PHYS 404

Lecture 5: Spherical, Modified and Third Kind Bessel Functions

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How they come up?

- When we try to solve the Helmholtz equation in spherical coordinates, the radial equation which comes out leads us to define the so called spherical Bessel equations:
- In the special case where the as we have seen $\lambda=l(l+1)$ (l=0, 1, 2, …) then Eq. 4.1 is written as

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{\left(l+1/2\right)^2}{x^2}\right)u=0 \qquad (5.1)$$

- The solutions of (5.1) are called **spherical Bessel** functions.

How they come up?

- In many physical problems we may find the so called **modified** Helmholtz equation which comes from the ordinary Helmholtz equation when k^2 -> - k^2, thus

$$\left(\nabla^2 - k^2\right)\Psi = 0$$

- In this case the corresponding radial equations become:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \left[k^2 + \frac{\lambda}{r^2}\right]R = 0 \quad \text{(Spherical)} \quad (5.2)$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) - \left[k^2 + \frac{\nu^2}{\rho^2}\right]R = 0 \quad \text{(Cylindrical)} \quad (5.3)$$

How they come up?

- By making the following substitutions:

$$R(r) = \zeta^{-1/2} I(\zeta), \quad \zeta = kr, \quad \lambda = \sqrt{\nu^2 - 1/4} \quad \text{(Spherical)}$$

$$R(\rho) = I(\zeta), \quad \zeta = k\rho \qquad \text{(Cylindrical)}$$

- Both equations are reduced to the so called **modified Bessel equation**.

$$\frac{d^2 I}{d\zeta^2} + \frac{1}{\zeta}\frac{dI}{d\zeta} + \left(1 + \frac{\nu^2}{\zeta^2}\right) I = 0 \qquad (5.4)$$

- This can be reduced to the Bessel equation once we put

$$\zeta = ix \text{ or } (\zeta = -ix), \; I(\zeta) = Cu(ix), \quad C = \text{constant}$$

Bessel Functions of Third Kind
Hankel Functions

In Bessel functions theory we introduce the so-called Hankel functions, which are defined as follows:

$$H_{\nu}^{(1)}\left(x\right) = J_{\nu}(x) + iN_{\nu}\left(x\right) \qquad x > 0 \qquad (5.5a)$$

$$H_{\nu}^{(2)}\left(x\right) = J_{\nu}(x) - iN_{\nu}\left(x\right) \qquad x > 0 \qquad (5.5b)$$

Because of the linear independence of the Bessel function of the first and second kind, the Hankel functions provide an alternative pair of solutions to the Bessel differential equation.

Hankel functions satisfy the same recurrence relations as the first and second kind Bessel functions. Why?

Modified Bessel Functions-a

- Modified Bessel functions are found as solutions to the modified Bessel equation

$$x^2 y_\nu'' + x y_\nu' - (x^2 - \nu^2) y_\nu = 0 \qquad (5.6)$$

which transforms to the original equation if x is replaced by ix. To avoid dealing with complex solutions in practical applications we have introduced the modified Bessel functions.

$$I_\nu(x) \equiv i^{-\nu} J_\nu\left(ix\right)$$

$$I_\nu(x) = \sum_{s=0}^{\infty} \frac{1}{s!\left(n+s\right)!}\left(\frac{x}{2}\right)^{\nu+2s}$$

Modified Bessel Functions-b

- The general solution of the modified Bessel function is expressed as follows with the help of the first and second order modified Bessel functions:

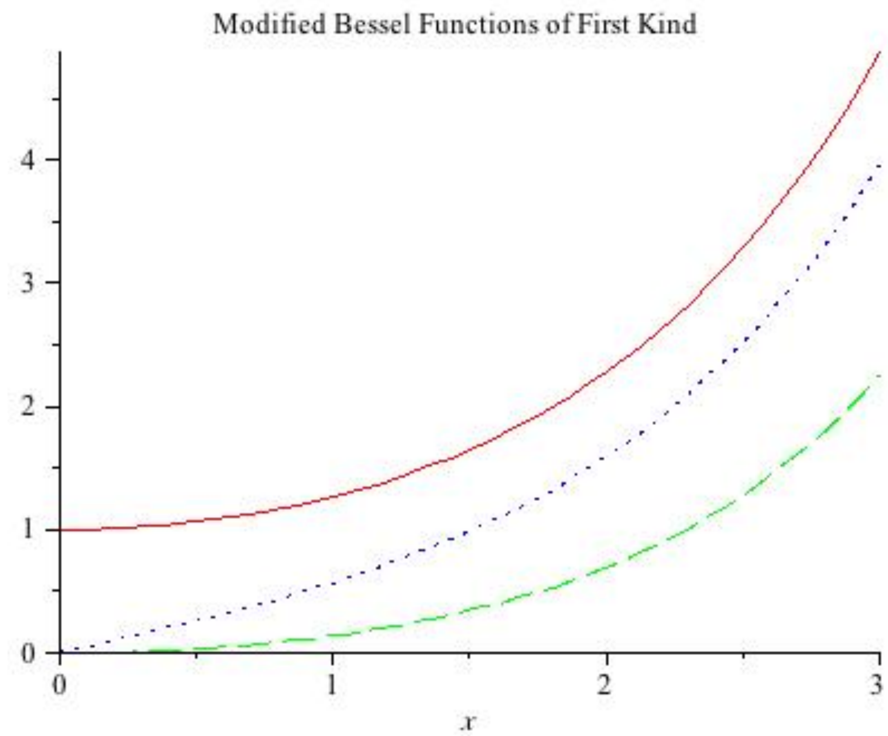
$$y(x) = CI_\nu(x) + DK_\nu(x) \qquad x > 0 \qquad (5.8)$$

- A solution for non-integer orders of ν is given by the modified Bessel functions of the second kind

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi} \qquad (5.9)$$

Modified Bessel Functions-c

- From the plot we see that the modified Bessel functions differ very much than the ordinary Besel functions.
- They do not have zeros (other than x=0 for some of them).
- They do not oscillate.
- They look like exponential functions for large values of *x*. That's why we call them **hyperbolic Bessel**.



Red solid: I_0
Blue dot: I_1
Green dash: I_2

Modified Bessel Functions
Recurrence relations

- We may show the following recurrence relation for the modified Bessel functions of first kind

$$I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_{\nu}(x) \qquad (5.10a)$$

$$I_{\nu-1}(x) + I_{\nu+1}(x) = 2I_{\nu}^{'}(x) \qquad (5.10b)$$

Modified Bessel Functions - Applications

Potential Applications

- Modified Bessel functions appear less frequently in applications, but can be found in transmission line studies, non-uniform beams, and the statistical treatment of a relativistic gas in statistical mechanics
- Problems involving the displacement of a vibrating membrane.
- Heat conduction in an annular fin of rectangular cross section attached to a circular base.

Zeroes of Bessel Functions

- The zeroes, or roots, of the Bessel functions are the values of x where value of the Bessel function becomes zero. Frequently the roots are given by in tabulated formats.
- Bessel functions of first and second kind have an infinite number of roots as x goes to infinity.
- The modified Bessel functions of the first kind have only one root at $x=0$, and the modified Bessel function of the second kind functions do not have zeroes.

Spherical Bessel Functions

When we try to solve the Helmholtz equation in spherical coordinates, the radial equation which comes out leads us to define the so called spherical Bessel equations:

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x), \quad n_n(x) = \sqrt{\frac{\pi}{2x}} N_{n+1/2}(x) = \left(-1\right)^{n+1} \sqrt{\frac{\pi}{2x}} J_{-n-1/2}(x)$$

$$h_n^{(1)}(x) = \sqrt{\frac{\pi}{2x}} H_{n+1/2}^{(1)}(x) = j_n(x) + in(x)$$

$$(5.11)$$

$$h_n^{(2)}(x) = \sqrt{\frac{\pi}{2x}} H_{n+1/2}^{(2)}(x) = j_n(x) - in(x)$$

Spherical Bessel Functions Orthogonality

- Spherical Bessel functions obey the following orthogonality relation:

$$\int_0^a j_n\left(a_{np}\frac{\rho}{a}\right) j_n\left(a_{nq}\frac{\rho}{a}\right)\rho^2 d\rho = \frac{a^3}{2}\left[j_{n+1}\left(a_{np}\right)\right]^2 \delta_{pq} \qquad (5.12)$$

- Where a_{np} and a_{nq} are roots of j_n

Spherical Bessel Functions Recurrence Relations

- Spherical Bessel obey the following recurrence relations:

$$f_{n-1}(x) + f_{n+1}(x) = \frac{2n+1}{x} f_n(x) \qquad (5.13a)$$

$$nf_{n-1}(x) - (n+1) f_{n+1}(x) = (2n+1) f_n'(x) \qquad (5.13b)$$

$$\frac{d}{dx}\left[x^{-n} f_n(x)\right] = -x^n f_{n+1}(x) \qquad (5.13c)$$

$$\frac{d}{dx}\left[x^{n+1} f_n(x)\right] = x^{n+1} f_{n-1}(x) \qquad (5.13d)$$

f stands for j, n, h

- These are useful for calculations of Bessel functions with small n.

Spherical Bessel Functions Recurrence Relations

- With the help of induction we can prove:

$$j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right) \qquad (5.14a)$$

$$n_n(x) = (-1)^{n+1} x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right) \qquad (5.14b)$$

$$h_n^{(1)}(x) = i(-1)^{n+1} x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{e^{ix}}{x}\right) \qquad (5.14c)$$

$$h_n^{(2)}(x) = i(-1)^{n+1} x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{e^{-ix}}{x}\right) \qquad (5.14d)$$

Bessel functions -Asymptotic expressions

- The Bessel functions satisfy the following asymptotic expressions:

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left[x - \left(n + \frac{1}{2}\right)\left(\frac{\pi}{2}\right)\right], \qquad \text{for } 8x >> 4n^2 - 1 \qquad (5.15\text{a})$$

$$j_n(x) \approx \frac{2^n n!}{(2n+1)!} x^n, \qquad x << 1 \qquad (5.15\text{b})$$

$$j_n(x) \approx \frac{1}{x} \sin\left(x - \frac{n\pi}{2}\right), \qquad x >> n(n+1)/2 \qquad (5.15\text{c})$$

Asymptotic expressions for spherical Bessel functions for large argument

- The spherical Bessel functions satisfy the following asymptotic expressions when $x >> n(n+1)/2$:

$$j_n(x) \approx \frac{1}{x}\sin\left(x - \frac{n\pi}{2}\right) \qquad n_n(x) \approx -\frac{1}{x}\sin\left(x - \frac{n\pi}{2}\right) \qquad (5.16\text{a})$$

$$h_n^{(1)}(x) \approx (-i)\frac{e^{i(x-n\pi/2)}}{x} \qquad h_n^{(1)}(x) \approx i\frac{e^{-i(x-n\pi/2)}}{x} \qquad (5.16\text{b})$$

- From these expressions we see that $j_n(x)$, $n_n(x)$ are appropriate for a description of **standing spherical waves**. While $h_n^{(1)}(x)$, $h_n^{(2)}(x)$ correspond to **traveling spherical waves**. If he time dependence is chosen to be $e^{-i\omega t}$ then $h_n^{(1)}(x)$ represents an outgoing traveling wave and , $h_n^{(2)}(x)$ represents an incoming traveling wave.