Introduction to Operations Research

- Mathematical programming
- Linear programming problems
- Model building

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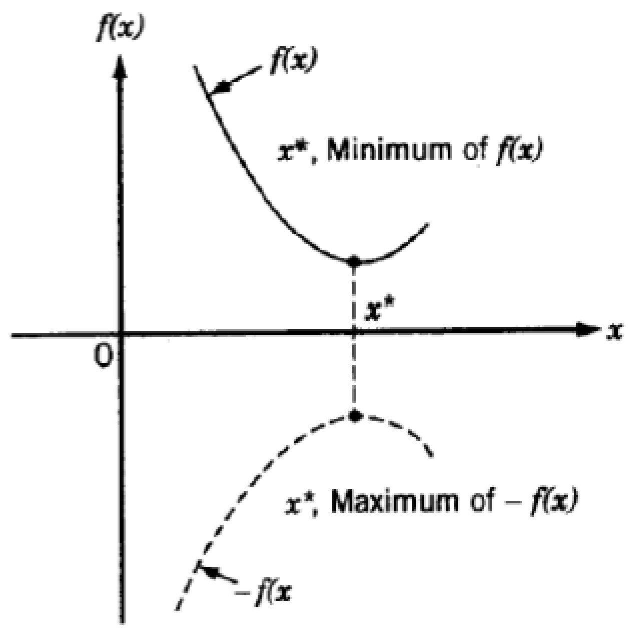
Definition of Operations Research

OR is a branch of mathematics which is concerned with the application of scientific methods and techniques to decision making problems and with establishing the best or optimal solutions.

The optimum seeking methods are also known as mathematical programming techniques and are generally studied as a part of OR.

Definitions

- Optimization is the process of finding the set conditions that give the maximum or minimum value of a function.
- This set of conditions is called "*the optimum solution*".
- If x^* is the minimum of $f(x)$, then x^* is the maximum of $-f(x)$.
- Therefore the terms minimization and maximization are interchangeable.



Statement of the optimization problem

- An optimization problem can be stated as:
- Find $\mathbf{X} = [x_1 \quad x_2 \quad \cdots \quad x_n]$
- that minimizes $f(\mathbf{X})$
- Subject to:

$$g_j(\mathbf{X}) \le 0, \qquad j = 1, 2, \ldots, m$$
$$h_i(\mathbf{X}) = 0, \qquad i = 1, 2, \ldots, p$$

- **X**: the design vector.
- the objective function.
- inequality and equality constraints
- The number of variables n and the number of constraints m and/or p need not be related in any way.

Classification of the Optimization Problems

- Classification based on the existence of constraints:
  - Constrained
  - Unconstrained
- Classification based on the nature of the equations involved:
  - Linear
  - Non-linear
- Classification based on the permissible values of the design variables:
  - Continuous
  - Integer

Linear Programming

Requirements for the linear programming problem:

- The objective function is a linear function of the decision variables.
- There must be a set of linear constraints that can be expressed as linear equalities or inequalities.
- The resources must be limited.
- The decision variables must be interrelated , continuous and non-negative

Example of a linear programming problem

- Maximize:

$$z = 4x_1 + 3x_2 + 6x_3$$

- Subject to:

$$2x_1 + 3x_2 + 2x_3 \leq 440$$

$$4x_1 + 3x_3 \leq 470$$

$$2x_1 + 5x_2 \leq 430$$

$$x_1, x_2, x_3 \geq 0$$

Example on the application of linear programming:

- A person wants to decide the constituents of a diet which will fulfill his daily requirements of proteins, fats, and carbohydrates at the minimum cost. The choice is to be made from four different types of food. The yields per unit of these foods are as given in table. Formulate the LP model for the problem.

Food type	Yield per unit			Cost/unit ¢
	Proteins	Fats	Carbohydrates	
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	7	65
Minimum requirement	800	200	700	

Solution

- Assume the diet is made up of x1, x2, x3 and x4 units of types 1, 2, 3 and 4 respectively.
- The objective is to minimize the cost function:

$$Z = 45x_1 + 40x_2 + 85x_3 + 65x_4$$

- The constraints on the fulfillment of the daily requirements:

$$\text{proteins}: 3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$$

$$\text{fats}: 2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$$

$$\text{carbohydrates}: 6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$$

- The non-negativity constraints:

$$x_1, x_2, x_3, x_4 \geq 0$$

- The LP program is:
- Minimize:

$$Z = 45x_1 + 40x_2 + 85x_3 + 65x_4$$

- Subject to

$$3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$$

$$2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$$

$$6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Example (2): The Transportation Problem

- A dairy firm has two milk plants with daily milk production of 6 million liters and 9 million liters respectively. Each day the firm must fulfill the needs of its three distribution centers which have milk requirements of 7, 5 and 3 million liters respectively. Cost of shipping 10,000 liters of milk from each plant to each distribution center is given in table below. Formulate the LP model to minimize the transportation cost.

Cost of transportation in L.E. per 1,000,000 liters

		Distribution Center			
		1	2	3	Supply
Plant	1	20,000	30,000	110,000	6,000,000
	2	10,000	90,000	60,000	9,000,000
Demand		7,000,000	5,000,000	3,000,000	

Solution

- Let x1 and x2 be the quantities of milk transported from plant 1 to distribution centers 1 and 2 respectively.
- We can represent the transportation of milk as shown below:

		Distribution center			
		1	2	3	Supply
Plant	1	x1	x2	6-x1-x2	6
	2	7-x1	5-x2	9-(7-x1)-(5-x2)	9
Demand		7	5	3	

- The objective is to minimize the transportation cost:

$$Z = 2x_1 + 3x_2 + 11(6 - x_1 - x_2)$$
$$+ 1(7 - x_1) + 9(5 - x_2) + 6[9 - (7 - x_1) - (5 - x_2)]$$

- The constraints are:

$$x_1, x_2 \geq 0$$
$$7 - x_1 \geq 0 \quad \text{or} \quad x_1 \leq 7$$
$$5 - x_2 \geq 0 \quad \text{or} \quad x_2 \leq 5$$
$$9 - (7 - x_1) - (5 - x_2) \geq 0 \quad \text{or} \quad x_1 + x_2 \geq 3$$

GRAPHICAL SOLUTION OF THE LP PROBLEMS

Example (1) on graphical solution

- A chemical firm produces two solvents using 3 machines. The different machining times required for each solvent, the machining times available on different machines, and the profit on each machine part are given in the following table. Determine the volumes of solvents I and II to be manufactured per week to maximize the profit.

Type of machine	Machining time required (min)		Maximum time available per week (min)
	Solvent I	Solvent II	
M1	10	5	2500
M2	4	10	2000
M3	1	1.5	450
Profit per unit	$50	$100	

Solution

- Let the number of solvents I and II manufactured per week be x and y, respectively.
- Constraints due to maximum time limitations on the various machines are given by:

$$M1: 10x + 5y \le 2500 \qquad \dots(E1)$$

$$M2: 4x + 10y \le 2000 \qquad \dots(E2)$$

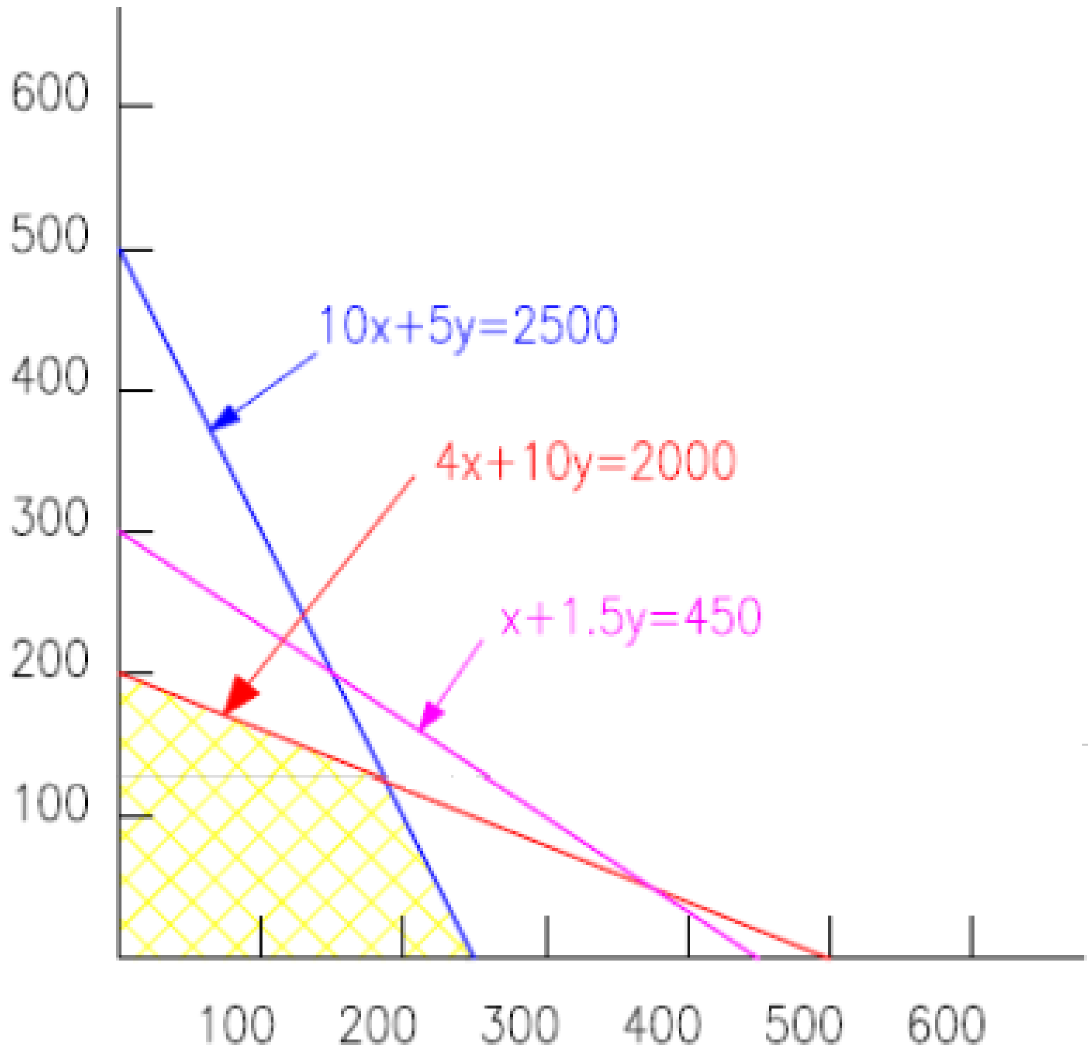
$$M3: x + 1.5y \le 450 \qquad \dots(E3)$$

- The number of products cannot be negative:

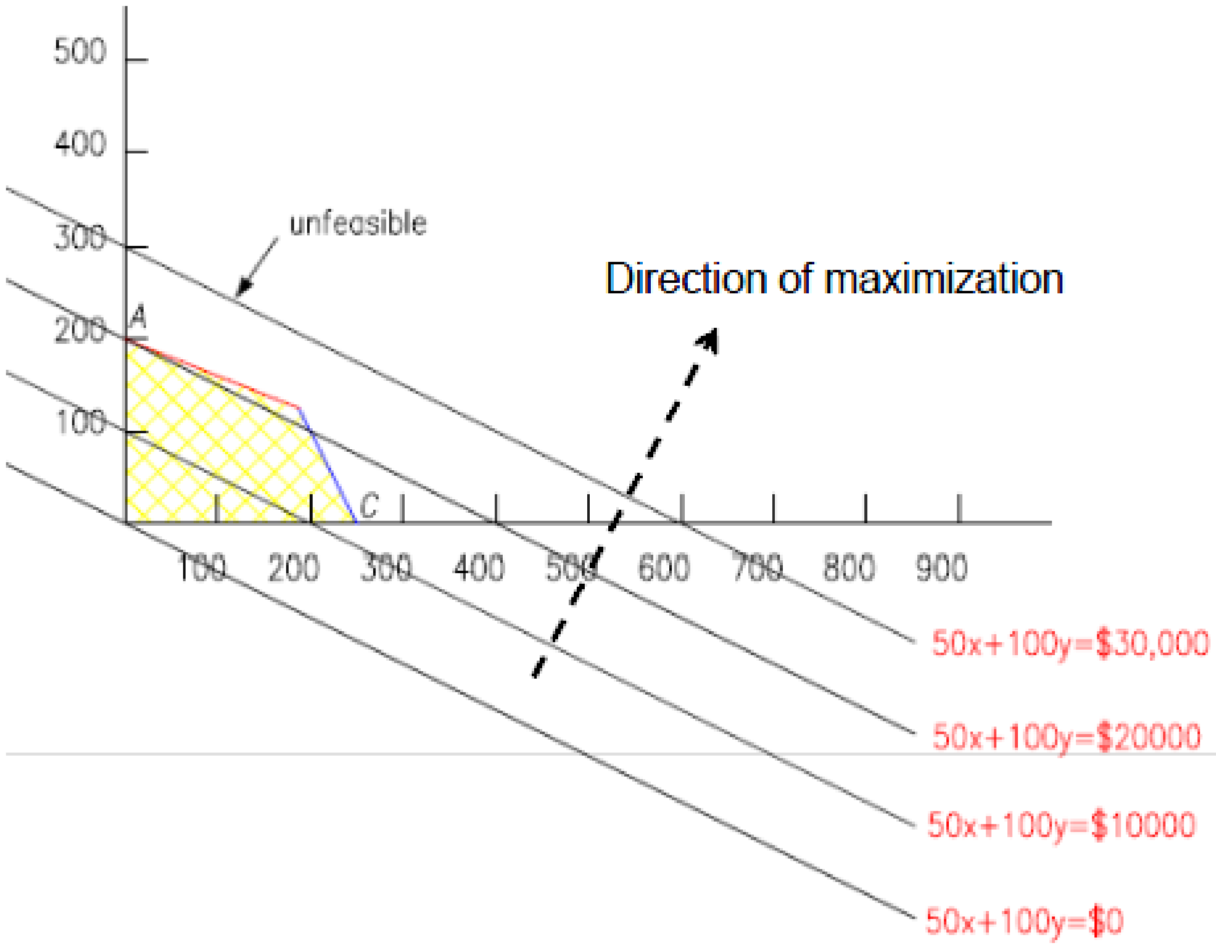
$$x \ge 0 \text{ and } y \ge 0 \qquad \dots(E4)$$

- The total profit is given by:

$$f(x, y) = 50x + 100y$$



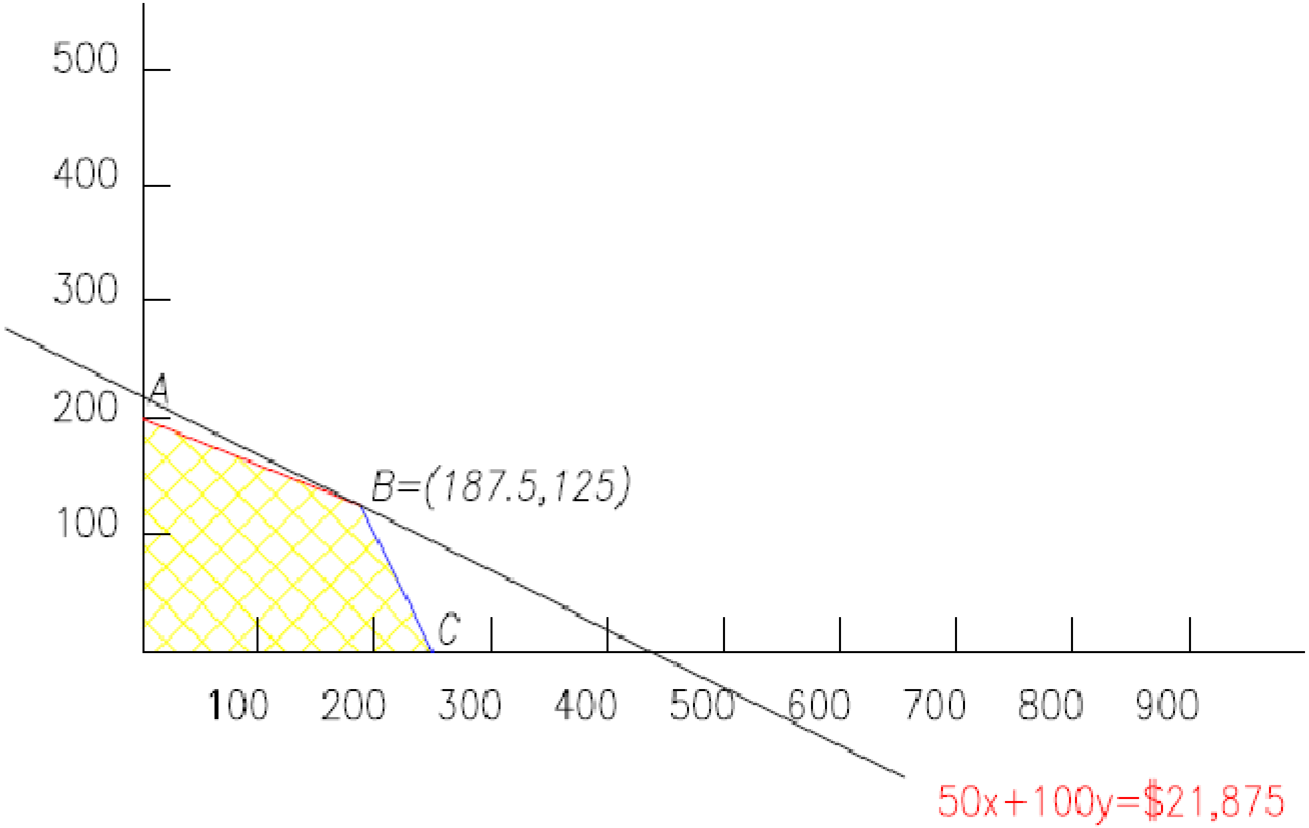
- Plot the inequalities (E1), (E2), (E3) and (E4) on the *xy* plane
- The feasible region is the hatched region



- The contours of the objective function, *f*, are defined by the linear equation:

$$50x + 100y = k$$

- As *k* varies, the objective function line moves parallel to it self



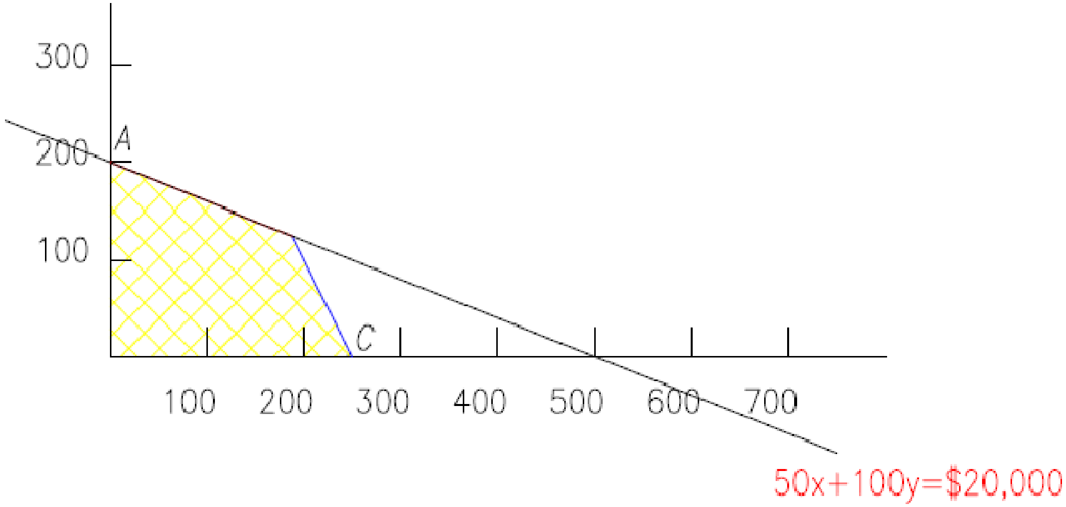
- The optimum is *B=(187.5, 125)* and the maximum profit is $21,875
- The problem has a unique optimal solution

Exceptional cases of the LP Problem

- In some cases the optimum solution may have:
  - An infinite number of optimal solutions
  - An unbounded solution
  - No feasible solution

Example (2) on LP with an infinite number of optimal solutions

- In the previous example: let the profit rates for the solvents I and II be \$40 and 100\$ instead of \$50 and \$100.
- The contours of the profit function will be parallel to line AB.
- Any point between A and B can be taken as an optimum solution with a profit value \$20,000



An infinite number of feasible solutions

Example (2): on unbounded solutions

- Maximize:

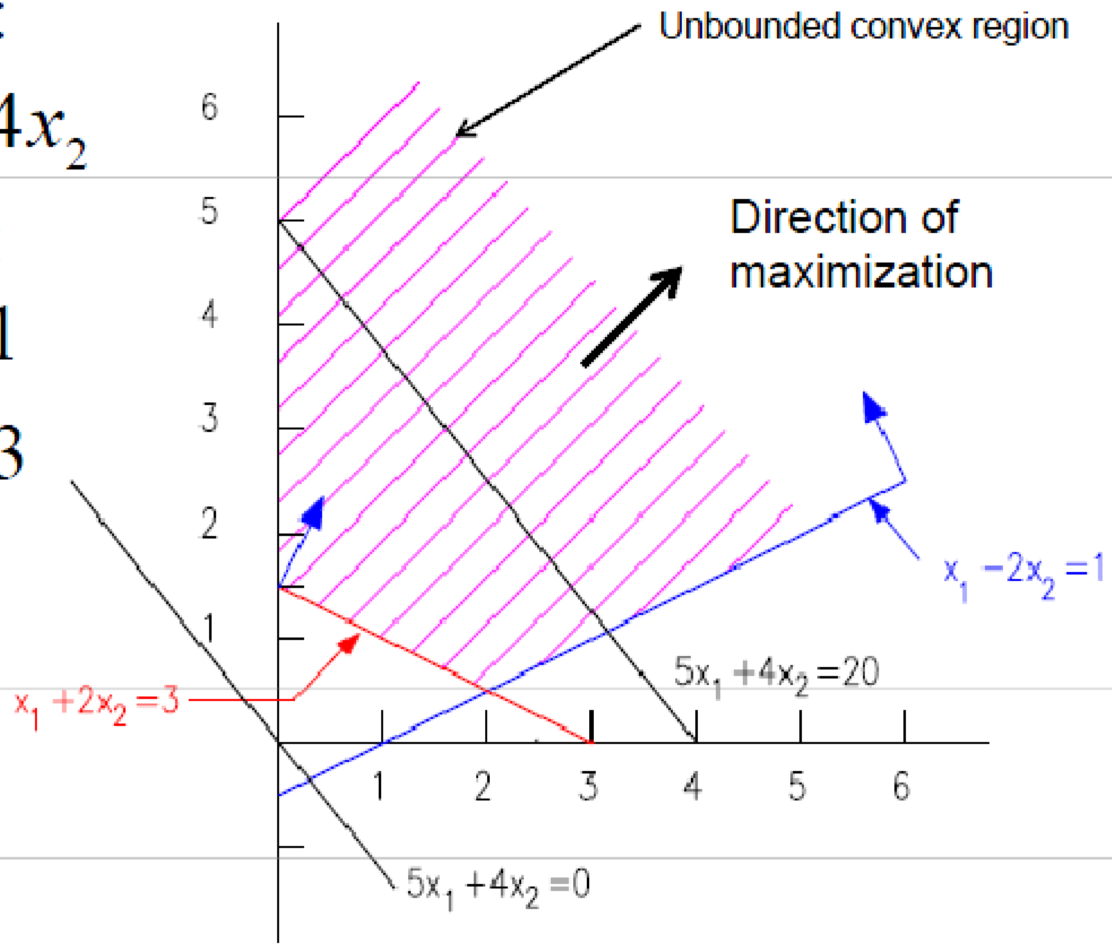
  $$Z = 5x_1 + 4x_2$$

- Subject to:

  $$x_1 - 2x_2 \le 1$$

  $$x_1 + 2x_2 \ge 3$$

  $$x_1, x_2 \ge 0$$



The problem has unbounded solution

Example (3) on LP problems with no feasible solutions

- Maximize:

  $$Z = 3x + 2y$$

- Subject to:

  $$-2x + 3y \le 9$$

  $$3x - 2y \le -20$$

  $$x, y \ge 0$$

- There is no point that satisfies both constraints
- The solution of the problem doesn't exist