KING SAUD UNIVERSITY

Math Department

First Semester 40/41

November 25th 2019

Time 90mn

Math106 Midterm2

Question 1(2+3+3)

a) Find $\lim_{x\to 0}(1+8x^2)^{\frac{1}{x^2}}$

b) Compute the integral $\int e^{4x} sinx dx$

c) Evaluate $\int (sinx)^2(cosx)^2 dx$

Question 2(3+3+2)

a) Evaluate the integral $\int \frac{\sqrt{x^2-25}}{x} dx$

b) Find $\int \frac{3x^2+7x+2}{(x+1)^2(x+3)} dx$

c) Compute $\int \frac{dx}{\sqrt{x(x+1)+1}}$

Question 3(3+3+3)

a) Find $\int \frac{dx}{x^{1/2}+x^{1/3}}$

b) Does the integral $\int_0^{\infty} \frac{xdx}{1+x^4}$ converge? Find its value if it does.

c) Compute the area of the region bounded by the curves: $y = x^2, y = x-1$

$y = 0$, and $y = 4$.

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Answer Sheet Math 106

Q1) a) $\lim_{x\to 0} (1+8x^2)^{\frac{1}{x^2}} = 1^{\infty}$ I.F

$\ln\left[(1+8x^2)^{\frac{1}{x^2}}\right] = \frac{1}{x^2}\ln(1+8x^2)$

$\lim_{x\to 0} \frac{\ln(1+8x^2)}{x^2} = \frac{0}{0}$ IF

$\lim_{t\to 0} \frac{\ln(1+8t)}{t}$ Hospital rule $t = x^2$ $= \lim_{t\to 0} \frac{\frac{8}{1+8t}}{1}$

$= 8$

So $\lim_{x\to 0} (1+8x^2)^{\frac{1}{x^2}} = e^8$

b) $I = \int e^{4x} \sin x \, dx$

Integration by parts

$u(x) = e^{4x}$ $\Rightarrow$ $u'(x) = 4e^{4x}$

$v'(x) = \sin x$ $\quad v(x) = -\cos x$

$I = -e^{4x}\cos x + 4\int e^{4x}\cos x \, dx$

$J = \int e^{4x} \cos x \, dx$

$u(x) = e^{4x}$ $\Rightarrow$ $u'(x) = 4e^{4x}$

$v'(x) = \cos x$ $\quad v(x) = \sin x$

$J = e^{4x}\sin x - 4\int e^{4x}\sin x \, dx$

$= e^{4x}\sin x - 4I$

So

$I = -e^{4x}\cos x + 4\left[e^{4x}\sin x - 4I\right]$

$I = -e^{4x}\cos x + 4e^{4x}\sin x - 16I$

$17I = e^{4x}\left[4\sin x - \cos x\right]$

$I = \int e^{4x}\sin x \, dx = \frac{1}{17}\left[e^{4x}(4\sin x - \cos x)\right] + C$

c) $M = \int \sin^2 x \cos^2 x \, dx$

$= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx$

$= \frac{1}{4}\int 1^2 - \cos^2(2x) \, dx$

$= \frac{1}{4}\int \sin^2(2x) \, dx$

$= \frac{1}{4}\int \frac{1-\cos 4x}{2} \, dx$

$= \frac{1}{8}\left[x - \frac{\sin(4x)}{4}\right] + C$

$= \frac{x}{8} - \frac{\sin(4x)}{32} + C$

Q2) a) $\int \frac{\sqrt{x^2-25}}{x} \, dx$

$x = 5\sec\theta$

$dx = 5\sec\theta\tan\theta \, d\theta$

$\sqrt{x^2-25} = \sqrt{25\sec^2\theta - 25}$

$= \sqrt{25(\sec^2\theta - 1)}$

$= \sqrt{5^2\tan^2\theta} = 5\tan\theta$

$\int \frac{\sqrt{x^2-25}}{x} \, dx = \int \frac{5\tan\theta}{\cancel{5}\cancel{\sec\theta}} \cancel{5\sec\theta}\tan\theta \, d\theta$

$= 5\int \tan^2\theta \, d\theta$

$= 5\int (\sec^2\theta - 1) \, d\theta$

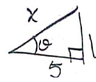
$= 5[\tan\theta - \theta] + C$

$= \cancel{5}\frac{\sqrt{25-x^2}}{\cancel{5}} - 5\sec^{-1}\left(\frac{x}{5}\right) + C$

As $x = 5\sec\theta$

$\Rightarrow \sec\theta = \frac{x}{5}$

$\theta = \sec^{-1}\left(\frac{x}{5}\right)$



$\cos\theta = \frac{5}{x}$

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b) $I = \int \frac{3x^2+7x+2}{(x+1)^2(x+3)}\,dx$

$= \int \left[\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}\right] dx$

$\stackrel{1}{=} A \ln|x+1| - \frac{\overset{-1}{B}}{x+1} + \underset{\underset{2}{\shortparallel}}{C} \ln|x+3| + cst$

Now we find A, B & C:

$f(x) = \frac{3x^2+7x+2}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$

$\forall x \in \mathbb{R} \setminus \{-1, -3\}$

$C = \lim_{x \to -3} (x+3) f(x) = \lim_{x \to -3} \frac{3x^2+7x+2}{(x+1)^2} = \frac{27-21+2}{4} = 2$

$B = \lim_{x \to -1} (x+1)^2 f(x) = \lim_{x \to -1} \frac{3x^2+7x+2}{x+3} = \frac{3-7+2}{2} = -1$

$f(0) = \frac{2}{3} = A + B + \frac{C}{3}$

$\cancel{\frac{2}{3}} = A - 1 + \cancel{\frac{2}{3}}$

$A = 1$

c) $I = \int \frac{dx}{\sqrt{x(x+1)+1}}$

$x(x+1)+1 = x^2 + x + 1$

$= \left(x+\frac{1}{2}\right)^2 - \frac{1}{4} + 1$

$= \left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 > 0$

$I = \int \frac{dx}{\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(x+\frac{1}{2}\right)^2}} = \sinh^{-1}\left(\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{3}}\right) + c$

$= \sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$

$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + c$

$\int_0^t \frac{x}{1+x^4}\,dx = \frac{1}{2}\tan^{-1}(t^2)$

$\lim_{t \to +\infty} \frac{1}{2}\tan^{-1}(t^2) = \pi/4$ So $\int_0^\infty \frac{x}{1+x^4}\,dx$

Converges and its value is $\pi/4$

Q3) (a) $\int \frac{dx}{x^{1/2} + x^{1/3}} = \int \frac{6u^5\,du}{u^3+u^2} = 6\int \frac{u^5}{u^2(u+1)}\,du$

$n = lcm(2,3) = 6$

We put $u^6 = x$

$6u^5\,du = dx$

$u^3 = x^{1/2}$

$u^2 = x^{1/3}$

$= 6\int \frac{u^3}{u+1}\,du$

$u^3 \;|\; u+1$

$-u^3 - u^2 \;|\; u^2 - u + 1$

$-u^2$

$u^2 - u$

u

$-u-1$

-1

$= 6\int \left[u^2 - u + 1 - \frac{1}{1+u}\right] du$

$= 6\left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln|1+u|\right] + c$

$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln|1+x^{1/6}| + c$

(b) $\int_0^\infty \frac{x}{1+x^4}\,dx = \lim_{t \to \infty} \int_0^t \frac{x}{1+x^4}\,dx$

$\int_0^t \frac{x}{1+x^4}\,dx = \int_0^t \frac{x\,dx}{1+(x^2)^2} \underset{u=x^2}{=} \frac{1}{2}\int_0^{t^2} \frac{du}{1+u^2}$

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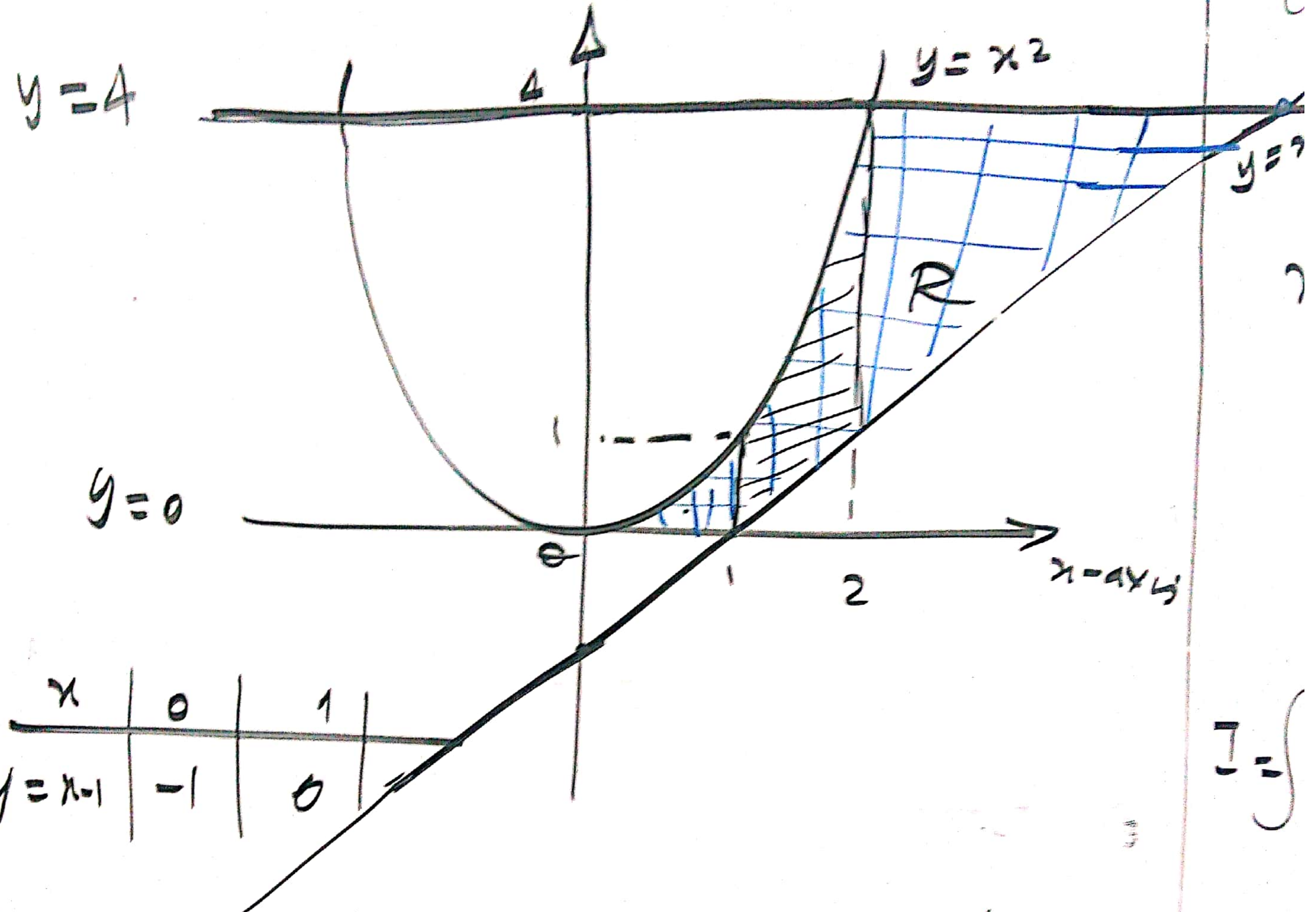
c) $y = x^2$; $y = x - 1$; $y = 0$; $y = 4$



x	0	1
$y = x-1$	-1	0

$$R_y = \left\{ (x,y) \middle/ \begin{array}{l} 0 \leq y \leq 4 \\ \sqrt{y} \leq x \leq y+1 \end{array} \right\}$$

The area is

$$A(R_y) = \int_0^4 \left[(y+1) - (\sqrt{y}) \right] dy$$

$$= \left[\frac{y^2}{2} + y - \frac{2}{3} y^{3/2} \right]_0^4$$

$$= 8 + 4 - \frac{2}{3}(2^2)^{3/2} = 12 - \frac{16}{3} = \frac{20}{3}$$