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Operations Research Letters 34 (2006) 264–268
Operations Research Letters
www.elsevier.com/locate/orl

A 1-center problem on the plane with uniformly distributed demand points

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Received 28 May 2004; accepted 30 April 2005
Available online 18 August 2005

Abstract

Center problems or minimax facility location problems are among the most active research areas in location theory. In this paper, we find the best unique location for a facility in the plane such that the maximum expected weighted distance to all random demand points is minimized.
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Keywords: Minimax facility location; Probabilistic facility location; Planar location

1. Introduction

Location theory is concerned with problems of determining optimal locations or optimal paths for one or more new service facilities so as to supply a given set of demand destinations (also called existing facilities). Depending on the application being modelled, the facilities and destinations may be nodes in a network or points in the real plane. Facility location problems can be divided into two main types: static facility location and moving facility location.

In this paper, we consider the problem of locating a new facility which interacts with a number of existing facilities (demand points) whose locations are not known; each demand point is assumed to have a bivariate uniform distribution in a given rectangle. The uniformity assumption provides a "building block" upon which to structure a system that is not adequately approximated directly. It plays a role analogous to the assumption of negative exponential service times in queueing systems. Our objective is the minimization of the maximum expected weighted distance to all demand points. Examples of such situation arise when locating a service facility which interacts with a number of potential demand points and it is not known which particular ones will request service.

There is a substantial literature on the minimax facility location problem but, to the best of our knowledge, the problem assessed here has not been studied in the literature. In this note we propose a solution to this problem. The weighted 1-median problem with straight-line distances (Euclidean distance) was

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doi:10.1016/j.orl.2005.04.011

introduced by Katz and Cooper [5,6]. The rectangular distance was discussed by Wesolowsky [7]. Berman et al. [2] have studied the weighted minimax (1-center) location problem in the plane with stochastic weights. Probabilistic demands for problems on the network were first introduced by Frank [3,4].

Following this introduction, the problem is analyzed in Section 2. The main results are described and illustrative examples are provided in this section.

2. Analysis

Let each of the n demand points Y_i have random coordinates (U_i, V_i) where U_i and V_i are independent random variables. Let $w_i > 0,\ i = 1, 2, \ldots, n$, be the demand for service. Assume that U_i (resp. V_i) are uniform random variables in the range $[a_i, b_i]$ with probability density function $f_{U_i}(\cdot)$ (resp. $f_{V_i}(\cdot)$), $i = 1, 2, \ldots, n$. We denote by $d_{\mathrm{R}}(Y_i, X)$ the rectilinear distance between demand point Y_i and unknown facility location $X = (x_1, x_2)$.

The problem can be stated as the following unconstrained nonlinear optimization problem:

$$\min_X F(X), \tag{1}$$

where

$$F(X) = \max_{1 \leqslant i \leqslant n} \{w_i E[d_{\mathrm{R}}(Y_i, X)]\},$$

and $E[\cdot]$ denotes the expected value.

2.1. The case when U_i and V_i, $1 \leqslant i \leqslant n$, are uniform over $[a, b]$

We start the analysis by considering the simple case when U_i and V_i are uniformly distributed over the interval $[a, b]$. The expected distance can be written as follows:

$$\begin{aligned} E[d_{\mathrm{R}}(Y_i, X)] &= E[|U_i - x_1| + |V_i - x_2|] \\ &= E[|U_i - x_1|] + E[|V_i - x_2|]. \end{aligned}$$

Now, we easily have

$$\begin{aligned} E[|U_i - x_1|] &= \int_{-\infty}^{\infty} |u - x_1| f_{U_i}(u)\,\mathrm{d}u \\ &= \frac{1}{b-a}\int_a^b |u - x_1|\,\mathrm{d}u \\ &= \frac{(x_1 - a)^2 + (x_1 - b)^2}{2(b-a)}, \end{aligned}$$

and similarly,

$$E[|V_i - x_2|] = \frac{(x_2 - a)^2 + (x_2 - b)^2}{2(b-a)},$$

which means

$$\begin{aligned} F(X) &= F(x_1, x_2) \\ &= \max_{1 \leqslant i \leqslant n} \left\{ w_i \left(\frac{(x_1 - a)^2 + (x_1 - b)^2}{2(b-a)} + \frac{(x_2 - a)^2 + (x_2 - b)^2}{2(b-a)} \right) \right\} \\ &= \max_{1 \leqslant i \leqslant n} \{w_i\} \times \left(\frac{(x_1-a)^2+(x_1-b)^2+(x_2-a)^2+(x_2-b)^2}{2(b-a)} \right). \end{aligned}$$

From this expression, it is clear that the optimal solution will be independent of $\{w_i\}$. Problem (1) therefore becomes

$$\begin{aligned} \min_{\{(x_1,x_2)\}} F(x_1, x_2) &= \max_{1 \leqslant i \leqslant n} \{w_i\} \\ &\times \left(\frac{(x_1-a)^2+(x_1-b)^2+(x_2-a)^2+(x_2-b)^2}{2(b-a)} \right), \end{aligned} \tag{2}$$

where $(x_1, x_2) \in [a, b] \times [a, b]$. The optimal solution to (2) is given by $x_1^* = (a+b)/2,\ x_2^* = (a+b)/2$ with $F(x_1^*, x_2^*) = ((b-a)/2)\max_{1 \leqslant i \leqslant n}\{w_i\}$. This result intuitively makes sense. Indeed, since the demand points are uniformly scattered in the square $[a, b] \times [a, b]$, it is plausible to have the facility located in the middle of this square.

2.2. *The case when U_i and V_i are uniform over $[a_i, b_i]$, $1 \leqslant i \leqslant n$*

As we will see shortly, the case when U_i and V_i are uniformly distributed over $[a_i, b_i]$ is not as straightforward as the previous case. First, we show that the optimal solution to problem (1) exists and is unique, and then we devise a simple procedure to determine this optimal solution. The expected distance is again given by

$$E[d_R(Y_i, X)] = E[|U_i - x_1|] + E[|V_i - x_2|]$$

and

$$\begin{aligned} E[|U_i - x_1|] &= \int_{-\infty}^{\infty} |u - x_1| f_{U_i}(u)\,\mathrm{d}u \\ &= \int_{a_i}^{b_i} \frac{|u - x_1|}{b_i - a_i}\,\mathrm{d}u \\ &= \frac{1}{b_i - a_i}\int_{a_i}^{b_i} |u - x_1|\,\mathrm{d}u. \end{aligned}$$

A logical interval for x_1 is $[\bar{a}, \bar{b}]$, where $\bar{a} = \min_i\{a_i\}$ and $\bar{b} = \max_i\{b_i\}$. Let us denote $E[|U_i - x_1|]$ (resp. $E[|V_i - x_2|]$) by the functional $f_i(x_1)$ (resp. $g_i(x_2)$) defined over $[\bar{a}, \bar{b}]$, $1 \leqslant i \leqslant n$. Then we have that

$$f_i(x_1) = \begin{cases} \dfrac{1}{b_i - a_i}\left(x_1 - \dfrac{a_i + b_i}{2}\right)^2 \\ \quad + \dfrac{b_i - a_i}{4}, & a_i \leqslant x_1 \leqslant b_i, \\ -x_1 + \dfrac{a_i + b_i}{2}, & \bar{a} \leqslant x_1 \leqslant a_i, \\ x_1 - \dfrac{a_i + b_i}{2}, & b_i \leqslant x_1 \leqslant \bar{b}, \end{cases} \tag{3}$$

and $g_i(x_2) = f_i(x_2)$.

Remark 2.1. The function $f_i(x_1)$ is strictly convex and differentiable over $\mathbb{R}$ and reaches its global minimum at $x_1^*(i) = (a_i + b_i)/2$.

Problem (1) can now be stated as follows:

$$\min_{\{(x_1, x_2)\}} F(x_1, x_2), \tag{4}$$

where $F(x_1, x_2) = \max_{1 \leqslant i \leqslant n}\{w_i(f_i(x_1) + f_i(x_2))\}$ and $(x_1, x_2) \in [\bar{a}, \bar{b}] \times [\bar{a}, \bar{b}]$. Clearly, $F(x_1, x_2)$ is strictly convex over $\mathbb{R}^2$. This implies that problem (4) has a unique global minimum $X^* = (x_1^*, x_2^*)$. We shall now show that $x_1^* = x_2^*$.

Table 1
Marginal distribution parameters of five points

i	a_i	b_i
1	2	10
2	7	9
3	7	12
4	3	15
5	1	14

Problem (4) can be transformed into an equivalent constrained nonlinear program given by

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & w_i(f_i(x_1) + f_i(x_2)) - z \leqslant 0, \\ & i = 1, 2, \ldots, n. \end{aligned} \tag{5}$$

Applying Kuhn–Tucker necessary conditions for problem (5) gives

$$1 - \sum_{i=1}^{n} u_i = 0, \tag{a}$$

$$\sum_{i=1}^{n} w_i u_i f_i'(x_1) = 0, \tag{b}$$

$$\sum_{i=1}^{n} w_i u_i f_i'(x_2) = 0, \tag{c}$$

$$(w_i(f_i(x_1) + f_i(x_2)) - z)u_i = 0, \quad i = 1, 2, \ldots, n, \tag{d}$$

$$w_i(f_i(x_1) + f_i(x_2)) - z \leqslant 0, \quad i = 1, 2, \ldots, n. \tag{e}$$

$$u_i \geqslant 0, \quad i = 1, 2, \ldots, n. \tag{f}$$

The u_i are the Lagrange multipliers. Conditions (b) and (c) and the fact that $\sum_{i=1}^{n} w_i u_i f_i'(x)$ is a strictly increasing function of x imply that at optimality $x_1 = x_2$. Therefore, in order to find the optimal solution to problem (4) it suffices to consider the minimization of $F(x_1, x_2)$ over the set of points satisfying $x_1 = x_2$. This leads to a univariate unconstrained optimization problem with a strictly convex function given by

$$\min_{x_1} G(x_1) = \max_{1 \leqslant i \leqslant n}\{w_i f_i(x_1)\}, \tag{6}$$

where $G(x_1) = F(x_1, x_1)$. Note that $G(x_1)$ is a piecewise nonlinear function. Solution to problem (6) can

Table 2
Steps in golden section

Iteration k	α_k	β_k	x_k	y_k	$G(x_k)$	$G(y_k)$
1	1	15	6.348	9.652	3.586	3.667
2	1	9.652	4.305	6.348	5.195	3.586
3	4.305	9.652	6.348	7.609	3.586	3.250
4	6.348	9.652	7.609	8.389	3.250	3.310
5	6.348	8.389	7.127	7.609	3.292	3.250
6	7.127	8.389	7.609	7.906	3.250	3.250
7	7.127	7.906	7.424	7.609	3.255	3.250
8	7.424	7.906	7.609	7.721	3.250	3.253
9	7.424	7.721	7.537	7.609	3.250	3.250
10	7.424	7.609	7.494	7.537	3.250	3.250
11	7.424	7.537	7.467	7.494	3.250	3.250
12	7.424	7.494				

be found using one of the well-known univariate search techniques based on function evaluations.

Remark 2.2. x_1^* lies on the interval $[\bar{a}, \bar{b}]$. Suppose on the contrary that $x_1^* < \bar{a}$ (a similar argument holds for the case $x_1^* > \bar{b}$). Since $G(x_1)$ is a piecewise nonlinear strictly convex function, therefore one of the pieces with positive first-order derivative (negative first-order derivative in case $x_1^* > \bar{b}$) forming the curve $G(x_1)$ and corresponding to some function $w_{i_0} f_{i_0}(x_1)$, $(1 \leqslant i_0 \leqslant n)$ will have its minimum point outside $[\bar{a}, \bar{b}]$. But by Remark 2.1, $w_{i_0} f_{i_0}(x_1)$ attains its global minimum at $x_1^*(i_0) = (a_{i_0} + b_{i_0})/2 \in [a_{i_0}, b_{i_0}] \subseteq [\bar{a}, \bar{b}]$. This is a contradiction. Therefore, the above assumption: $x_1^* < \bar{a}$ is not true and hence $x_1^* \in [\bar{a}, \bar{b}]$.

Example 2.1. Consider five demand points with random coordinates $Y_i = (U_i, V_i)$ specified by a bivariate uniform distribution over square regions $[a_i, b_i] \times [a_i, b_i]$, $1 \leqslant i \leqslant 5$. Table 1 gives the data for the example. To simplify the computations, all the weights w_i have been taken equal to 1. The golden section search technique [1] is applied to problem (6) with initial interval of uncertainty $[\alpha_1, \beta_1] = [\bar{a}, \bar{b}] = [1, 15]$ and the final length of uncertainty $l = 0.1$. At each iteration k, the interval of uncertainty is denoted by $[\alpha_k, \beta_k]$ and the readings by $[x_k, y_k]$. Table 2 shows the steps of the algorithm as applied to find the minimum of $G(x_1)$. After eleven iterations involving twelve observations, the interval of uncertainty is [7.424, 7.494], so that the minimum x_1^* could be estimated to be the midpoint 7.459 with the objective value equal to 3.250. This gives $x_2^* = 7.459$ and the optimal facility location that solves problem (4) is $(x_1^*, x_2^*) = (7.459, 7.459)$.

2.3. The case when U_i is uniform over $[a_i, b_i]$ and V_i is uniform over $[c_i, d_i]$, $1 \leqslant i \leqslant n$

In this case and in the light of the previous section, problem (1) can be stated as follows:

$$\min_{\{(x_1, x_2)\}} F(x_1, x_2), \tag{7}$$

where

$$F(x_1, x_2) = \max_{1 \leqslant i \leqslant n} \{w_i(f_i(x_1) + g_i(x_2))\},$$

and $(x_1, x_2) \in [\bar{a}, \bar{b}] \times [\bar{c}, \bar{d}]$, where $\bar{c} = \min_i\{c_i\}$ and $\bar{d} = \min_i\{d_i\}$. Also, $f_i(x_1)$ is given by (3) and $g_i(x_2)$ is given by

$$g_i(x_2) = \begin{cases} \dfrac{1}{d_i - c_i}\left(x_2 - \dfrac{c_i + d_i}{2}\right)^2 \\ \quad + \dfrac{d_i - c_i}{4}, & c_i \leqslant x_2 \leqslant d_i, \\ -x_2 + \dfrac{c_i + d_i}{2}, & \bar{c} \leqslant x_2 \leqslant c_i, \\ x_2 - \dfrac{c_i + d_i}{2}, & d_i \leqslant x_2 \leqslant \bar{d}. \end{cases} \tag{8}$$

Remark 2.3. The function $F(x_1, x_2)$ is strictly convex, being the maximum of strictly convex functions.

Table 3
Marginal distribution parameters of three points

i	a_i	b_i	c_i	d_i
1	1	4	3	11
2	2	10	4	9
3	7	12	1	4

Table 4
Steps in Hooke and Jeeves method

Iteration k	$(x_1^{(k)}, x_2^{(k)})$	$F(x_1^{(k)}, x_2^{(k)})$
1	(6.5, 6.0)	6.50
2	(6.7, 5.8)	6.38
3	(6.7, 5.8)	6.38
4	(6.5, 5.8)	6.30
5	(6.5, 5.8)	6.30
6	(6.4, 5.5)	6.18
7	(6.2, 5.3)	6.10
8	(6.1, 5.1)	6.05
9	(5.9, 4.9)	6.00
10	(5.8, 4.7)	5.96
11	(5.6, 4.5)	5.90
12	(5.4, 4.2)	5.88

This implies that problem (7) has a unique global minimum.

The solution to problem (7) can be found using some multi-dimensional search technique. In the following example, we use the method of Hooke and Jeeves, with discrete steps [1] to solve problem (7).

Example 2.2. Consider three demand points with random coordinates $Y_i = (U_i, V_i)$ specified by a bivariate uniform distribution over rectangular regions $[a_i, b_i] \times [c_i, d_i]$, $1 \leqslant i \leqslant 3$. Table 3 gives the data for the example. For simplicity, the weights w_i have again been taken equal to 1. Table 4 shows the steps in the method of Hooke and Jeeves. Here the subscript k denotes the iteration number and $(x_1^{(k)}, x_2^{(k)})$ are the iterates generated by the algorithm. After twelve iterations the optimal solution $(5.4, 4.2)$ is reached and the corresponding objective function value is 5.88.

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