

King Saud University

Department of Mathematics

First Midterm Exam(ANSWERS)**Course Title:** Math 111 (Integral Calculus)**Date:** Wednesday 14 October 2015; 3–4 $\frac{1}{2}$ pm**Instructions:**

- Students are NOT allowed to use CALCULATORS.

Name ID

Question	Grade
Q1	
Q2	
Q3	
Q4	
Total	

Question 1	(a)	(b)	(c)	(d)	(e)
Answer	(ii)	(ii)	(ii)	(i)	(iii)

Question 1

Choose the correct answer (write it down on the table above):

(a) $F(x) = (x + 1)^2$ is an antiderivative of

- (i) $x^2 + 2x + 1$
- (ii) $2(x + 1)$
- (iii) $\frac{1}{3}(x + 1)^3 + c$
- (iv) None.

Solution: (ii) $F'(x) = f(x) \implies f(x) = 2(x + 1)$.

(b) $\sum_{i=3}^7 i(i - 1)$ equals

- (i) 35
- (ii) 110
- (iii) 40
- (iv) None.

Solution: (ii)

$$\sum_{i=3}^7 i(i-1) = \sum_{i=1}^7 (i^2 - i) - \sum_{i=1}^2 (i^2 - i) = \left[\frac{7(8)(15)}{6} - \frac{7(8)}{2} \right] - \left[\frac{2(3)(5)}{6} - \frac{2(3)}{2} \right] = 112 - 2 = 110.$$

(c) If $f(x) = x$ and $g(x) = \sqrt{x}$, then

- (i) $\int_1^2 \sqrt{x} dx \geq \int_1^2 x dx$
- (ii) $\int_1^2 \sqrt{x} dx \leq \int_1^2 x dx$
- (iii) $\int_1^2 \frac{1}{x} dx \geq \int_1^2 \sqrt{x} dx$
- (iv) None.

Solution: (ii) Since $\sqrt{x} \leq x$ on the interval $[1, 2]$ then $\int_1^2 \sqrt{x} dx \leq \int_1^2 x dx$.

(d) $\int_0^{2\pi} \sin x dx$ equals

- (i) 0
- (ii) 1
- (iii) 2
- (iv) None.

Solution: (i) $\int_0^{2\pi} \sin x dx = -\cos x \Big|_0^{2\pi} = -(1 - 1) = 0$

(e) If $\int_1^6 g(x)dx = 3$ and $\int_1^9 g(x)dx = 10$, then $\int_6^9 g(x)dx$ equals

- (i) 3
- (ii) -7
- (iii) 7
- (iv) None

Solution: (iii) $\int_1^6 g(x)dx + \int_6^9 g(x)dx = \int_1^9 g(x)dx \implies \int_6^9 g(x)dx = \int_1^9 g(x)dx - \int_1^6 g(x)dx = 10 - 3 = 7$

Question 2

- (a) If $f(x)$ is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)dt$, prove that $F'(x) = f(x)$ on $[a, b]$.

Solution: From the book.

- (b) If $F(x) = (\sin x)^2 \int_0^x \sqrt{t}dt$

- (i) Find $F'(x)$.

Solution: Let $g(x) = (\sin x)^2$ and $h(x) = \int_0^x \sqrt{t}dt$, then

$$\begin{aligned} F(x) &= g(x)h(x) && \text{"Product of two functions"} \\ &= g'(x)h(x) + g(x)h'(x) \end{aligned}$$

Since $g'(x) = 2 \cos x \sin x$ and by The Fundamental Theorem of Calculus $h'(x) = \sqrt{x}$, Then

$$F'(x) = 2 \cos x \sin x \int_0^x \sqrt{t}dt + (\sin x)^2 \sqrt{x}.$$

- (ii) Find $F'(0)$.

Solution: $F'(0) = 2 \cos(0) \sin(0) \int_0^0 \sqrt{t}dt + (\sin(0))^2 \sqrt{0} = 0 + 0 = 0$

Question 3

(a) Find $\int_0^3 (x^2 + 1)dx$ using limit of Riemann sum and right endpoints. **Solution:**

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i)\Delta x$$

$$a = 0, \quad b = 3, \quad f(x) = (x^2 + 1).$$

$$1. \quad \Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

$$2. \quad c_i = x_i = x_{i-1} + \Delta x$$

$$x_0 = a = 0, \quad x_1 = \frac{3}{n}, \quad x_2 = \frac{3}{n} + \frac{3}{n} = 2\frac{3}{n}, \quad x_3 = \frac{3}{n} + 2\frac{3}{n} = 3\frac{3}{n}, \dots, \quad x_n = n\frac{3}{n} = 3 = b$$

this implies that $c_i = x_i = i(\frac{3}{n})$.

3.

$$\begin{aligned} \int_0^3 (x^2 + 1)dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(\frac{3i}{n}\right) \cdot \frac{3}{n} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\left(\frac{3i}{n}\right)^2 + 1\right) \cdot \frac{3}{n} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{9i^2}{n^2} + 1\right) \cdot \frac{3}{n} \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \left[\sum_{i=1}^n \frac{9}{n^2} i^2 + \sum_{i=1}^n 1 \right] \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \left[\frac{9}{n^2} \sum_{i=1}^n i^2 + \sum_{i=1}^n 1 \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{3}{n} \frac{9}{n^2} \frac{n(n+1)(2n+1)}{6} + \frac{3}{n} n \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{9}{n^2} \frac{(n+1)(2n+1)}{2} + \frac{3}{n} n \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{9(2n^2 + 3n + 1)}{2n^2} + 3 \right] \\ &= \lim_{n \rightarrow \infty} \frac{18n^2 + 27n + 9}{2n^2} + \lim_{n \rightarrow \infty} 3 \\ &= \frac{18}{2} + 3 = 9 + 3 = 12. \end{aligned}$$

- (b) Find the value of c that satisfies the Integral Mean Value Theorem for $f(x) = 4x^3 + 1$ on $[0, 1]$.

Solution:

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx \quad (1)$$

$a = 0$, $b = 1$ and

$$\int_0^1 f(x) dx = \int_0^1 (4x^3 + 1) dx = \left(4 \cdot \frac{x^4}{4} + x\right) \Big|_0^1 = \left(x^4 + x\right) \Big|_0^1 = (1^4 + 1) - (0^4 + 0) = 2$$

Substitute in (1)

$$\begin{aligned} f(c) = 4c^3 + 1 = 2 &\implies 4c^3 = 1 \\ &\implies c^3 = \frac{1}{4} \\ &\implies c = \sqrt[3]{\frac{1}{4}} \in (0, 1) \end{aligned}$$

Question 4

Evaluate the following integrals:

$$(i) \int \sqrt{x} \sqrt[3]{x} dx = \int (x)^{\frac{1}{2}} (x)^{\frac{1}{3}} dx = \int x^{\frac{1}{2} + \frac{1}{3}} dx = \int x^{\frac{5}{6}} dx = \frac{x^{\frac{5}{6} + 1}}{\frac{5}{6} + 1} + c = \frac{6}{11} x^{\frac{11}{6}} + c$$

$$(ii) \int \frac{1}{x \ln x} dx$$

$$\text{Let } u = \ln x \text{ then } \frac{du}{dx} = \frac{1}{x} \implies du = \frac{1}{x} dx$$

Substitute in the integral

$$\int \frac{1}{u} du = \ln |u| + c = \ln |\ln x| + c$$

$$(iii) \int_{-1}^1 f(x) dx, \text{ where } f(x) = \begin{cases} 2x & x > 0 \\ 1 & x \leq 0 \end{cases}$$

$$\int_{-1}^1 f(x) dx = \int_{-1}^0 dx + \int_0^1 2x dx = x \Big|_{-1}^0 + x^2 \Big|_0^1 = (0 - (-1)) + (1 - 0) = 1 + 1 = 2.$$

$$(iv) \int \frac{\cos x - \sin x}{\sin x + \cos x} dx.$$

Let $u = \sin x + \cos x$ then

$$\frac{du}{dx} = \cos x - \sin x \implies \int \frac{\cos x - \sin x}{\sin x + \cos x} dx = \int \frac{1}{u} \cdot \frac{du}{dx} dx = \int \frac{1}{u} du = \ln |u| + c = \ln |\sin x + \cos x| + c$$