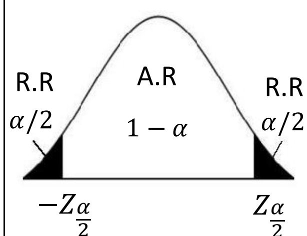
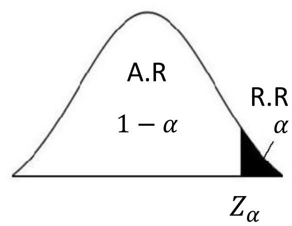
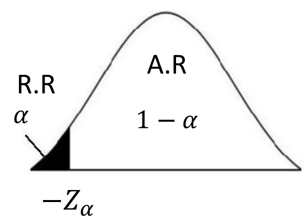
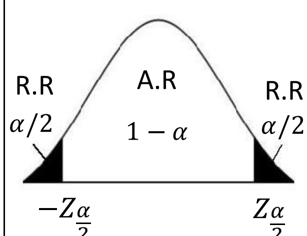
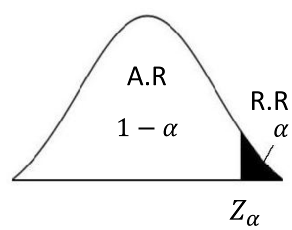
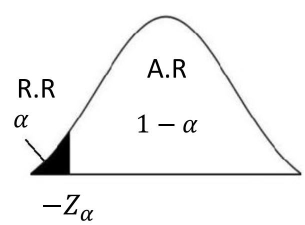
Tests of Hypotheses:

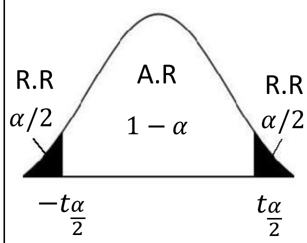
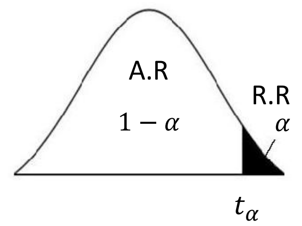
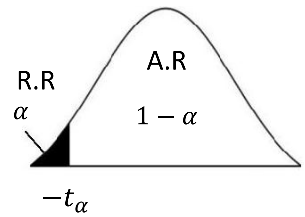
	H_0 **is true**	H_0 **is false**
Fail to reject H_0	Correct Decision	Type II error, β
Reject H_0	Type I error, α	Correct Decision

- $\boldsymbol{\alpha = P\,(Type\ I\ error) = P\,(rejecting\ H_0\ |H_0\ is\ true)}$.
- $\boldsymbol{\beta = P\,(Type\ II\ error)\ =\ P\,(accepting\ H_0\ |H_0\ is\ false)}$.

1. **Test Hypotheses for the population Mean (μ):**

Test Procedures:

Hypotheses	$H_0: \mu = \mu_0$ $H_a: \mu \neq \mu_0$	$H_0: \mu = \mu_0$ $H_a: \mu > \mu_0$	$H_0: \mu = \mu_0$ $H_a: \mu < \mu_0$
First Case	$\boldsymbol{\sigma^2}$ **is known; Normal or Non-normal Distribution**		
Test Statistic (T.S.)	$\boldsymbol{Z = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)}$		
Rejection Region (R.R) & Acceptance Region (A.R) of H_0			
Critical Value	$\boldsymbol{-Z_{\frac{\alpha}{2}}\ and\ \ Z_{\frac{\alpha}{2}}}$	$\boldsymbol{Z_\alpha}$	$\boldsymbol{-Z_\alpha}$
Decision	**We reject H_0 at the significant level α if:**		
	$\boldsymbol{Z < -Z_{\frac{\alpha}{2}}\ \ or\ \ Z > Z_{\frac{\alpha}{2}}}$	$\boldsymbol{Z > Z_\alpha}$	$\boldsymbol{Z < -Z_\alpha}$
Second Case	$\boldsymbol{\sigma^2}$ **is unknown; n large (n>30)**		
Test Statistic (T.S.)	$\boldsymbol{Z = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}} \sim N(0,1)}$		
Rejection Region (R.R) & Acceptance Region (A.R) of H_0			
Critical Value	$\boldsymbol{-Z_{\frac{\alpha}{2}}\ and\ \ Z_{\frac{\alpha}{2}}}$	$\boldsymbol{Z_\alpha}$	$\boldsymbol{-Z_\alpha}$
Decision	**We reject H_0 at the significant level α if:**		
	$\boldsymbol{Z < -Z_{\frac{\alpha}{2}}\ \ or\ \ Z > Z_{\frac{\alpha}{2}}}$	$\boldsymbol{Z > Z_\alpha}$	$\boldsymbol{Z < -Z_\alpha}$

Hypotheses	$H_0: \mu = \mu_0$ $H_a: \mu \neq \mu_0$	$H_0: \mu = \mu_0$ $H_a: \mu > \mu_0$	$H_0: \mu = \mu_0$ $H_a: \mu < \mu_0$
third Case	**σ^2 is unknown; Normal , n<30 small**		
Test Statistic (T.S.)	$t = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}} \sim N(0,1)$		
Rejection Region (R.R) & Acceptance Region (A.R) of H_0			
Critical Value	$-t_{\frac{\alpha}{2}}$ and $t_{\frac{\alpha}{2}}$	t_α	$-t_\alpha$
Decision	**We reject H_0 at the significant level α if:**		
	$t < -t_{\frac{\alpha}{2}}$ or $t > t_{\frac{\alpha}{2}}$	$t > t_\alpha$	$t < -t_\alpha$

Steps:

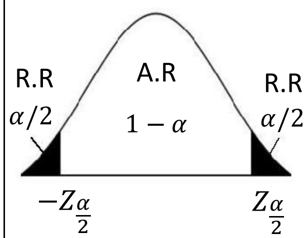
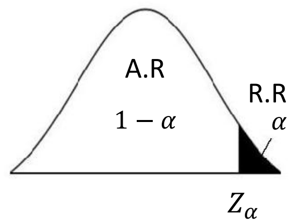
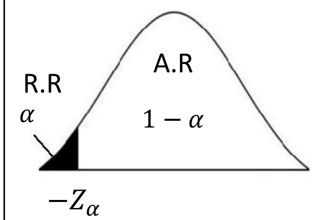
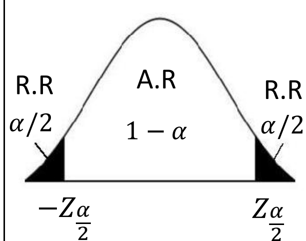
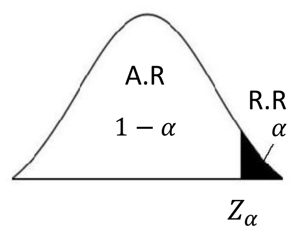
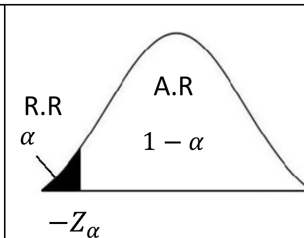
1. **Hypotheses**
2. **Assumptions**
3. **Test Statistic**
4. **Rejection region**
5. **Conclusion (Decision)**

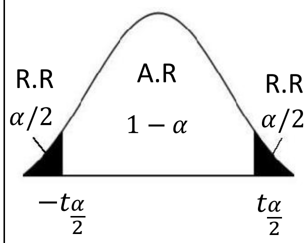
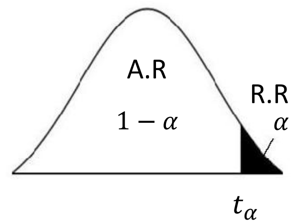
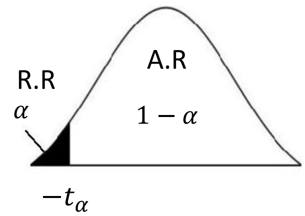
P-value:

Alternative Hypothesis	$H_a: \theta \neq \theta_0$	$H_a: \theta > \theta_0$	$H_a: \theta < \theta_0$
P-value =	$2P\ (Z\ > \lvert Z_{TS} \rvert)$	$P\ (Z > Z_{TS})$	$P\ (Z < Z_{TS})$
Significance Level	α		
Decision	Reject H_0 if a $P - value \leq \alpha$		

Z_{TS}: The Test Statistic

2. **<u>Test Hypotheses for the difference between two population Means ($\mu_1 - \mu_2$):</u>**
   Test Procedures:

Hypotheses	$H_0: \mu_1 - \mu_2 = 0$ $H_a: \mu_1 - \mu_2 \neq 0$	$H_0: \mu_1 - \mu_2 = 0$ $H_a: \mu_1 - \mu_2 > 0$	$H_0: \mu_1 - \mu_2 = 0$ $H_a: \mu_1 - \mu_2 < 0$
First Case	**$\sigma_1^2 \& \sigma_2^2$ are known; Normal or Non-normal Distribution**		
Test Statistic (T.S.)	$Z = \dfrac{\bar{X} - \bar{Y}}{\sqrt{\dfrac{\sigma_1^2}{m} + \dfrac{\sigma_2^2}{n}}} \sim N(0,1)$		
Rejection Region (R.R) & Acceptance Region (A.R) of H_0			
Critical Value	$-Z_{\frac{\alpha}{2}}$ and $Z_{\frac{\alpha}{2}}$	Z_α	$-Z_\alpha$
Decision	**We reject H_0 at the significant level α if:**		
	$Z < -Z_{\frac{\alpha}{2}}$ or $Z > Z_{\frac{\alpha}{2}}$	$Z > Z_\alpha$	$Z < -Z_\alpha$
Second Case	**$\sigma_1^2 \& \sigma_2^2$ are unknown; m, n large (m>30&n>30)**		
Test Statistic (T.S.)	$Z = \dfrac{\bar{X} - \bar{Y}}{\sqrt{\dfrac{s_1^2}{m} + \dfrac{s_2^2}{n}}} \sim N(0,1)$		
Rejection Region (R.R) & Acceptance Region (A.R) of H_0			
Critical Value	$-Z_{\frac{\alpha}{2}}$ and $Z_{\frac{\alpha}{2}}$	Z_α	$-Z_\alpha$
Decision	**We reject H_0 at the significant level α if:**		
	$Z < -Z_{\frac{\alpha}{2}}$ or $Z > Z_{\frac{\alpha}{2}}$	$Z > Z_\alpha$	$Z < -Z_\alpha$

Hypotheses	$H_0: \mu_1 - \mu_2 = 0$ $H_a: \mu_1 - \mu_2 \neq 0$	$H_0: \mu_1 - \mu_2 = 0$ $H_a: \mu_1 - \mu_2 > 0$	$H_0: \mu_1 - \mu_2 = 0$ $H_a: \mu_1 - \mu_2 < 0$
third Case	**σ_1^2&σ_2^2 are unknown; Normal; m, n small (m<30&n<30)**		
Test Statistic (T.S.)	$\boldsymbol{t = \dfrac{\bar{X} - \bar{Y}}{S_p\sqrt{\dfrac{1}{m} + \dfrac{1}{n}}} \sim N(0,1)}$		
Rejection Region (R.R) & Acceptance Region (A.R) of H_0			
Critical Value	$\boldsymbol{-t_{\frac{\alpha}{2}}\ and\ \ t_{\frac{\alpha}{2}}}$	$\boldsymbol{t_\alpha}$	$\boldsymbol{-t_\alpha}$
Decision	**We reject H_0 at the significant level α if:**		
	$\boldsymbol{t < -t_{\frac{\alpha}{2}}\ \ or\ \ t > t_{\frac{\alpha}{2}}}$	$\boldsymbol{t > t_\alpha}$	$\boldsymbol{t < -t_\alpha}$

Note:

- $\boldsymbol{S_p = \sqrt{\frac{(m-1)s_1^2 + (n-1)s_2^2}{m+n-2}}}$
- $\boldsymbol{df = v = m + n - 2}$

Example (replacement for example 9.4):

To compare the income level in two cities, two samples are selected randomly. The first sample of size 50 families from the first city has average income 64 thousand dollar per year and variance 6 thousand dollar and a sample of size 60 families from the second city has an average income 66 thousand dollar with variance 5 thousand dollar. Is there exist a significant difference between the average income for families in the two cities at $\boldsymbol{\alpha = 0.05}$.

Solution:

First city: $m = 50,\ \bar{X} = 64,\ s_1^2 = 6.$

Second city: $n = 60,\ \bar{Y} = 66,\ s_2^2 = 5$

1. **Hypotheses:** $H_0: \mu_1 = \mu_2 \quad Vs. \quad H_a: \mu_1 \neq \mu_2$

   Or $H_0: \mu_1 - \mu_2 = 0 \quad Vs. \quad H_a: \mu_1 - \mu_2 \neq 0$

2. **Assumptions:** $\sigma_1^2, \sigma_2^2\ unknown,\ \ Normal,\ \ m\&n < 30\ small.$

3. **Test statistic:**

$$\boldsymbol{Z} = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}} = \frac{64 - 66}{\sqrt{\frac{6}{50} + \frac{5}{60}}} = -4.435$$

4. **Rejection region: R.R**

   Reject H_0 if: $Z > Z_{\frac{\alpha}{2}}$ or $Z < -Z_{\frac{\alpha}{2}}$

$$Z_{\frac{\alpha}{2}} = Z_{\frac{0.05}{2}} = Z_{0.025} = 1.96$$

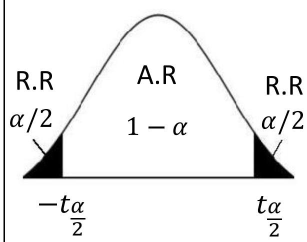
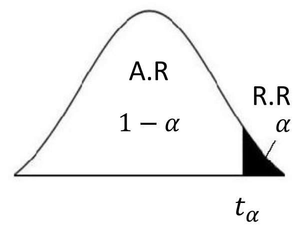
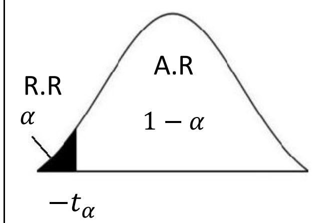
5. **Decision:**

   Since $Z = -4.435 < -1.96 = -Z_{\frac{\alpha}{2}}$, we reject H_0 at $\alpha = 0.05$.

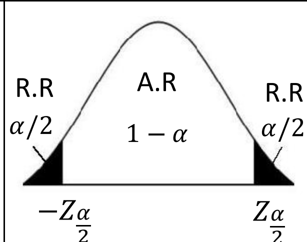
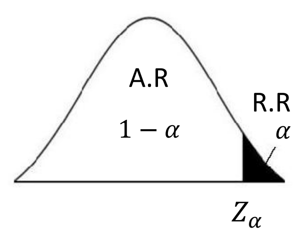
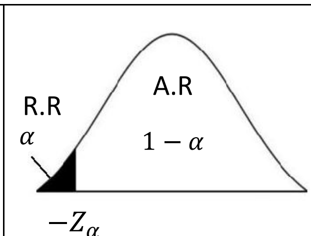
   Thus, there is a significant difference between the average incomes for families in the two cities.

3. <u>Paired t Test:</u>

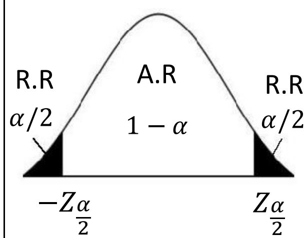
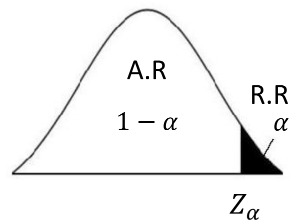
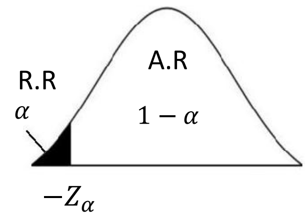
Test Hypotheses for the difference between two population Means (for dependent populations) ($\boldsymbol{\mu_D = \mu_1 - \mu_2}$):

<table>
<tr><td>Hypotheses</td><td>$H_0: \mu_D = 0$
$H_a: \mu_D \neq 0$</td><td>$H_0: \mu_D = 0$
$H_a: \mu_D > 0$</td><td>$H_0: \mu_D = 0$
$H_a: \mu_D < 0$</td></tr>
<tr><td>Assumptions</td><td colspan="3">Normal populations, n<30</td></tr>
<tr><td>Test Statistic (T.S.)</td><td colspan="3">$$\boldsymbol{t = \frac{\bar{D}}{s_D/\sqrt{n}}}$$</td></tr>
<tr><td>Rejection Region (R.R) & Acceptance Region (A.R) of H_0</td><td>
</td><td>
</td><td>
</td></tr>
<tr><td>Critical Value</td><td>$\boldsymbol{-t_{\frac{\alpha}{2}}\ and\ \ t_{\frac{\alpha}{2}}}$</td><td>$\boldsymbol{t_\alpha}$</td><td>$\boldsymbol{-t_\alpha}$</td></tr>
<tr><td rowspan="2">Decision</td><td colspan="3">We reject $\boldsymbol{H_0}$ at the significant level $\boldsymbol{\alpha}$ if:</td></tr>
<tr><td>$\boldsymbol{t < -t_{\frac{\alpha}{2}}\ or\ t > t_{\frac{\alpha}{2}}}$</td><td>$\boldsymbol{t > t_\alpha}$</td><td>$\boldsymbol{t < -t_\alpha}$</td></tr>
</table>

4. <u>**Test Hypotheses for the population Proportion ($\boldsymbol{P}$):**</u>

<table>
<tr><td>Hypotheses</td><td>$H_0: P = p_0$
$H_a: P \neq p_0$</td><td>$H_0: P = p_0$
$H_a: P > p_0$</td><td>$H_0: P = p_0$
$H_a: P < p_0$</td></tr>
<tr><td>Test Statistic (T.S.)</td><td colspan="3">$$\boldsymbol{Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \sim N(0,1)}$$</td></tr>
<tr><td>Rejection Region (R.R) & Acceptance Region (A.R) of H_0</td><td>
</td><td>
</td><td>
</td></tr>
<tr><td>Critical Value</td><td>$\boldsymbol{-Z_{\frac{\alpha}{2}}\ and\ \ Z_{\frac{\alpha}{2}}}$</td><td>$\boldsymbol{Z_\alpha}$</td><td>$\boldsymbol{-Z_\alpha}$</td></tr>
<tr><td rowspan="2">Decision</td><td colspan="3">We reject $\boldsymbol{H_0}$ at the significant level $\boldsymbol{\alpha}$ if:</td></tr>
<tr><td>$\boldsymbol{Z < -Z_{\frac{\alpha}{2}}\ or\ Z > Z_{\frac{\alpha}{2}}}$</td><td>$\boldsymbol{Z > Z_\alpha}$</td><td>$\boldsymbol{Z < -Z_\alpha}$</td></tr>
</table>

5. **<u>Test Hypotheses for Two Population Proportions:</u>**

<table>
<tr><td>Hypotheses</td><td>$H_0: P_1 = P_2$
$H_a: P_1 \neq P_2$</td><td>$H_0: P_1 = P_2$
$H_a: P_1 > P_2$</td><td>$H_0: P_1 = P_2$
$H_a: P_1 < P_2$</td></tr>
<tr><td>Test Statistic (T.S.)</td><td colspan="3">$\boldsymbol{Z = \dfrac{\widehat{p}_1 - \widehat{p}_2}{\sqrt{\widehat{p}\widehat{q}\left(\frac{1}{m}+\frac{1}{n}\right)}} \sim N(0,1)\ \ where \quad \widehat{p} = \dfrac{X+Y}{m+n}}$</td></tr>
<tr><td>Rejection Region (R.R)
& Acceptance Region
(A.R) of H_0</td><td>
</td><td>
</td><td>
</td></tr>
<tr><td>Critical Value</td><td>$\boldsymbol{-Z_{\frac{\alpha}{2}}\ \ and\ \ Z_{\frac{\alpha}{2}}}$</td><td>$\boldsymbol{Z_{\alpha}}$</td><td>$\boldsymbol{-Z_{\alpha}}$</td></tr>
<tr><td rowspan="2">Decision</td><td colspan="3">We reject $\boldsymbol{H_0}$ at the significant level $\boldsymbol{\alpha}$ if:</td></tr>
<tr><td>$\boldsymbol{Z < -Z_{\frac{\alpha}{2}}\ \ or\ \ Z > Z_{\frac{\alpha}{2}}}$</td><td>$\boldsymbol{Z > Z_{\alpha}}$</td><td>$\boldsymbol{Z < -Z_{\alpha}}$</td></tr>
</table>

6. **<u>Test Hypotheses for the population Variance (σ^2):</u>**

<table>
<tr><td>Hypotheses</td><td>$H_0: \boldsymbol{\sigma^2 = \sigma_0^2}$
$H_a: \boldsymbol{\sigma^2 \neq \sigma_0^2}$</td><td>$H_0: \boldsymbol{\sigma^2 = \sigma_0^2}$
$H_a: \boldsymbol{\sigma^2 > \sigma_0^2}$</td><td>$H_0: \boldsymbol{\sigma^2 = \sigma_0^2}$
$H_a: \boldsymbol{\sigma^2 < \sigma_0^2}$</td></tr>
<tr><td>Test Statistic (T.S.)</td><td colspan="3">$\chi^2 = \dfrac{\boldsymbol{(n-1)s^2}}{\boldsymbol{\sigma_0^2}}$</td></tr>
<tr><td>Critical Value</td><td>$\boldsymbol{\chi^2_{\frac{\alpha}{2},n-1}\ \ and\ \ \chi^2_{1-\frac{\alpha}{2},n-1}}$</td><td>$\boldsymbol{\chi^2_{\alpha,n-1}}$</td><td>$\boldsymbol{\chi^2_{1-\alpha,n-1}}$</td></tr>
<tr><td rowspan="2">Decision</td><td colspan="3">We reject $\boldsymbol{H_0}$ at the significant level $\boldsymbol{\alpha}$ if:</td></tr>
<tr><td>$\chi^2 < \boldsymbol{\chi^2_{1-\frac{\alpha}{2},n-1}}\ \ \boldsymbol{or}$
$\chi^2 > \boldsymbol{\chi^2_{\frac{\alpha}{2},n-1}}$</td><td>$\chi^2 > \boldsymbol{\chi^2_{\alpha,n-1}}$</td><td>$\chi^2 < \boldsymbol{\chi^2_{1-\alpha,n-1}}$</td></tr>
</table>

Example: **Chemically evaluated irrigation water samples from 14 Qatif wells. The percent of Na cations in the water was measured:**
43, 47,40,45, 45,48, 47,47,46,52,50,50,51,49
Assuming a normal distribution, test whether the variance of the total Na cations is less than 10 at $\alpha = 0.05$.
Solution:
$n = 14,\ s_1 = 3.2548, \alpha = 0.05, \sigma_0^2 = 10.$

1. **Hypotheses:** $H_0: \sigma^2 = 10 \quad Vs. \quad H_a: \sigma^2 < 10$
2. **Test statistic:**

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{13(3.2548)^2}{10} = 13.77$$

3. **Rejection region: R.R**

   Reject H_0 if: $\chi^2 < \chi^2_{1-\alpha,n-1}$

   $\chi^2_{1-\alpha,n-1} = \chi^2_{0.95,13} = 5.892$

4. **Decision:**

   Since $\chi^2 = 13.77 \nless 5.892 = \chi^2_{1-\alpha,n-1}$, we cannot reject H_0 at $\alpha = 0.05$.

7. **Test Hypotheses for Equality of Variances:**

Hypotheses	$H_0: \sigma_1^2 = \sigma_2^2$ $H_a: \sigma_1^2 \neq \sigma_2^2$	$H_0: \sigma_1^2 = \sigma_2^2$ $H_a: \sigma_1^2 > \sigma_2^2$	$H_0: \sigma_1^2 = \sigma_2^2$ $H_a: \sigma_1^2 < \sigma_2^2$
Test Statistic (T.S.)	$f = s_1^2/s_2^2$		
Critical Value	$\chi^2_{\frac{\alpha}{2},n-1}$ and $\chi^2_{1-\frac{\alpha}{2},n-1}$	$\chi^2_{\alpha,n-1}$	$\chi^2_{1-\alpha,n-1}$
Decision	**We reject H_0 at the significant level α if:**		
	$F < F_{1-\frac{\alpha}{2},m-1,n-1}$ or $F > F_{\frac{\alpha}{2},m-1,n-1}$	$F > F_{\alpha,m-1,n-1}$	$F < F_{1-\alpha,m-1,n-1}$

Note:

$$F_{1-\alpha,m-1,n-1} = \frac{1}{F_{\alpha,n-1,m-1}}$$