King Saud University
Department of Mathematics

218
Final Exam, December 2021
1 pm - 4 pm

NAME: Answer sheet

Group Number/Instructors's name:

ID:

Number:

Question	I	II	III	IV	V	Total
Grade	9	8	9	5	9	40

I) A) Find the absolute maximum and minimum values of $f(x) = 12 + 4x - x^2$, for $x \in [0, 5]$.

$f'(x) = 4 - 2x$

$f'(x) = 0 \Leftrightarrow 4 - 2x = 0 \Leftrightarrow x = 2$

$f(0) = 12$
$f(2) = 16$
$f(5) = 7$

1,5

x	0		2		5
$f'(x)$		+	0	−	
$f(x)$	12	$\nearrow$	16	$\searrow$	7

$f(2) = 16$ is the absolute maximum value.
$f(5) = 7$ is the absolute minimum value.

B) Compute the area of the triangle determined by the points $A(1, 2, 3)$, $B(2, 1, 3)$ and $C(3, 1, 2)$.

$\overrightarrow{AB} = (1, -1, 0)$
$\overrightarrow{AC} = (2, -1, -1)$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 2 & -1 & -1 \end{vmatrix} = i + j + k$$

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Area of triangle (ABC) is $\frac{1}{2} \| \overrightarrow{AB} \times \overrightarrow{AC} \| = \frac{\sqrt{3}}{2}$.

C) Show that $\cosh^2 x - \sinh^2 x = 1$.

let $x \in \mathbb{R}$,

$\cosh x = \frac{e^x + e^{-x}}{2}$; $\sinh x = \frac{e^x - e^{-x}}{2}$

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$\cosh^2 x - \sinh^2 x = (\cosh x - \sinh x)(\cosh x + \sinh x)$

$= e^x \cdot e^{-x}$

$\cosh^2 x - \sinh^2 x = e^{x-x} = e^0 = 1$

D) Let $f(x) = \begin{cases} -2x^2+2 & , \quad x<1 \\ (x-1)^2 & , \quad x>1 \\ 2 & , \quad x=1 \end{cases}$

i) Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

0,5

- $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (-2x^2+2) = 0$
- $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1)^2 = 0$

ii) Does $\lim_{x\to 1}$ f(x) exist? Justify your answer.

$\lim_{x\to 1} f(x) = 0$ since $\lim_{1^-} f = \lim_{1^+} f = 0$

1,5

iii) Is f continuous at $x=1$? Justify your answer.

As $\lim_{x\to 1} f(x) = 0 \neq f(1) = 2$

So f is not continuous at $x=1$.

E) Let $z = 1+i$

i) Plot z in a system of coordinates.

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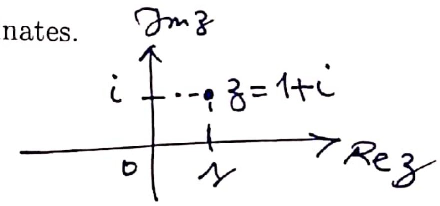

ii) Compute $|z|$, $\overline{z}$ and z^2.

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- $|z| = \sqrt{z\overline{z}} = \sqrt{2}$
- $\overline{z} = 1-i$
- $z^2 = (1+i)^2 = 2i$

iii) Write z in polar form.

$r = |z| = \sqrt{2}$

1

$\tan\theta = y/x = 1 \Rightarrow \theta = \pi/4$

$z = \sqrt{2}\left(\cos(\pi/4) + i\sin(\pi/4)\right) = \sqrt{2}\, e^{i\pi/4}$.

iv) Compute z^{400}.

3/4

$z^{400} = (\sqrt{2})^{400}\left(e^{i\pi/4}\right)^{400} = (\sqrt{2})^{400}\underbrace{\left(e^{i\pi}\right)^{100}}_{=1}$

$z^{400} = 2^{200}$.

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II) Evaluate the following limits, provided they exist.

0,5 i) $\lim_{x\to 1} \frac{x+5}{x-2} = \frac{1+5}{1-2} = \frac{4}{-1} = -4.$

1 ii) $\lim_{x\to \infty} \frac{x+4x^2}{1-2x^2} = \lim_{x\to\infty} \frac{4x^2}{-2x^2} = -2.$

1 iii) $\lim_{x\to 4} \frac{x^2-2x-8}{x-4} = \lim_{x\to 4} \frac{\cancel{(x-4)}(x+2)}{\cancel{(x-4)}} = 6.$ 1

2 iv) $\lim_{x\to 0} \frac{\sin 5x}{17x} = \frac{0}{0} \overset{\text{Hospital}}{=} \lim_{x\to 0} \frac{5\cos(5x)}{17x} = \frac{5}{17}.$

2,5 v) $\lim_{x\to \infty} \frac{\ln x}{\sqrt{x}} = \frac{\infty}{\infty} \overset{\text{Hospital}}{=} \lim_{x\to +\infty} \frac{1/x}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to+\infty} \frac{2}{\sqrt{x}} = 0.$

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III) Differentiate the following functions:

i) $f(x) = x^4 - 5x^2 + 2\sqrt{x}$

1,5 $f'(x) = 4x^3 - 10x + 2 \cdot \frac{1}{2\sqrt{x}} = 4x^3 - 10x + \frac{1}{\sqrt{x}}$

ii) $f(x) = e^x(x + x^2)$

$f'(x) = (e^x)'(x+x^2) + e^x(x+x^2)'$

1,5 $= e^x(x+x^2) + e^x(1+2x)$

$f'(x) = e^x(x^2+3x+1)$

iii) $f(x) = \dfrac{\sin x}{1 + \cos x}$

$f'(x) = \dfrac{\cos x(1+\cos x) + \sin x \sin x}{(1+\cos x)^2} = \dfrac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$

2 $f'(x) = \dfrac{1+\cos x}{(1+\cos x)^2} = \dfrac{1}{1+\cos x}$

iv) $f(x) = 10^{x \ln x}$

$f'(x) = 10^{x\ln x}(\ln 10)\frac{d}{dx}(x \ln x)$

2,5 $f'(x) = 10^{x\ln x}(\ln 10)(1+\ln x)$

v) $f(x) = [\ln(\cos x)]^2$

$f'(x) = 2\ln(\cos x)\frac{d}{dx}\ln((\cos x))$

1,5 $f'(x) = 2\ln(\cos x)\frac{1}{\cos x}(-\sin x) = -2\ln(\cos x)\tan x$

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IV) A) Use Gauss-Jordan elimination or Gaussian elimination to find the solution of the system of equations

$$\begin{cases} x + 8y = 2 \\ 3x + 14y = 16 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 8 & 2 \\ 3 & 14 & 16 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 8 & 2 \\ 0 & -10 & 10 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 8 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & -1 \end{array}\right] \Rightarrow \begin{array}{l} x = 10 \\ y = -1 \end{array}$$

B) Use Cramer's method to find y (**only** y) for the system

$$\begin{cases} x - z = 3 \\ 2x - y = 1 \\ 3x + y - z = 2 \end{cases}$$

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 0 \\ 3 & 1 & -1 \end{pmatrix} \quad ; \det A = -4.$$

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$$A_y = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & -1 \end{pmatrix} \quad ; \quad \det(A_y) = 4.$$

$$\text{So } y = \frac{\det(A_y)}{\det(A)} = -1$$

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V) Let $f(x) = \dfrac{x-1}{x^2}$.

i) Find the domain of f.

ii) Find the x-intercepts and the y-intercepts (if any).

iii) Find the horizontal asymptotes (if any).

iv) Find the vertical asymptotes (if any).

v) Find the intervals on which f is increasing and the intervals on which f is decreasing.

vi) Find the local minimum and the local maximum values (if any).

vii) Sketch the graph of f.

(i) $D_f = \mathbb{R} \setminus \{0\}$

(1) (ii) $\cdot\, x \neq 0$; no y-intercetpts

$\cdot\, f(x) = 0 \Leftrightarrow x - 1 = 0 \Leftrightarrow x = 1$ so $(1,0)$ is the x-intercept.

(iii) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x-1}{x^2} = 0$

(1.5) $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x-1}{x^2} = 0$ So $y=0$ is the horizontal asymptote.

(iv) $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} \dfrac{x-1}{x^2} = -\infty$

(1.5) $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} \dfrac{x-1}{x^2} = -\infty$ So $x=0$ is the vertical asymptote.

(v) $f'(x) = \dfrac{d}{dx}\left(\dfrac{x-1}{x^2}\right) = \dfrac{1x^2 - (x-1)(2x)}{x^4} = \dfrac{-x^2+2x}{x^4} = \dfrac{2-x}{x^3}$

(2) $f'(x) = 0 \Leftrightarrow x = 2$

x	$-\infty$		0		2		$+\infty$
$f'(x)$		$-$	‖	$+$	0	$-$	
$f(x)$	0	↘ $-\infty$	‖	$-\infty$ ↗	$\frac{1}{4}$	↘	0

(1) $\cdot\, f$ is increasing on $(0,2)$

$\cdot\, f$ is decreasing on $(-\infty,0) \cup (2,+\infty)$

(vi) $f(2) = 1/4$ is a local maximum value.

(1) f has no local minimum value.

vii)

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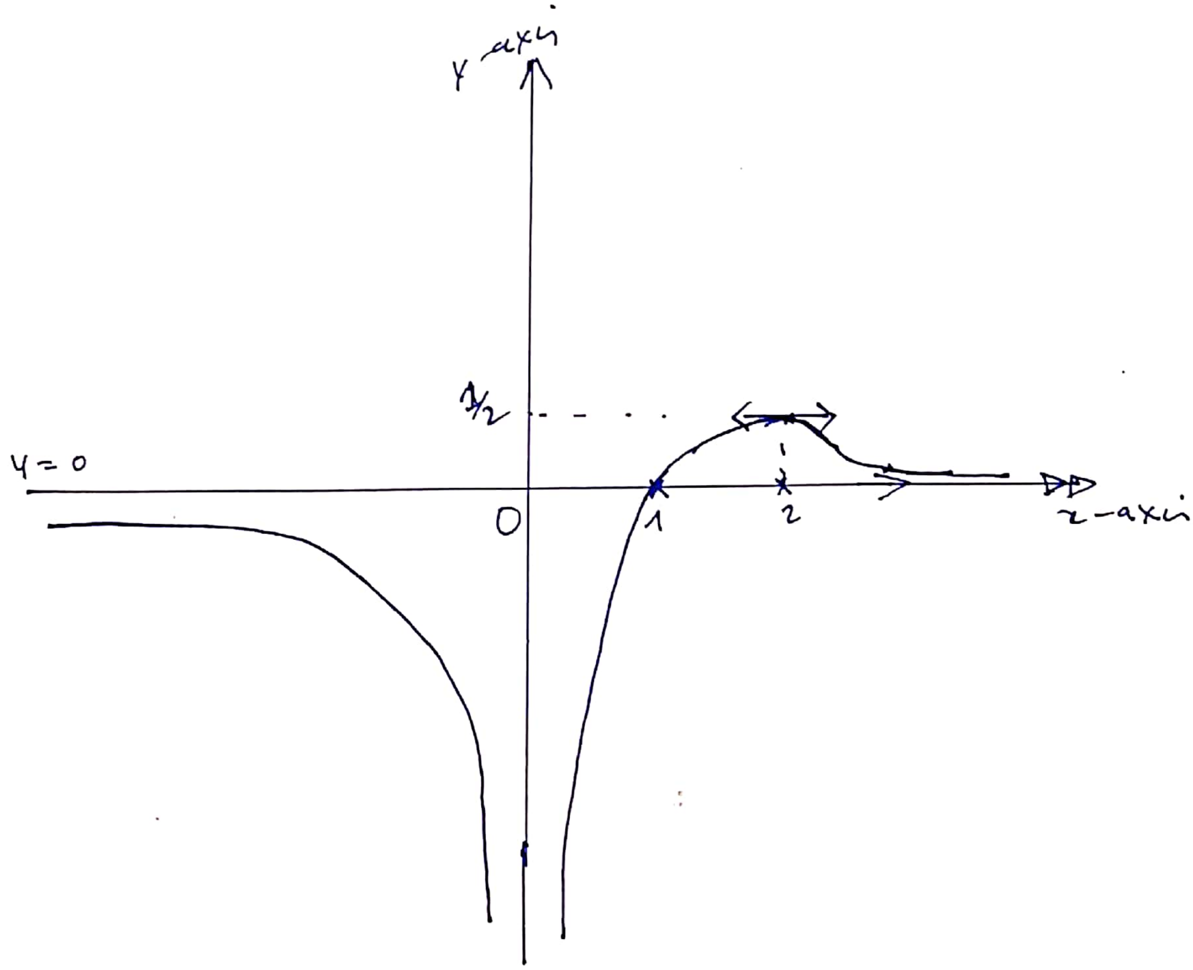

Graph of $f(x) = \frac{x-1}{x^2}$