

$$S_X(x) = (\cancel{1+}\sqrt{1+\theta x}\cancel{-1})^{-2\alpha}$$

$$\therefore S_X(x) = \left((\sqrt{1+\theta x})^2\right)^{-\alpha} = (1+\theta x)^{-\alpha} \Rightarrow \text{pareto distn}$$

d) The survival fn $S_X(x) = (1+\theta x)^{-\alpha}$ can be considered as a mixed survival fn of gamma survival fn of frailty distn of Λ and the exponential survival fn of conditional distn of $X|\Lambda$.

Thus, if $S_{X|\Lambda} = e^{-\lambda x}$ (exp. distn)

$$\Rightarrow A(x) = x$$

$$\therefore S_X(x) = M_\Lambda[-A(x)] = M_\Lambda(-x)$$

$$\therefore S_X(x) = (1+\theta x)^{-\alpha}$$

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For gamma distn (α, θ)
$M_\Lambda(z) = (1-\theta z)^{-\alpha}$
$\Rightarrow M_\Lambda(-x) = (1+\theta x)^{-\alpha}$

Revise Lecture 15

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Derive the mean and the variance of the gamma distribution

Ans:

The pdf for gamma distn is given by

$$f(x;\theta) = \frac{\theta^{-\alpha} x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)}$$

$$f(x;\theta) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} \frac{e^{-1/\theta \cdot x}}{\theta^{\alpha}} = \frac{p(x)\, e^{r(\theta)x}}{q(\theta)}$$

where $r(\theta) = -1/\theta$, $q(\theta) = \theta^{\alpha}$ and $p(x) = x^{\alpha-1}/\Gamma(\alpha)$

i.e the gamma distn is a member of the linear exp. family

i.e the mean, $E(X) = \mu(\theta) = \dfrac{q'(\theta)}{r'(\theta)\, q(\theta)}$

$$= \frac{\alpha\, \theta^{\alpha-1}}{1/\theta^2 \cdot \theta^{\alpha}} = \alpha\theta$$

Revise Lecture 18

& Variance $Var(X) = \sigma(\theta) = \dfrac{\mu'(\theta)}{r'(\theta)}$

$$= \frac{\alpha}{1/\theta^2} = \alpha\theta^2$$

H.W "Assignment" * #(2) Mid I

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