Ex Suppose that an investor will receive payments given in the following table

Year	1	2	3	4	5	6
Payments	465	233	632	365	334	248

If the interest rate is ~~39.9~~ 3.99% compounded monthly. What is the present value of the investment?

Answ

1 2 3 4 5 6

$$465\left(1+\frac{0.0399}{12}\right)^{-12}$$

$$+ \quad 233\left(1+\frac{0.0399}{12}\right)^{-24}$$

$$+ \quad 632\left(1+\frac{0.0399}{12}\right)^{-36}$$

$$+ \quad 365\left(1+\frac{0.0399}{12}\right)^{-48}$$

$$334\left(1+\frac{0.0399}{12}\right)^{-60} \quad -72$$

Hom Suppose you have the choice of investing $1000 in just one of 2 ways. Each investment will pay you an amount given by the following table

Year	1	2	3	4	5
Investment A	225	215	250	225	205
Investment B	220	225	250	250	210

a) using the present value of the investment to make the decision, which investment would you choose? we assume that the annual interest rate is 4.33 %

b) Using the rate of return per year of the investment to make the decision, which investment would you choose?

The rate of return is a profit on an investment over a period of time.

If someone ~~boor~~ invest P now and receive a sequence of positive payoffs $\{A_1, A_2, \dots, A_n\}$ at regular intervals.

In this case the rate of return per period is the interest rate such that the present value of the sequence of payoffs is equal to the amount invested.

In this case:

$$P = \sum_{i=1}^{n} A_i \, (1+r)^{-i}$$

$\Rightarrow$ r will be a solution of the equation

$$f(r) = 0 \quad \text{where} \quad f(r) = -P + \sum_{i=1}^{n} A_i (1+r)^{-i}$$

<u>Note</u> r is called zero of f.

We can see that:

- $f(r)$ is continuous on $(-1, \infty)$
- $\lim_{r \to -1^+} f(r) = \infty$, $\lim_{r \to \infty} f(r) = -P < 0$

By the intermediate value theorem, there exist r^*, with $-1 < r^* < \infty$ such that $f(r^*) = 0$

<u>Note</u> r^* is unique

<u>Note</u> The return can be used to compute interest rate (compounded):

$$K(0, 1/m) = r/m$$

3) For continuous compounded interest

$$K(s,t) = e^{r(t-s)} - 1$$

<u>Remark</u> For continuous and discrete compounded interest, the return fails to be additive

Example: $K(0,1) = e^r - 1$ $\quad K(1,2) = e^r - 1$

$K(0,2) = (e^r)^2 - 1$

We have $K(0,1) + K(1,2) \neq K(0,2)$

It will be more convenient to introduce the logarithmic return:

$$K(s,t) = \ln\left[\frac{V(t)}{V(s)}\right]$$

<u>Proposition</u> The logarithmic return is additive:

$$K(s,t) + K(t,u) = K(s,u)$$

<u>Proof</u> $K(s,t) + K(t,u) = \ln \frac{V(t)}{V(s)} + \ln \frac{V(u)}{V(t)}$

$= \ln \frac{V(u)}{V(s)}$

$= K(s,u)$

Hom

You want to endow a fund which pays out a scholarship of \$1000 every year in perpetuity. The first scholarship will be paid out in five years' time. Assume that the interest rate is 7%, how much do you need to pay into the fund.

5) Return - Rate of Return

The return on an investment commencing at time s and terminating at t is defined by:

$$K(s,t) = \frac{V(t) - V(s)}{V(s)} = \frac{V(t)}{V(s)} - 1$$

Note $\quad V(t) = V(s) + K(s,t)\, V(s)$

Examples

1) For simple interest:

$$K(s,t) = \frac{\cancel{P}(1+rt)}{\cancel{P}(1+rs)} - 1 = \frac{r(t-s)}{1+rs}$$

2) In the case of compounding interest

$$K(s,t) = \left(1+r/m\right)^{(t-s)m} - 1$$

4) Perpetuity

A perpetuity is an infinite sequence of payments of a fixed amount C occuring at the end of each year.

The present value of a perpetuity can be obtaind as follows:

$$P_v = \sum_{i=1}^{\infty} c\left(1+\frac{r}{1}\right)^{-i.1}$$

$$\Rightarrow P_v = c \sum_{i=1}^{\infty} (1+r)^{-i} \quad : \text{Infinite geometric series}$$

We recall that:

if $\sum_{i=1}^{\infty} a^i$, then $S_n = \sum_{i=1}^{n} a^i$

$$= a\frac{(1-a^n)}{1-a}$$

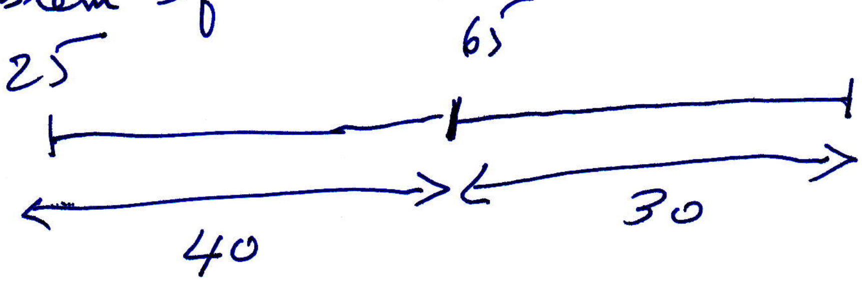
and if $|a|<1$, then $a^n \xrightarrow[n \to \infty]{} 0$.

Hence $\sum_{i=1}^{\infty} a^i = \frac{a}{1-a}$

In our Case $a = \frac{1}{1+r} \Rightarrow \frac{a}{1-a} = \frac{1}{r}$

$$\Rightarrow \boxed{P_v = \frac{c}{r}}$$

Problem of retirement



$$r = 10\%$$

x : deposit amount per month

x ?

x will be calculated by the equation :

$$\underline{P}_V\left(\underset{25 \to 65}{40 \text{ years}}\right) = P_V\left(\underset{65 \to 95}{30 \text{ years}}\right)$$

$$\sum_{i=1}^{480} x\left(1+\frac{0.1}{12}\right)^{-i} = x\,\frac{12}{0.1}\left[1-\left(1+\frac{0.1}{12}\right)^{-480}\right]$$

$$\|$$

$$\sum_{i=481}^{840} 1500\left(1+\frac{0.1}{12}\right)^{-i} = 1500\,\frac{12}{0.1}\left[\left(1-\left(1+\frac{0.1}{12}\right)^{-360}\right)\right]$$

$\Rightarrow \quad x = 27.03$ \$ per month

$$S = a + \cancel{a^2} + \cancel{a^3} + \cdots + \cancel{a^n}$$

$$-aS = \cancel{a^2} + \cancel{a^3} \cdots \cancel{+} a^{n+1}$$

$$(1-a)\, S = a - a^{n+1}$$

$$\boxed{S = \frac{a\left(1-a^n\right)}{1-a}}$$

$$P_V = \sum_{i=1}^{mt} x\left(1+r/m\right)^{-i} =$$

$$= x\left(1+r/m\right)^{-1} \frac{\left[1-\left(1+r/m\right)^{-mt}\right]}{1-\underbrace{\left(1+\frac{r}{m}\right)}^{-1}}$$

$$\left(\frac{m+r}{m}\right)^{-1}$$

$$1-\frac{m}{m+r}$$

$$\frac{\cancel{m}+r-\cancel{m}}{m+r}$$

$$\frac{r}{m+r}$$

$$= \frac{r}{m}\left(\frac{1}{1+r/m}\right)$$

The monthly payments can be calculated as follows:

x : the fixed monthly payments

$$P = \begin{array}{l} x\left(1+\frac{r}{m}\right)^{-1} \\ x\left(1+r/m\right)^{-2} \end{array}$$

$$P_v = \sum_{i=1}^{mt} x\left(1+r/m\right)^{-i}$$

$$S = 1 + \cancel{a} + \cancel{a^2} + \cancel{a^3} + a^4 + \dots + a^n$$

$$- \quad aS = \cancel{a} + \cancel{a^2} + \cancel{a^3} \dots + a^{n+1}$$

$$(1-a)S = 1 - a^{n+1}$$

$$S = \frac{1-a^{n+1}}{1-a}$$

$$S = a + a^2 + \dots + a^n$$

$$\Rightarrow S = \frac{a\left(1-a^n\right)}{1-a}$$

$$\boxed{P_V = x\,\frac{m}{r}\left[1-\left(1+\frac{r}{m}\right)^{-mt}\right]}$$

Annuity can be use to ~~calculate~~ the concept of retirement

<u>Ex</u> Suppose a person is 25 years today plan to retire at age 65 years. For the next 40 years he plan to invest a portion of her monthly incomes in securities which earn interest at the rate of 10% compounded monthly. After retirement the person plans to receive \$ 1500 per month for 30 years. What will be the monthly deposit of the person.

$\Rightarrow \quad P_V = \$ 2003.31$

Note $\quad P_V \neq$ Sum of payments $\$ 2277$

4) Annuities

An annuity is a sequence of finitely many payments of a fixed amount due at equal time intervals.

Suppose that some-one borrows P from the bank with interest r compounded m times per year

Ans 1) $P_V = \left(1 - \frac{1}{12} 0.08\right) 6000 = 5960$

2) $5960 = (1-d)^{1/12} \cdot 6000$

$\Rightarrow d = 0.077131$

rate of discount : $d \simeq 7.71\ \%$

3) The rate of compound interest r follows from $\frac{1}{1+r} = 1-d \Rightarrow$

$r = 0.0836 \simeq 8.36\%$

4) The rate of simple interest:

$5960 = \left(1 + \frac{1}{12} r\right)^{-1} 6000$

$\Rightarrow r = 8.05\ \%$

Hom The commercial rate of discount per annum is 18 %.

(a) We borrow a certain amount. The loan is settled by a payment of \$ 1000 after 3 months. Compute the amount borrowed and the effective annual rate of discount

(b) Now, the loan is settled by a payment of \$ 1000 after ~~nine~~ 9 months.

Answer the Same question.

Question Which amount x accumulates to C in n year? (with r as compounding interest)

Ans $(1+r)^n x = C \Rightarrow x = \frac{C}{(1+r)^n} = v^n C$

$x = (1-d)^n C$

This is called compound discounting, analogous the compound interest.

There is another method called simple discounting (analogous to simple interest) or commercial discounting defined as follows:

The present value of a payment of C due in n years at a rate of simple discount d is $(1-nd)C$

Note that simple discount is not the same as simple interest.

Example What is the present value of \$ 6000 due in a month assuming 8% per annum as a simple discount? What is the corresponding rate of compound discount? and the rate of compound interest? and the rate of simple interest?

Now if interest have to be paid one year earlier, the equivalent amount is:

Ans 1 : $70 \cdot \frac{1}{1+0.07} = \$65.42$

Ans 2 $C - vC = 1000 - \frac{1}{1+0.07} \times 1000$

$= 1000 - 934.58 = \$65.42$

Proposition The rate of discount d is:

$$d = rv$$

Proof

$$v = \frac{1}{1+r}$$

$$d = 1 - v = \frac{r}{1+r} = rv$$

We summarize the previous discussion:

- rate of interest : interest paid at the end of a time unit
- rate of discount : interest paid at the begining of the time
- discount factor : the amount of money one needs to invest to get one unit of capital after 1 time unit.

In the previous question:

$$r = 4.25\%$$

$$v = \frac{1}{1+r} = 0.95923$$

Note that $v < 1$, therefore people often use the rate of discount d:

$$d = 1 - v$$ (usually expressed as a percentage)

In our example, the rate of discount

$$d = 1 - 0.95923 = 0.04077$$
$$= 4.077\%$$

Note There are some situations in which the interest is paid in advance, the rate of discount is useful in these situations

Example

Suppose that you borrow \$ 1000 for a year with $r = 7\%$ and you have to pay interest at the start ~~interest~~ of the year.
How much do you have to pay?

Ans We recall that if interest were to be paid in arrears, then:

$$I = 1000 \times 0.07 \times 1 = \$\,70$$

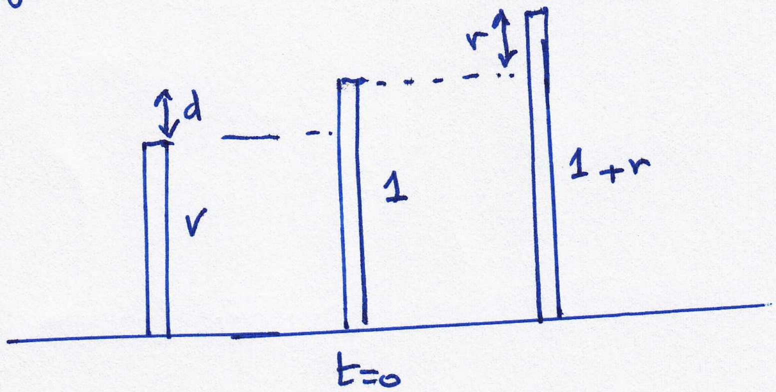
Discounting - Accumulating

Question : How much do you need to invest now to get \$ 2000 after 5 years if the rate of interest is 4.25 %

Ans $(1+0.0425)^5 \, C = 2000$

$C = \$\, 1624.24$

- When you move a payment forward, it accumulates
- When you move it backward, it is discounted



$v =$ discount factor of 1 unit capital (per year)

$d =$ rate of discount

. Generally, we need to invest $\frac{1}{1+r} C$ to get the capital C after 1 time unit (here 1 year)

. The factor $v = \frac{1}{1+r}$ is known as the discount factor. It is the factor with which you have to multiply a payment to shift it backward by one year.

Solution of Ex

r_A ?

$$\underbrace{1000 - \frac{225}{1+r} - \frac{215}{(1+r)^2} - \frac{250}{(1+r)^3} - \frac{225}{(1+r)^4} - \frac{205}{(1+r)^5}}_{f(r)} = 0$$

Newton's Method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$f'(r) = \frac{225}{(1+r)^2} + \frac{430}{(1+r)^3} + \frac{750}{(1+r)^4} + \frac{900}{(1+r)^5} + \frac{1025}{(1+r)^6}$$

We choose $x_1 = 0$

$$x_2 = 0 - \frac{f(0)}{f'(0)}$$

$$\simeq 0.03604$$

$$x_3 = 0.03604 - \frac{f(0.03604)}{f'(0.03604)}$$

$$x_3 = 0.0393299$$

We can say that $r_A \simeq 0.03 \simeq 3\%$

We compute r_B by choosing $x_1 = 0$ also.

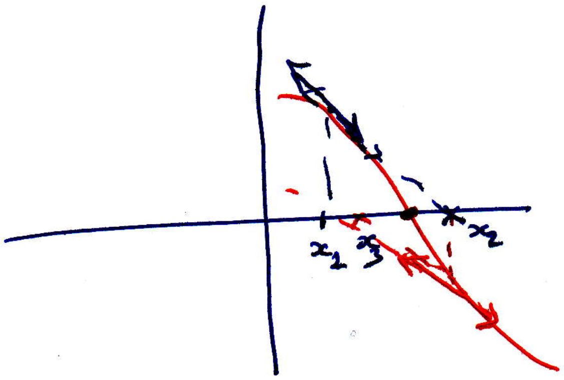
we obtain $r_B \simeq 4\%$. Then:

We choose Investment B Since

$$r_B > r_A.$$

Newton's Method to approximate a solution of

$f(r) = 0$



$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$\vdots$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Ex $\quad f(x) = x^7 - 1000$

$f(2) > 0$
$f(3) < 0$ $\Rightarrow x_0 \in [2, 3]$

$$x_{n+1} = x_n - \frac{x_n^7 - 1000}{7x_n^6}$$

$x_1 = 3$

$$x_2 = 3 - \frac{3^7 - 1000}{7\ 3^6}$$

$= 2.76739173$

$$x_3 = 2.76739173 - \frac{(2.76739173)^7 - 1000}{7(2.76\ldots)^6}$$

$\simeq 2.69008741$

$x_4 \simeq 2.68275645$
$x_5 \simeq 2.68269580$
$\simeq 2.68269580$

$\Rightarrow$ An approximate solution is $x_6 = 2.68269580$