King Saud University

College of Sciences

Department of Mathematics

1432 - 2011

On Normed Spaces

and

Inner Product Spaces

Submitted in partial fulfillment of the requirements for the degree of Bachelor of Sience in mathematics.

Prepared by;

Asma Hasan Althagafi

Dalal Alzahrani

Supervised by;

Dr.Fatmah Baker Jamjoom

بسم الله الرحمن الرحيم

الحمد لله الذي ما كنا لنهتدي لولا هداه

والصلاة على خير البرية محمد المصطفى عليه افضل الصلاة واتم التسليم

إهداء

اخط كلمات مدادها دم قلبي وإحساسي الصادق ، كلمات ملؤها شكر وعرفان

تفيض حب وامتنان

***إلى من أنا قطرة في بحرها ونجمة في سمائها**

إلى من أنا لها شيء وهي لي كل شيء

إليك..أمي الحبيبة ..إليك أهدي تعبي وجهدي وفرحتي وحياتي كلها

إليك عهدي بأن أبرك ما دام الدم يسري في شراييني

***إلى سندي وساعدي ،إلى الشعاع الذي أنار دربي ،إلى من علمني الصبر والثبات والصمود مهما تبدلت الظروف ..إلى أبي الغالي...**

***إلى جزأي الذي لا يتجزأ،إلى عزوتي وملوك وجداني..إلى الأحبة أخوتي وأخواتي**

***إلى المنار الوضاء إلى من صاغت عبارات من ذهب إلى من تعلمنا منها علما أثمن من علم الكتب ..إلى الدكتورة الرائعة فاطمة جمجوم ..**

لك منا سفينة شكر يحملها بحر الاحترام

***إلى من تجولنا في رحابها لنقطف من بستان العلم زهوره ...إلى جامعتي الحبيبة**

***إلى كل من يحب العلم ويسعى لتحصيله**

Contents

Introduction

Particularly useful and important metric spaces are obtained if we take a vector space and define on it a metric by means of a norm .The resulting space is called a normed space. If it is a complete metric space, it is called a Banach space. The theory of normed spaces, in particular Banach spaces, and the theory of linear operators defined on them are the most highly developed parts of functional analysis.

Inner product spaces are special normed spaces, as we shall see. Historically they are older than general normed spaces. Their theory is richer and retains many features of Euclidean spaces, a central concept being orthogonality. In fact, inner product spaces are probably the most natural generalization of Euclidean spaces, The whole theory was initiated by the work of D. Hilbert (1912)on integral equations. The currently used geometrical notation and terminology is analogous to that of Euclidean geometry and was coined by E. Schmidt (1908), who followed a suggestion of G. Kowalewski. These spaces have been, up to now, the most useful spaces in practical applications of functional analysis.

CHAPTER 1

1.1 Metric Space

In calculus we study functions defined on real line **R** . A little reflection shows that in limit processes and many other considerations we use the fact that on R we have available a distance function, call it d, which associates a distance $d(x,y)=|x-y|$ for every pair of point $x, y \in \mathbf{R}$

1.1-1 Definition (Metric space , metric).

A metric space is a pair (X,d),where X is a set and d is a metric on X space (or distance function on X), that is, a function defined on $X \times X$ such that, for all $x, y, z \in X$ we have:

(M1) d is real-valued, finite and nonnegative .

(M2) $d(x,y)=0$ if and only if $x=y$.

(M3) $d(x,y)=d(y,x)$ (symmetry).

(M4) $d(x,y) \le d(x,z)+d(z,y)$ (Triangle inequality)

Examples:

1.1-2 Real line R. This is the set of all real numbers, taken with the usual metric defined by

$$d(x, y)=|x-y| , \quad x, y \in \boldsymbol{R}$$

1.1-3 Euclidean plane R^2. The metric space R^2 ,space called the Euclidean plane, is obtained if we take the set of ordered pairs (ξ_1,ξ_2) of real numbers, Then $d:R^2\times R^2\to R$ is defined by

$$d(x,y)=\sqrt{(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2}$$

Where $x=(\xi_1,\xi_2)$, $y=(\eta_1,\eta_2)$

1.2 (Hőlder inequality).

Let P > 1 , and define $q\in R$ such that ; $\frac{1}{p}+\frac{1}{q}=1$ then,

$$l^p=\left\{x=(\xi_i)_{i=1}^{n}:\xi_i\in C;\sum|\xi_i|^p<\infty\right\}$$

$$then\ \sum_{i=1}^{\infty}|\xi_i\eta_i|\le(\sum_{i=1}^{\infty}|\xi_i|^p)^{1/p}\cdot(\sum_{i=1}^{\infty}|\eta_i|^q)^{1/q}$$

Proof:

Let $(\xi_i)\in l^p,(\eta)\in l^q$, and assume $\sum_{i=1}^{\infty}|\xi_i|^p=1,\sum_{i=1}^{\infty}|\eta_i|^q=1$

Note that for any $\alpha,\beta>0;\ \alpha\beta\le\frac{\alpha^p}{p}+\frac{\beta^q}{q}$

So for each i = 1,2,…

Putting $\alpha_i=|\tilde{\xi}_i| and\quad \beta_i=|\tilde{\eta}_i|$, we have $i\in N$

So , $|\tilde{\xi}_i\tilde{\eta}_i|=|\tilde{\xi}_i||\tilde{\eta}_i|\le\frac{|\tilde{\xi}_i|^p}{p}+\frac{|\tilde{\eta}_i|^q}{q}$, for each $i\in N$

Hence, for each $i\in$ □ and $n\in$ □ we have,

$$\sum_{i=1}^{n}\left|\tilde{\xi}_i\tilde{\eta}_i\right| \le \sum_{i=1}^{n}\left(\frac{\left|\tilde{\xi}_i\right|^p}{p}+\frac{\left|\tilde{\eta}_i\right|^q}{q}\right)=\sum_{i=1}^{n}\frac{\left|\tilde{\xi}_i\right|^p}{p}+\sum_{i=1}^{n}\frac{\left|\tilde{\eta}_i\right|^q}{q} \le \frac{1}{p}\sum_{i=1}^{\infty}\left|\tilde{\xi}_i\right|^p+\frac{1}{q}\sum_{i=1}^{\infty}\left|\tilde{\eta}_i\right|^q$$

Since $\sum_{i=1}^{\infty}\left|\tilde{\xi}_i\right|^p=1, \sum_{i=1}^{\infty}\left|\tilde{\eta}_i\right|^q=1, then \sum_{i=1}^{\infty}\left|\tilde{\xi}_i\tilde{\eta}_i\right| \le \frac{1}{p}+\frac{1}{q}=1$

Now, let $(\xi_i)\in l^p, (\eta_i)\in l^q$, and put

$$\tilde{\xi}_i=\frac{\xi_i}{\left(\sum_{i=1}^{\infty}\left|\xi_i\right|^p\right)^{1/p}} \quad and \quad \tilde{\eta}_i=\frac{\eta_i}{\left(\sum_{i=1}^{\infty}\left|\eta_i\right|^q\right)^{1/q}}$$

We note that the definition of $\tilde{\xi}_i, \tilde{\eta}_i$ are both satisfy the condition.

since $\sum_{i=1}^{\infty}\left|\tilde{\xi}_i\right|^p=1, \sum_{i=1}^{\infty}\left|\tilde{\eta}_i\right|^q=1,$

$$\sum_{i=1}^{\infty}\left|\frac{\xi_i}{\left(\sum_{i=1}^{\infty}\left|\xi_i\right|^p\right)^{1/p}}\right|^p=\frac{\sum_{i=1}^{\infty}\left|\xi_i\right|^p}{\left(\left(\sum_{i=1}^{\infty}\left|\xi_i\right|^p\right)^{1/p}\right)}=\frac{\sum_{i=1}^{\infty}\left|\xi_i\right|^p}{\sum_{i=1}^{\infty}\left|\xi_i\right|^p}=1$$

$$and \quad \sum_{i=1}^{\infty}\left|\frac{\eta_i}{\left(\sum_{i=1}^{\infty}\left|\eta_i\right|^q\right)^{1/q}}\right|^q=\frac{\sum_{i=1}^{\infty}\left|\eta_i\right|^q}{\left(\left(\sum_{i=1}^{\infty}\left|\eta_i\right|^q\right)^{1/q}\right)^q}=\frac{\sum_{i=1}^{\infty}\left|\eta_i\right|^q}{\sum_{i=1}^{\infty}\left|\eta_i\right|^q}=1$$

Hence $\sum_{i=1}^{\infty}\left|\tilde{\xi}_i\tilde{\eta}_i\right|=\sum_{i=1}^{\infty}\left(\frac{\left|\xi_i\right|}{\left(\sum_{i=1}^{\infty}\left|\xi_i\right|^p\right)^{1/p}}\cdot\frac{\left|\eta_i\right|}{\left(\sum_{i=1}^{\infty}\left|\eta_i\right|^q\right)^{1/q}}\right)\le 1$

$$\Rightarrow \frac{1}{\left(\sum_{i=1}^{\infty}|\xi_i|^p\right)^{1/p}\left(\sum_{i=1}^{\infty}|\eta_i|^q\right)^{1/q}}\sum_{i=1}^{\infty}|\xi_i\eta_i| \le 1$$

$$\Rightarrow \sum_{i=1}^{\infty}|\xi_i\eta_i| \le (\sum_{i=1}^{\infty}|\xi_i|^p)^{1/p}.(\sum_{i=1}^{\infty}|\eta_i|^q)^{1/q}$$

This inequality is called Hőlder inequality.

If p = 2 , then q =2. This inequality yields the Cauchy – Schwarz inequality.

$$\sum_{i=1}^{\infty}|\xi_i\eta_i| \le (\sum_{i=1}^{\infty}|\xi_i|^2)^{1/2}.(\sum_{i=1}^{\infty}|\eta_i|^2)^{1/2}$$

1.3 (Minkowski inequality)

For any $(\xi_i),(\eta_i) \in l^p$, p >1. We have:

$$(\sum_{i=1}^{\infty}|\xi_i+\eta_i|^p) \le (\sum_{i=1}^{\infty}|\xi_i|^p)^{1/p} + (\sum_{i=1}^{\infty}|\eta_i|^p)^{1/p}$$

Proof:

Put $\omega_i = \xi_i + \eta_i, i \in N$

$$|\omega_i|^p = |\omega_i||\omega_i|^{p-1} = |\xi_i + \eta_i||\omega_i|^{p-1} \le (|\xi_i| + |\eta_i|)|\omega_i|^{p-1}$$

Then for each $n \in N$

$$\sum_{i=1}^{n}|\omega_i|^p \le \sum_{i=1}^{n}((|\xi_i||\eta_i|)|\omega_i|^{p-1} = \sum_{i=1}^{n}|\xi_i||\omega_i|^{p-1} + \sum_{i=1}^{n}|\eta_i||\omega_i|^{p-1}$$

Note that $\sum_{i=1}^{n}|\xi_i||\omega_i|^{p-1} \le (\sum_{i=1}^{n}|\xi_i|^p)^{1/p}(\sum_{i=1}^{n}(|\omega_i|^{p-1})^q)^{1/q}$ (From Hőlder inequality)

Where $q \in R$ and $\frac{1}{p} + \frac{1}{q} = p$

since $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{q+p}{qp} = 1 \Rightarrow q + p = qp \Rightarrow (p-1) = p$,

we have

$$\sum_{i=1}^{n} |\xi_i||\omega_i|^{p-1} \leq (\sum_{i=1}^{n} |\xi_i|^p)^{1/p} (\sum_{i=1}^{n} |\omega_i|^p)^{1/q}$$

Also, $\sum_{i=1}^{n} |\eta_i||\omega_i|^{p-1} \leq (\sum_{i=1}^{n} |\eta_i|^p)^{1/p} (\sum_{i=1}^{n} |\omega_i|^{p-1})^{1/q}$, form Hőlder inequality

This implies that $\sum_{i=1}^{n} |\eta_i||\omega_i|^{p-1} \leq (\sum_{i=1}^{n} |\eta_i|^p)^{1/p} (\sum_{i=1}^{n} |\omega_i|^p)^{1/q}$.

Therefore $\sum_{i=1}^{n} |\omega_i|^p \leq (\sum_{i=1}^{n} |\omega_i|^p)^{1/q} [(\sum_{i=1}^{n} |\xi_i|^p)^{1/p} + \sum_{i=1}^{n} |\eta_i|^p)^{1/p}]$

, for each n$\in N \Rightarrow \sum_{i=1}^{n} |\omega_i|^p \leq (\sum_{i=1}^{n} |\xi_i|^p)^{1/p} + (\sum_{i=1}^{n} |\eta_i|^p)^{1/p}$

, for each n$\in N \Rightarrow (\sum_{i=1}^{n} |\xi_i + \eta_i|^p)^{1/p} \leq (\sum_{i=1}^{n} |\xi_i|^p)^{1/p} + (\sum_{i=1}^{n} |\eta_i|^p)^{1/p}$

Since $\sum_{i=1}^{\infty} |\xi_i|^p < \infty$ *and* $\sum_{i=1}^{\infty} |\eta_i|^p < \infty$; we have

$$(\sum_{i=1}^{\infty} |\xi_i + \eta_i|^p) \leq (\sum_{i=1}^{\infty} |\xi_i|^p)^{1/p} + (\sum_{i=1}^{\infty} |\eta_i|^p)^{1/p}$$

CHAPTER 2

2.1: Normed spaces, Banach Spaces.

We first introduce the concept of a norm (definition below), which uses the algebraic operations of vector spaces. Then we employ the norm to obtain a metric d that is of the desired kind

2.1-1.Definition:

A norm on a vector space X (over K)

a scalar field K is a real valued function, $\|.\|: X \to R$ which satisfies the following properties:

1) $\|x\| \geq 0, \forall x \in X$

2) $\|x\| = 0 \Leftrightarrow x = 0, \forall x \in X$

3) $\|\alpha x\| = \|\alpha\| \|x\|, \forall x \in X ; \alpha \in K$ be any scalar.

4) $\|x + y\| \leq \|x\| + \|y\| . \forall x, y \in X$

A Banach space is a complete normed space it is complete in the metric defined by the norm $d(x, y) = \|x - y\|$

2.1.2. Lemma: The norm is continuous function.

Let X be a norm space and note that for any $x, y \in X$.

$$\left\| \|x\| - \|y\| \right\| \leq \|x - y\|$$

Proof: Let $x, y \in X$ *, then*

$$\|y\| = \|y - x + x\| \le \|y - x\| + \|x\|$$

$$\|x\| = \|x - y + y\| \le \|x - y\| + \|y\|$$

$$\forall x, y \in X \quad \|x\| \le \|x - y\| + \|y\|$$

$$\text{---------}\ (1) \Rightarrow \quad \|x\| - \|y\| \le \|x - y\|$$

Replacing x by y we have

$$\|y\| - \|x\| \le \|y - x\| = \|x - y\|$$

i.e. $\|y\| - \|x\| \le \|x - y\|$

$$-(\|x\| - \|y\|) \le \|x - y\| \Rightarrow$$

$$\text{----}\ (2)\ \|x\| - \|y\| \ge -\|x - y\| \Rightarrow$$

$$-\|x - y\| \le \|x\| - \|y\| \le \|x - y\|$$

Hence $\big|\|x\| - \|y\|\big| \le \|x - y\|$

Examples:

1) Consider the space

$=\{ x = (\xi_i)_{i=1}^{n} : \xi_i \in C ; \sum |\xi_i|^p < \infty \}\ \ell^p$

Define $\|.\| : \ell^p \to$ R by

$$\|\mathbf{x}\| = \left(\sum_{j=1}^{\infty} |\xi_j|^p \right)^{\frac{1}{p}} \quad (*)$$

Then $(\ell^p, \|.\|)$ is a normed space.

Proof:

first, we have to prove that **(*)** is well defined. So,

let $x = (\xi_j) \in \ell^p$

$$\Rightarrow \sum_{j=1}^{\infty} |\xi_j|^p < \infty$$

Since the sum of a convergent series is unique ,
(*) is well defined

$$\alpha = \beta$$

1) Since $|\xi_1|, |\xi_2|, |\xi_3|, \ldots\ldots, |\xi_n|, \ldots\ldots \geq 0$

$$\Rightarrow |\xi_1| + |\xi_2| + |\xi_3| + \ldots\ldots\ldots + |\xi_n| + \ldots\ldots \geq 0$$

$$\Rightarrow \|x\| \geq 0$$

2) $\|x\| = 0 \Leftrightarrow \left(\sum_{j=1}^{\infty} |\xi_j|^p\right)^{\frac{1}{p}} = 0 \Leftrightarrow \xi_j = 0 \ \forall j$

$$\Leftrightarrow (\xi_j)_{j=1}^{\infty} = 0 \Leftrightarrow x = 0$$

3) $\|\alpha x\| = \left(\sum_{j=1}^{\infty} |\alpha \xi_j|^p\right)^{\frac{1}{p}} = \left(|\alpha|^p \sum_{j=1}^{\infty} |\xi_j|^p\right)^{\frac{1}{p}} = |\alpha| \|x\|$

4) $\|x + y\| = \left(\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p\right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^{\infty} |\xi_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{\infty} |\eta_j|^p\right)^{\frac{1}{p}}$

(by Minkowski inequality)

$$\leq \|x\| + \|y\|$$

2)Space ℓ^{∞} : This space is a Banach Space with norm given by $\|x\| = \sup_j |\xi_j|$

We have to prove is well defined

$$\ell^{\infty} = \left\{ x = (\xi_j) : \xi_j \in \mathbb{R}, (\xi_j) \, is \, a \, bounded \, sequans \right\}$$

is a bounded sequence $\Rightarrow \exists M > 0 \ni |\xi_j| \leq M \quad \forall j \in \mathbb{N} \quad (\xi_j)$

is a bounded subset of $\mathbb{R}$ $A = \{ |\xi_j| : j \in \mathbb{N} \}$

supA exist $\Rightarrow$

$\sup_{j \in \mathbb{N}} |\xi_j|$ exist and is uniqe $\Rightarrow$

To each $x = (\xi_j) \in \ell^{\infty}$, $\sup_{j \in \mathbb{N}} |\xi_j|$ is uniqe

Now, Let $x = (\xi) \in l^{\infty}$ then

1) $\|x\| \geq 0 \quad since \quad |\xi_i| \geq 0 \Rightarrow \sup|\xi_i| \geq 0$

2) $\|x\| = 0 \quad \Leftrightarrow \sup \quad |\xi_i| = 0 \Leftrightarrow |\xi_i| = 0 \quad \forall i = 1, 2, \ldots$
$\Leftrightarrow \xi_i = 0 \; \forall i = 1, 2, \ldots \Leftrightarrow x = 0$

3) For any $\alpha \in R$

$$\|\alpha x\| = \sup|\alpha \xi_i| = \sup|\alpha||\xi_i| = |\alpha| \sup|\xi_i| = |\alpha| \|x\|$$

4) Let $y = (\eta_i) \in l^{\infty}$

$$\|x + y\| = \sup|\xi_i + \eta_i|$$

$$\leq \sup(|\xi_i| + |\eta_i|)$$
$$\leq \sup|\xi_i| + \sup|\eta_i| = \|x\| + \|y\|$$

Hence l^∞ is a normed space.

2.2. Some properties of normed spaces.

2.2-1 Definition:

A subspace Y of a normed space X is a subspace of X considered as a vector space, with the norm obtained by restricting norm on X to the subset Y.

If Y is closed in X , then Y is called a closed subspace of X.

2.2-2: Definition (convergence of sequences)

(i) A sequence (x_n) in a normed space X is said to be convergent if X contains an x such that

$$\lim_{n\to\infty} \|x_n - x\| = 0$$

Then we write $(x_n) \to x$ and call x the limit of (x_n).

(ii) A sequence (x_n) in a normed space X is Cauchy if for every $\varepsilon > 0$ there is an N such that

for all m, n >N $\|x_m - x_n\| < \varepsilon$

2.2-3: Definition (infinite series) .

A series $\sum_{k=1}^{\infty} x_k = x_1 + x_2 + ...$ in normed space (X, $\|.\|$) is said to be convergent if the sequence (S_n) of the partial sums convergent ,where $S_n = \sum_{i=1}^{n} x_i$

In this case $S = \sum_{k=1}^{\infty} x_k = x_1 + x_2 + ...$

is said to be absolutely convergent, if $\sum_{n=1}^{\infty} \|x_n\|$ is $\sum_{n=1}^{\infty} x_n$ convergent.

Lemma (2.2.4) Let X is a Banach space if $\sum x_n$ is absolutely convergent then $\sum x_n$ is convergent.

Proof:

Suppose $\sum x_n$ is absolutely convergent.

is convergent$\Rightarrow \sum \|x_n\|$

the sequence (t_n) of partial sums of $\sum \|x_n\|$ is convergent, $\Rightarrow$

where $t_n = \sum_{j=1}^{n} \|x_j\|$

is a Cauchy sequence.$\Rightarrow t_n$

Let $\varepsilon > 0$ be given

Since (t_n) is a Cauchy sequence $\exists \quad N_\varepsilon \in N \ni: \forall n, m \geq N_\varepsilon$.

hence $|t_n - t_m| < \varepsilon$

, $n > m \forall n, m \leq N$

$$\leq \sum_{j=m+1}^{n} \|x_j\| = \left\| \sum_{j=m+1}^{n} x_j \right\| \|S_n - S_m\| = \left\| \sum_{j=1}^{n} x_j - \sum_{j=1}^{m} x_j \right\|$$

$$\leq \sum_{j=N_\varepsilon+1}^{\infty} \|x_j\| < \varepsilon \leq \sum_{j=m+1}^{\infty} \|x_j\|$$

is a Cauchy sequence in X$\Rightarrow (S_n)$

Since X is complete

convergence in X$\Rightarrow (S_n)$

is convergence$\Rightarrow \sum x_n$

2.2.5. Definition:

Let X be a normed space. The space X is said to be complete if every Cauchy sequence in X converges.

Remark:

If a normed space X contains a sequence (e_n) with the property that for every x$\in$X there is a unique sequence of scalars (α_n) such that

as $n \to \infty \qquad \|x - (\alpha_1 e_1 + \ldots.. \alpha_n e_n)\| \to 0$

Then (e_n) is called a Schuder basis for X

Then we write, $x = \sum_{i=1}^{\infty} \alpha_i e_i$

2.2.6. Definition : (Dense set, separable space)

A subset M of a normed space X is said to be dense in X if $\overline{M} = X$,where $\overline{M}$ is the closure of M.

X is said to be separable if it is has a countable subset which is dense in X.

2.2.7: Theorem (complete subspace):

A subspace M of a complete metric space X is itself complete if and only if the set M is closed in X.

Proof:

Suppose M be complete $\Rightarrow$ every Cauchy sequence in M is Convergent

Let $x \in \overline{M} \Rightarrow \exists$ a sequence $(x_n) in\ M\ s.t\ x_n \to x$

Since (x_n) is convergent $\Rightarrow x_n$ is Cauchy sequence in M.

Since M is complete $\Rightarrow (x_n)$ converges in M, say $(x_n) \to y_0 \in M$

By the uniqueness of the limit $x = y_0 \in M \Rightarrow \overline{M} \subseteq M$ --(1)

----(2) (Clearly by definition) $M \subseteq \overline{M}$

From (1) and (2), we have $\overline{M} = M$,hence M is closed.

Conversely, suppose M is closed, and let (x_n) be a Cauchy sequence in M.

is a Cauchy sequence in X.$\Rightarrow (x_n)$

Since X is complete $\Rightarrow (x_n)$ convergence to $x_0 \Rightarrow x_0 \in \overline{M}$

Since M is closed $\Rightarrow x_0 \in \overline{M} = M$

every Cauchy sequences in M convergent in $M \Rightarrow$

M is complete$\Rightarrow$

2.2-8 Theorem (Completion).

Let $X = (X, \|\cdot\|)$ be a normed space. Then there is banach space $\hat{X}$ and an isomerty . A from X onto a subspace W of $\hat{X}$ which is dense in $\hat{X}$. The space $\hat{X}$ is unique, except for isometries.

2.3.Linear Operators

2.3-1 Definition (Linear Operators)

A linear operators T is an operator such that

(i) the domain D(T) of T is a vector space real or complex
and the range R(T) lies in a vector space over the same field

(ii) for all x, y ∈D(T) and any scalars α,

T(x +y)=T x +Ty
T(α x)= α T x (1)

By definition , the null space of T is the set of all x ∈ D(T) such that T x = 0

Clearly , (1) is equivalent to

T(α x+ β y) = α T x + β Ty, $\forall x, y \in D(T)$ and $\alpha, \beta \epsilon K(R \text{ or } C)$

Examples:

2.3-2: Identity operator .

Let X be a vector space over K(R or C).

The identity operator I: X → X is defined by Ix = x for all x∈ X

For all x, y ∈ $X, \alpha, \beta\varphi \in K$

I(α x + β y) = α x + β y

= α I(x)+β I(y)

Hence , I is an operator

2.3-3:Zero operator.

The zero operator O:X → X is defined by Ox =0 for all x∈ X

O(α x +β y) = 0

= 0 + 0

= α Ox + β O y

Hence, O is an operator .

2.3-4: Integration.

The function space C[a, b], as a set X we take the set of all real-valued functions x, y, …which are functions of independent real variable t and are defined and continuous on a given closed bound interval J=[a, b]

Now, A linear operator T from C[a, b]into itself can be defined by

T x(t) = $\int_a^t x(t)\ dt$ t∈ [a, b]

Proof:

Since any continuous function on [a , b] is integrable on [a , b]

⇒ T is well-defined

T((α x +β y)(t)) = $\int_a^t (\alpha x + \beta y)(t) \quad dt$

$= \int_a^t \alpha x(t) + \beta y(t) \quad dt$

$= \int_a^t \alpha x(t) \quad dt + \int_a^t \beta y(t) \quad dt$

$= \alpha \int_a^t x(t) \quad dt + \beta \int_a^t y(t) \quad dt$

=α T x(t) + β Ty(t)

Hence , T is an operator

2.3-5: Multiplication by t.

Another linear operator T from C[a , b] into itself is defined by

(T x)(t)=T x(t) = t x(t) , ∀ t ∈[a , b] (*)

Proof:

I want proof (*) is well-defined

Let x, y C[a ,b] s.t x = y

⇒ x(t) = y(t) for all t ∈ [a , b]

⇒ t x(t) = t y(t)

⇒ T x(t) + Ty(t)

Hence , (*) is well defined

Now I want to prove (*)is an operator

T(α x +β y)(t) = t(α x +β y)(t)

= t(α x(t)) + t (β y(t))

=α (t x(t)) + β (t y(t))

= α T x(t) + β Ty(t)

Hence , T is an operator

2.3-5 Theorem (range and null space).

Let T be a linear operator .

Then:

(a)The range R(T) is a vector space .

(b)If dim D(T) =n $< \infty$, then dim R(T) $\leq$ n .

(c)The null space N(T) is a vector space .

proof :

(a)We take any y_1, $y_2 \in R(T)$ and show that $\alpha y_1+\beta y_2 \in R(T)$ for any scalars α, β

Since $y_1, y_2 \in R(T)$, we have $y_1=Tx_1$, $y_2=Tx_2$ for some $x_1, x_2 \in D(T)$ and $\alpha x_1+\beta x_2 \in D(T)$ (since D(T) is a vector space).

The linearity of T yields

$$T(\alpha x_1+\beta x_2) = \alpha\ Tx_1 + \beta\ Tx_2$$
$$=\alpha y_1+\beta y_2$$

Hence $\alpha y_1+\beta y_2 \in R(T)$. (since y_1, $y_2 \in R(T)$ were arbitrary and so were the scalar)

(b) We choose n+1 elements $y_1, y_2, \ldots, y_{n+1}$ of R(T) in an arbitrary Fashion.

Then we have $y_1 = Tx_1, \ldots, y_{n+1} = T x_{n+1}$ for some $x_1, \ldots, x_{n+1}$ in D(T).

Since dim D(T) = n, this set $\{x_1, \ldots, x_{n+1}\}$ must be linearly dependent. Hence $\alpha_1 x_1 + \ldots + \alpha_{n+1} x_{n+1} = 0$ for some scalars $\alpha_1, \ldots, \alpha_{n+1}$, not all zero.

Since T is linear and T0 = 0, application of T on both sides gives

$T(\alpha_1 x_1 + \ldots + \alpha_{n+1} x_{n+1}) = \alpha_1 y_1 + \ldots + \alpha_{n+1} y_{n+1} = 0$

This shows that $\{y_1, \ldots, y_{n+1}\}$ is a linearly dependent set .
(since the α_i s are not all zero).

Remembering that this subset of R(T) was chosen in an arbitrary fashion , we conclude that R(T) has no linearly independent subsets of n+1 or elements .By the definition this means that dim $R(T) \leq n$

(c)We take any $x_1, x_2 \in N(T)$. then $Tx_1 = Tx_2 = 0$.

Since T is linear , for any α, β we have

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 = 0$$

This shows that $\alpha x_1 + \beta x_2 \in N(T)$.Hence N(T) is a vector space.

2.3-6 Theorem (Inverse operator).

Let X,Y be vector spaces, both domain D(T) complex. Let $T: D(T) \rightarrow Y$ be a linear operator with domain D(T) X

and range R(T) Y. then:

(a)The inverse $T^{-1}: R(T) \rightarrow D(T)$ exists if and only if

$Tx = 0 \Rightarrow x = 0$

(b)If T^{-1} exist, it is a linear operator.

(c)If dim $D(T) = n < \infty$ and T^{-1} exists, then dim R(T) = dim D(T)

Proof:

$(\Leftarrow)$ I want to prove T^{-1} is exist $\Leftrightarrow$ T is 1-1 (a)

Now , suppose $T(x) = 0 \Rightarrow x = 0$

Let $T(x_1) = T(x_2)$

$\Rightarrow$ $T(x_1) - T(x_2) = 0$

$\Rightarrow T(x_1 - x_2) = 0$ (since T is a linear operator)

$\Rightarrow x_1 - x_2 = o$ (from given)

$\Rightarrow x_1 = x_2$

T is $1 - 1 \Rightarrow$

T^{-1} is an exist $\Rightarrow$

($\Rightarrow$) I want to prove if $T(x) = 0 \Rightarrow x = 0$

Let T^{-1} is an exist then, $T(x_1) = T(x_2) \Rightarrow x_1 = x$

Take $x_2 = 0$, $T(x_1) = T(0) \Rightarrow x_1 = 0$

$\Rightarrow T(x_1) = 0 \Rightarrow x_1 = 0$ (since $T(0) = 0$)

This completes the proof of (a)

(b)We assume that T^{-1} exists and show that T^{-1} is linear.

$T^{-1}: R(T) \to D(T)$

$Y_1 = Tx_1$ and $y_2 = Tx_2$, where $x_1, x_2 \in D(T)$

Then $x_1 = T^{-1}y_1$ and $x_2 = T^{-1} y_2$

T is linear, so that for any scalars α and β we have

$\alpha y_1 + \beta y_2 = \alpha Tx_1 + \beta Tx_2 = T(\alpha x_1 + \beta x_2)$

Since $x_1 = T^{-1} y_2$, this implies

$T^{-1}(\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2 = \alpha T^{-1} y_1 + \beta T^{-1} y_2$

Hence, T^{-1} is linear.

(c) we have $\dim R(T) \leq \dim D(T)$ (1)(by theorem 2.3-5)

And $T^{-1}:R(T) \rightarrow D(T)$

$\Rightarrow \dim R(T^{-1})=\dim D(T) \leq \dim D(T^{-1}) = \dim R(T)$ (2)

Then from (1) , (2) $\dim R(T) = \dim D(T)$

2.3-7 Lemma (inverse of product).

Let $T:X \rightarrow Y$ and $S:Y \rightarrow Z$ be bijective linear operators, where X,Y,Z are vector spaces .Then the inverse $(ST)^{-1}:Z \rightarrow X$ of the product ST exists, and $(ST)^{-1}=T^{-1}S^{-1}$

Applications

Application(1):Let $T:D(T) \rightarrow Y$ be a linear operator whose inverse exists. If $\{x_1,\dots, x_n\}$ is a linearly independent set in D(T), Then the set $\{Tx_1,\dots,T x_n\}$ is an linearly independent.

Suppose $\alpha_1 Tx_1+\dots+\alpha_n T x_n=0$ for some scalars $\alpha_1,\dots\alpha_n$

$T(\alpha_1 x_1+\dots\alpha_n x_n)= T(0) =0$ (since T is linear) $\Rightarrow$

$\alpha_1 x_1+\dots+\alpha_n x_n= 0$ (since T is 1-1) $\Rightarrow$

But $x_1,\dots,x_n$ are linear independent

$\alpha_1=\dots=\alpha_n=0 \Rightarrow$

Hence the set $\{Tx_1,\dots,T x_n\}$is linearly independent

Application(2):Let $T:X \rightarrow Y$ be a linear operator and $\dim X=\dim Y=n<\infty$ Show that R(T)=Y if and only if

T^{-1}exist

Suppose $T:X \rightarrow Y$ is onto $T(X)=Y$, $\dim X=\dim Y=n$

$E=\{e_1,\dots,e_n\}$is a basis for X

Let y∈Y =T(X)

y=T x for some x∈X

x∈X=span$\{e_1,\ldots,e_n\}$

⇒ x=$\sum_{i=1}^{n}\alpha_i\ e_i$, for some $\alpha_1,\ldots,\alpha_n$

⇒ y=T x=T($\sum_{i=1}^{n}\alpha_i e_i$)=$\sum_{i=1}^{n}\alpha_i Te_i$

⇒ $\{Te_1,\ldots,Te_n\}$generates Y

⇒ $\{Te_1,\ldots,Te_n\}$ is a basis for Y

Now, let x∈ X ∋: $Tx = 0$,writing x=$\sum_{i=1}^{n}\alpha_i e_i$

0=T x= $\sum_{i=1}^{n}\alpha_i Te_i$ ⇒ $\alpha_1=\ldots=\alpha_n=0$

Since $\{T e_i: i =1,..,n\}$is linearly independent

⇒ T *is* 1_1⇒ T^{-1} :T(X)=Y→X exists

Conversely**,** Suppose T^{-1}:R(T)→ **X ex**ists

We have to prove R(T)=Y

Since T:X→R(T), T^{-1}:R(T)→X

⇒ dim R(T)≤dim X and dim X ≤dim R(T)

⇒ dim $R(T) = dimX$=n=dimY

Hence, R(T)=Y

2.4 Bounded and continuous linear operators.

2.4-1. Definition: Let X and Y be normed spaces and $T: X \to Y$

linear a operoter . The operator T is said to **be bounded** if there

is a number c such that for all $x \epsilon X$ $\|Tx\| \leq c\|x\|$ (1)

Hence, $\frac{\|Tx\|}{\|x\|} \leq c, x \neq 0 \Longrightarrow \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \leq c$

The number $\sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$ is denoted by $\|T\|$

From (1) we have $\|Tx\| \leq \|T\|\|x\|$

2.4-2. Lemma (Norm)

Let T be abounded linear operator

(a) $\|T\| = \sup_{\substack{x \in X \\ \|x\| = 1}} \|Tx\|$

(b) $\|T\| = \sup_{\substack{x \in X \\ X \neq 0}} \frac{\|Tx\|}{\|x\|}$ Satisfies the properties of the norm

Proof:

(a)Let $T: X \to Y$ be abounded linear operator

$\Longrightarrow c > 0 \ni: \|Tx\| \leq c\|x\| \ \ \forall x \in X$

$$\|\boldsymbol{T}\| = \mathop{Sup}_{\substack{x \in X \\ x \neq 0}} \frac{\|\boldsymbol{Tx}\|}{\|\boldsymbol{x}\|}$$

we want to prove $\|T\| = \sup_{\substack{x\epsilon X \\ \|x\|=1}} \|Tx\|$.

Let $\|x\|=\alpha$ and y= $(1/\alpha)$y , $x\neq 0$ $\|y\|=1$

And since T is linear and (1) is given $\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\alpha}$

$$= \sup_{\substack{x\in X \\ x\neq 0}} \left\|T(\frac{1}{\alpha}x)\right\| = \sup_{\substack{y\in X \\ \|y\|=1}} \|Ty\|$$

(b)1) since $\|Tx\| \geq 0$ and $\|x\| \geq 0$

$$\Rightarrow \quad \|T\| = \sup_{\substack{x\in X \\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in X \\ \|x\|=1}} \|\mathrm{Tx}\| \geq 0$$

2) Suppose $\|T\| = 0 \Rightarrow \sup_{\substack{x\in X \\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = 0$

$$\Rightarrow \frac{\|Tx\|}{\|x\|} = 0 \Rightarrow \|Tx\| = 0 \quad \forall x \in X, x \neq 0$$

$\Rightarrow Tx = 0 \quad \forall x \in X, x \neq 0$, hence T=0

3) $\|\alpha T\| = \sup_{\substack{x\in X \\ x\neq 0}} \frac{\|\alpha Tx\|}{x}$

$$= \sup_{\substack{x \in X \\ x \neq 0}} \frac{|\alpha| \|Tx\|}{\|x\|}$$

$$= |\alpha| \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = |\alpha| \|T\|$$

4)Let $T_1 : X \to Y$ *and* $T_2 : X \to Y$ are bounded linear Operator,

then :

$$\|T_1 + T_2\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|(T_1 + T_2)(x)\|}{\|x\|} = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|T_1(x) + T_2(x)\|}{\|x\|}$$

$$\leq \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|T_1(x)\|}{\|x\|} + \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|T_2(x)\|}{\|x\|}$$

$$= \|T_1\| + \|T_2\|$$

Examples:

2.4-3. (Identity operator) the identity operator $I : X \to X$ on a normed space X is bounded and has normed $\|I\| = 1$

$\|Ix\| = \frac{\|x\|}{\|x\|} = 1$, Hence I is bounded

2.4-4. (zero operator) :the zero operator $O: X \rightarrow Y$ on a normed space X is bounded and has

norm$\|O\| = 0$

2.4-5. (differentiation operator)

Let X be the normed space of all polynomials on J=[a, b] with norm given $\|x\| = \max|x(t)|, t \in J$.

A differentiation operator T:X→ Y is defined on x by

(T(x))(t)=$x`(t)$. T is well defined , since every polynomial x is differentiable and the derivative is unique, and x' is polynomial on $[0,1]$, let $x, y \in X$, then for any $t \in [0,1]$,

(T(x+y))(t)=(x+y)`(t)=x`(t)+y`(t)
=(Tx)(t)+(Ty)(t)=(Tx+Ty)(t)

T(x+ y)=T x+ Ty …(1).⇒

Now, let $\alpha \in$,then

(αTx)(t)=(αx)`(t)=$\alpha$(x)`(t)=(αTx)(t)

(αTx)=α(Tx)......(2)

From(1) and (2)T is linear

Now ,Let $x_n(t) = t^n$

$\Rightarrow x'_n(t) = nt^{n-1} \Rightarrow \|Tx_n\| = \|x'_n\| = \max |nt^{n-1}| = n$

$\|x_n\| = max|t^n| = 1 \Rightarrow \frac{\|Tx_n\|}{\|x_n\|} = n\,,\ n \in \boldsymbol{N}$…(*)

Suppose that T is bounded

$\Rightarrow \exists \quad some \quad c > 0 \ni$:

…..(**) $\|Tx\| \leq c\,\|x\| \qquad \forall x \in X$

Since c>0, by the Archimedes property $\exists n_c \in \boldsymbol{N}$

$\ni: n_c > c$

From (**),$\forall n \in \boldsymbol{N}\ n = \|Tx_n\| \leq c\|x_n\| = n$

$\Rightarrow n_c \leq c < n_c$ this contrary

$\Rightarrow \quad T\ is\ not\ bounded.$

2.4-6 Lemma (linear combinations).

Let $\{x_1,\ldots,x_n\}$ be a linearly independent set of vector in a normed space X . then there is a number c>0 such that for every choice of scalars $\alpha_1,\ldots,\alpha_n$ we have

$$\|\alpha_1 x_1+\ldots+\alpha_n x_n\| \geq c(|\alpha_2|+\ldots+|\alpha_n|)$$

2.4-7 Theorem (finite dimension).

If a normed space X is finite dimensional , then every linear operator on X is bounded.

Proof :

Let dim X=n , $\{e_1,\ldots,e_n\}$a basis for X , let $T:X\rightarrow Y$ be linear operator , Y is a normed space

Let $x=\sum_{i=1}^{n} \alpha_i\, e_i$, $\alpha_i \in K, i = 1, \ldots, n,$ and let $M=\underset{1\leq i\leq n}{Max} \|Te_i\|$.

$$\|Tx\| = \|T(\sum_{i=1}^{n} \alpha_i e_i \quad)\| = \|\sum_{i=1}^{n} \alpha_i\,(Te_i)\| \leq$$

$$\sum_{i=1}^{n} |\alpha_i| \|Te_i\| \leq \sum_{i=1}^{n} |\alpha_i| \left(\underset{1\leq i\leq n}{Max} \|Te_i\|\right) = M \sum_{i=1}^{n} |\alpha_i| \ ,$$

Since$\{e, \ldots, e_n\}$ Is linear independent ,then by lemma 2.4-6

$\Rightarrow \exists c>0 \ni$

$$\|x\| = \|\alpha_1 e_1 + \dots + \alpha_n e_n\| > c\sum_{i=1}^{n} |\alpha_i|$$

$$\Rightarrow \|Tx\| \le M\sum_{i=1}^{n} |\alpha_i| \le \frac{m}{c}\|\alpha_1 e_1 + \dots + \alpha_n e_n\| = \frac{m}{c}\|x\|$$

Hence, T is bounded.

Remark:

Let $T : X \to Y$ be any operator , not necessarily linear , where X an Y are normed spaces , the operator. T is continuous at an $x_0 \in X$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$\|Tx - Tx_0\|$ for all $x \in X$ satisfying $\|x - x_0\| < \delta$

2.4-8.thearem (continuity and boundedness):

let $T : X \to Y$ be linear operator , where X,Y are normed spaces , then :

(a)T is continuous if and only if T is bounded . (a

(b) If T is continuous at a single point , it is continuous on X.

Proof:

(a) suppose T is bounded

$\Rightarrow \exists c > 0 \ni: \quad \|Tx\| \leq \|T\| \|x\| \forall x \in X$

To show that T is continuous , we show that T is continuous at every point $x \in X$

So , let x_0 be arbitrary point in X , and let $\varepsilon > 0$ be given we need to find $\delta > 0 \ni:$

$$\text{if } \|x - x_0\| < \delta \text{ then } \|Tx - Tx_0\| < \varepsilon \quad , x \in X$$

Now, $\|Tx - Tx_0\| = \|T(x - x_0)\|$ (since T is linear)

$$\leq \|T\| \|x - x_0\| \quad \text{(since T is bounded)}$$

By taking $\delta = \dfrac{\varepsilon}{2\|T\|}$

if $\|x - x_0\| < \delta$

$$\Rightarrow \|Tx - Tx_0\| \leq \|T\| \|x - x_0\| \leq \|T\| \frac{\varepsilon}{2\|T\|} < \varepsilon$$

Since $x_o \in X$ was arbitrary , this shows that T is continuous.

Conversely , assume that T is continuous at an arbitrary $x_o \in X$

then :

for given $\varepsilon > 0 \; \exists \delta = \delta_\varepsilon > 0 \;\; \ni :$ if $\|x - x_0\| < \delta$

then $\|Tx - Tx_0\| < \varepsilon$

We want to show T is bounded . So,

i.e. $\exists c > 0 \;\; \ni : \|Tx\| \le c\|x\| \quad \forall x \in X$

let x be any element in X , $x \neq 0$

$$z = x_o + \frac{\delta}{2\|x\|}.x$$

$$\Rightarrow \|z - x_o\| = \left\| \frac{\delta}{2\|x\|}.x \right\| = \frac{\delta}{2}\frac{\|x\|}{\|x\|} < \delta$$

$$\Rightarrow \|Tz - Tx_o\| < \varepsilon$$

that is

$$\left\| Tx_o + \frac{\delta}{2\|x\|} Tx - Tx_o \right\| < \varepsilon$$

$$\Rightarrow \frac{\delta}{2\|x\|} \|Tx\| < \varepsilon \Rightarrow \|Tx\| < \frac{2\varepsilon}{\delta} \|x\|$$

$$take \quad c = \frac{2\varepsilon}{\delta}$$

$$\Rightarrow T \quad is \quad bounded$$

(b) Continuity of T at a point implies boundedness of T by the second part of the proof of (a), which in turn implies continuity of T by (a) .

2.4-9. corollary (continuity, null space)

let T be a bounded linear operator then :

a) $x_n \to x$ where x_n, x in X , implies $Tx_n \to Tx$

b) the null space N(T)is closed subspace of X.

proof :

let T be a bounded linear operator , and let $(x_n) \to x \ , \forall n \in N$

then:

$$\|Tx_n - Tx\| = \|T(x_n - x)\| \qquad \text{(since T is linear)}$$

$$\leq \|T\|\|x_n - x\| \qquad (*)(\text{since T is bounded })$$

Now, let $\varepsilon > 0$ be given , since $(x_n) \to x$, for

$\frac{\varepsilon}{2\|T\|}, \exists k_2 \in N \ni: \|x_n - x\| < \frac{\varepsilon}{2\|T\|} \qquad n \geq k_\varepsilon \qquad (1)$

Hence, where $n \geq k_\varepsilon$, from (*)

$$\Rightarrow \|Tx_n - Tx\| \leq \|T\| \|x_n - x\| \leq \|T\| \frac{\varepsilon}{2\|T\|} = \frac{\varepsilon}{2} < \varepsilon$$

Therefore, $Tx_n \to Tx$

(b)The null space $N(T) = \{x \in X \ : Tx = 0\}$, we want to (b)

prove N(T) is closed, So let $x \in \overline{N(T)}$

$$x \in \overline{N(T)} \Rightarrow \exists (x_n) \ \ in \ \ N(T) \ni: x_n \to x \ \ (by \ \ theorem)$$
$$\Rightarrow Tx_n \to Tx \ \ by \ \ part \ \ (a)$$

But $Tx_n = 0 \qquad (since \ \ x_n \in N(T))$

$\Rightarrow \quad \overline{N(T)} \subset N(T)$, Tx=0 $\Rightarrow$ $x \in N(T)$

since $N(T) \subset \overline{N(T)}$

$\Rightarrow$ $N(T) = \overline{N(T)}$

$\Rightarrow$ N(T) is closed

2.4-10.theorem (bounded linear extension):

let $T: D(T) \rightarrow Y$be abounded linear operator , where D(T)$\subset$X

and Y are a Banach space , then T has an extension

$\tilde{T}: \overline{D(T)} \rightarrow Y$

Where $\tilde{T}$ is abounded linear operator of norm $\|\tilde{T}\| = \|T\|$.

Proof :

Let $x \in \overline{D(T)} \Rightarrow \exists$ sequence (x_n) in X $\ni: x_n \rightarrow x$

$\Rightarrow \| x_n - x \| \rightarrow 0$

Define $\tilde{T}: \overline{D(T)} \rightarrow Y \left(Tx_n\right)_{n=1}^{\infty}$ is a sequence in Y.

$\tilde{T}(x) = \lim(Tx_n)_{n=1}^{\infty}$, to show that $\tilde{T}$ is will defined

Since $(x_n) \rightarrow x \quad \Rightarrow \quad (x_n)$ is Cauchy sequence in X ,

let $\varepsilon > 0$ be given

$$\exists k_\varepsilon \in N \ni: \|x_n - x_m\| < \varepsilon / \|T\| \qquad , \forall n, m \geq K_\varepsilon \to (1)$$

$$now \quad \|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \leq \|T\| \|x_n - x_m\|$$

$$\leq \|T\| \varepsilon / \|T\|$$

$$< \varepsilon$$

is Cauchy sequence in Y, since Y is a Banach space $\Rightarrow (Tx_n)$

converges $\Rightarrow \lim(Tx_n)$ exist. $\Rightarrow (Tx_n)$

We show that this definition is independent of the particular

choice of a sequence in D(T) converging to *x*. suppose that

$(x_n), (z_n)$ are two sequences in D(T) which convergence to *x*

and let (V_n) sequence D(T) in defined by

$(V_n) = (x_1, z_1, x_2 . z_2, \ldots\ldots\ldots\ldots)$, let $\varepsilon > 0$ be given

Since $(x_n), (z_n)$ converges to *x*,

$$\Rightarrow \exists k_1, k_2 \in N \ni : \|x_n - x\| < \varepsilon, \|z_n - x\| < \varepsilon \quad \forall n$$

$$Let\ k = max\{k_1, k_2\} \Rightarrow \|v_n - x\| < \varepsilon$$

$\Rightarrow \quad (v_n)$ a sequence convergence in D(T)

Since T is bounded linear operator

is converges $\Rightarrow (Tv_n)$

exist , since (Tx_n) and (Tz_n) are subsequence of $\Rightarrow \lim_{n\to\infty}(Tv_n)$

$(Tv_n) \Rightarrow$ they are converges to the same limit $\Rightarrow \lim(Tx_n) =$

$\lim(Tz_n) = \lim(Tv_n)$

To show

$\tilde{T}|_X = T\ ,\ let\ x \in D(T)$

$\Rightarrow$ The sequence $(x, x, \ldots, x)$ convergence to x

$\tilde{T}(x) = \lim(Tx, Tx, \ldots) = Tx \Rightarrow \tilde{T}|_X = T$

We want to show T is linear , let $x_1, x_2 \in X, \alpha \in K$

$\Rightarrow \exists (x_n), (x`_n)$ in D(T) $\ni: (x_n) \to x_1, (x`_n) \to x_2$

$$\tilde{T}(\alpha x_1 + x_2) = \lim T(\alpha(x_n) + (x`_n))$$
$$= \alpha \lim T(x_n) + \lim T(x`_n) = \alpha \tilde{T}(x_1) + \tilde{T}(x_2)$$

$$\|\tilde{T}\| = \sup \frac{\|\tilde{T}x\|}{\|x\|} \geq \sup \frac{\|\tilde{T}x\|}{\|x\|} = \|T\|$$

$$\|\tilde{T}x\| = \|\lim(Tx_n)\| = \|\lim T(x_n)\| \leq \lim \|T\|\|x_n\|$$
$$= \|T\|\|x\|$$

$$\Rightarrow \|\tilde{T}\| = \sup \frac{\|\tilde{T}x\|}{\|x\|} \leq \|T\| \Rightarrow \|\tilde{T}\| = \|T\|$$

Applications

Application(1): let X and be normed spaces , a linear operator T:X→ Y is bounded if and only if T maps bounded sets in X into bounded sets in Y

First recall that subset A of a metric space is said to be bounded

if its diameter $\delta(A)$ is finite number, where

$$\delta(A)=\sup_{x,y\in A}\|x-y\|$$

$$\delta(A)=\sup_{x,y\in A} d(x,y)<\infty$$

If $A\subseteq X$,X is normed space, then

Suppose that T is a bounded linear operator , and A be bounded

subset of X

$$\Rightarrow \sup_{x,y\in A}\|x-y\|=M<\infty$$

$$\Rightarrow \forall x,y\in A \qquad claim\, T(A)\, is \quad bounded$$

$$\|Tx-Ty\|=\|T(x-y)\|\le\|T\|\|x-y\|(\text{since} \quad T \quad is \quad bounded)$$

$$\le\|T\|M$$

$$\Rightarrow \delta(A)=\sup_{x,y\in A}\|Tx-Ty\|\le\|T\|M$$

$$\Rightarrow T(A) \quad is \quad bounded$$

Conversely ,suppose that T maps bounded sets into bounded in set Y , note that $A = \{x \in X : \|x\| \le 1\}$ is bounded subset of X

$\Rightarrow T(A)$ is bounded

, since $T(A)$ is bounded $let \quad x \in X, x \neq 0 \Rightarrow \frac{x}{\|x\|} \in A$

$\Rightarrow \exists M > 0 \ni: \|Tx - Ty\| \le M \qquad \forall x, y \in A$, since $0 \in A$, and T is linear $\Rightarrow T(0) = 0$,we have $\|Tx\| = \|Tx - T0\| \le M \qquad \forall x \in X$

Now, let x be any non-zero element in X , then

$$\frac{x}{\|x\|} \in A \Rightarrow \left\|T\left(\frac{x}{\|x\|}\right)\right\| \le M \Rightarrow \|Tx\| \le M \qquad \forall x \in X$$

Hence, T is bounded

Application(2): Let T be a bounded linear operator from a normed space X onto a normed space Y, if there is appositive b such that $\|Tx\| \ge b\|x\|$ for all $x \in X$, show that the $T^{-1}: Y \to X$ exists and is bounded.

I want to prove $T^{-1}:Y \to X$ $exist$

T^{-1} is exists

$\Leftrightarrow T$ is $one-to-one \Leftrightarrow N(T)=\{0\}, so$ let $x \in N(T)$

$\Rightarrow Tx=0$ $, x \in X,$ $since$ $\|Tx\| \geq b\|x\| \Rightarrow 0=\|0\|=$

$\|Tx\| \geq \|b\|\|x\| \Leftrightarrow 0 \leq \|x\| \leq 0 \Leftrightarrow \|x\|=0 \Leftrightarrow x=$

0 $, since$ $x \in N(T)$ was an $arbitrary \Rightarrow N(T)=\{0\}$

$\Rightarrow T$ is $one-to-one$

$hence$ $T^{-1} exist$, To show T^{-1} is bounded i.e.$\exists M>$

0 $, and$ $\forall y \in Y$ $\|T^{-1}y\| \leq M\|y\|.$ $since$ T $onto \Rightarrow \forall y \in$

Y $\exists x \in X$ $\ni: Tx=y, x=T^{-1}y$.

Hence,$\|x\|=\|T^{-1}y\| \leq \frac{1}{b}\|Tx\|=\frac{1}{b}\|y\|$ $since$ $\|Tx\| \geq b\|x\|$,

and $b \neq 0$

take M= $\frac{1}{b}>0 \Rightarrow \|T^{-1}y\| \leq M\|y\|$ $\forall y \in Y$

Therefore $T^{-1}$is bounded

2.5.linear functional

2.5-1 definition (linear functional)

A linear functional f is a linear operator with domain in a vector space X and range in the scalar field **K** (□ or □) of X , thus $f: X \to \boldsymbol{K}$.

2.5-2 definition (Bounded linear functional)

A bounded linear functional f is a bounded linear operator with range in the scalar field of the normed. Thus there exists a real number c such that for $|f(x)| \le c\,\|x\|$.

Furthermore , the norm of f is $sup \frac{|f(x)|}{\|x\|}$, or

$$\|f\| = sup_{\substack{x \in X \\ \|x\|=1}} |f(x)|$$

This implies , $|f(x)| \le \|f\|\|x\|$

2.5-3.Example:(define integral),

let $f: C[a,b] \to$ □ $, f(x) = \int_a^b x(t)dt$ $, \forall x \epsilon C[a,b], t \epsilon [a,b]$ .Then:

f is a bounded linear functional on C[a, b].

proof:

Let x ,yϵC$[a,b]$ *and Let* $\alpha\epsilon$ □ **,**then

f(αx + y) = $\int_a^b (\alpha x + y)\,(t)dt = \int_a^b \big(\alpha x(t) + y(t)\big)\,dt$

= $\alpha \int_a^b x(t)$dt+ $\int_a^b y(t)\,dt = \alpha f(x) + f(y)$

$\Rightarrow$ f is linear .

$$|f(x)| = \left|\int_a^b x(t)dt\right| \le \int_a^b |x(t)|dt \le \int_a^b \max|x(t)|dt$$

$$= \int_a^b \|x\|dt = \|x\|(b-a)$$

That is , $\|f\| = sup_{\underbrace{x\epsilon[a,b]}_{x\neq 0}} \frac{|f(x)|}{\|x\|} \le b - a$ …… (1)

note that, $x_0 : [a,b] \rightarrow$ □ $, x_0(t) = 1, \|x_0\| = 1$

$\|f\| = sup \frac{|f(x)|}{\|x\|} \ge \frac{|f(x_0)|}{\|x_0\|} = b - a$ ……(2)

From (1),(2) $\|f\| = b - a$

Examples:

2.5-4. Let $t_0 \in [a, b]$ be a fixed point, and define $f: C[a, b] \to \mathbb{R}$ by

$$f(x) = x(t_0) \qquad , x \in c[a, b].$$

Then f is a bounded linear functional on C[a, b], and $\|f\| = 1$

let x ,y$\epsilon C[a, b], \alpha, \beta \epsilon$ ▯ or ▯

$f(\alpha x + \beta y) = (\alpha x + \beta y)(t_0)$

$= \alpha x(t_0) + \beta y(t_0) = \alpha f(x) + \beta f(y)$

Hence, f is linear.

Now, I want to prove $f_1$is bounded and has norm $\|f\| = 1$

$|f(x)| = |x(t_0)| \le \max_{t \in [a,b]} |x(t)| = ▯ x ▯$

$\|f\| = sup \frac{|f(x)|}{\|x\|} \le 1$ …..(1) =>f is bounded

For $x_0 = 1, x_0 : [a, b] \to$ ▯ $, x_0(t) = 1 \ \forall \ t \epsilon [a, b]$

$\|f\| = sup \frac{|f(x)|}{\|x\|} \ge \frac{|f(x_0)|}{\|x_0\|} = 1$ …. (2)

from (1) and (2) $\|f\| = 1$

Applications

Application(1):let $f \neq 0$be any linear functional an a vector space X, and

$x_0$any fixed element of X$-N(f)$, where is the null space of f . Then each $x \in X$ has a unique representation$x = \alpha x_0 + y$,where y$\epsilon N(f)$.

Proof:

Let $x \in X$, and note that $\left(x - \frac{f(x)}{f(x_o)} . x_o\right) \in N(f)$

Since $f\left(x - \frac{f(x)}{f(x_0)} . x_0\right) = f(x) - \frac{f(x)}{f(x_0)} . f(x_0) = 0$

$$\Rightarrow f\left(x - \frac{f(x)}{f(x_o)} . x_o\right) = 0$$

for some $y \in N(f) \Rightarrow x - \frac{f(x)}{f(x_o)} . x_o = y$

(for the uniqueness) $\Rightarrow x = \frac{f(x)}{f(x_o)} . x_o + y \qquad \rightarrow (*)$

Let $x \in X$, suppous

$$x = \alpha_1 x_o + y_1 = \alpha_2 x_o + y_2 \quad y_1, y_2 \in N(f)$$

$$\Rightarrow f(x) = \alpha_1 f(x_o) = \alpha_2 f(x_o)$$

$$\Rightarrow \alpha_1 = \alpha_2 \rightarrow y_1 = y_2$$

Hence , the representation in (*) is unique

Application(2): Let $f: X \to K$ be a linear function, then either $f \equiv 0$ on X or $f(X) = K$.

Suppose $f \neq 0$ and suppose on the contrary that $f(X) \neq K$

$\Rightarrow \quad \exists \alpha \epsilon K \ni \alpha \notin f(X)$

Since $f \neq 0 \quad \Rightarrow y \in X \ni f(y) \neq 0$

Hence, $\frac{\alpha}{f(y)} y \in X$ and $f(\frac{\alpha}{f(y)} y) \in f(X)$

But $\alpha = \frac{\alpha}{f(y)} f(y) = f\left(\frac{\alpha}{f(y)} y\right) \epsilon f(X)$

Our assumption that $f(X) \neq K$ is false , and we must have $f(X) = K$

Chapter (3)

3.1:Inner product spaces, Hilbert spaces:

The spaces to be considered in this chapter are defined as follows.

3.1-1:Definition:

An inner product space on a vector space X(over $\mathbb{R}$ or $\mathbb{C}$) is a real-valued function, $\langle , \rangle : X \times X \to \mathbb{R}$,

Which is satisfies the following properties :

Let x,y and z be any vectors , and a scalar α .

(1) $\langle x , x \rangle \geq 0$

(2) $\langle x , x \rangle = 0 \Leftrightarrow x = 0$

(3) $\langle x + y , z \rangle = \langle x , z \rangle + \langle y , z \rangle$

(4) $\langle \alpha x , y \rangle = \alpha \langle x , y \rangle$

(5) $\langle x , y \rangle = \overline{\langle y , x \rangle}$

The complete of inner product space with the metric induced by inner product is called a Hilbert space.

We define a norm and a metric in an inner product space by,

$\| x \| = \sqrt{\langle x , x \rangle}, \quad \forall x \in X$.

So $d(x , y) = \sqrt{\langle x - y , x - y \rangle}$.

Hence inner product spaces are normed spaces, and Hilbert spaces are Banach spaces.

3.1-2:Remarks:

1-$\langle \alpha x + \beta y , z \rangle = \alpha \langle x , z \rangle + \beta \langle y , z \rangle$

2-$\langle x , \alpha y \rangle = \overline{\alpha} \langle x , y \rangle$

3-$\langle x , \alpha y + \beta z \rangle = \overline{\alpha} \langle x , y \rangle + \overline{\beta} \langle x , z \rangle$

3.1-3:Defination: An element x of an inner product space X is said to be orthogonal to an element $y \in X$ if $\langle x , y \rangle = 0$

Examples:

3.1-4: The Unitary space

$$\mathbb{C}^n = \{x : x = (\xi_1, \xi_2, ..., \xi_n), \xi_i \in \mathbb{C} \ \forall i = 1, 2, ..., n\}$$

is an inner product space with the inner product defined by

$\langle x, y \rangle = \sum_{i=1}^{n} \xi_i \overline{\eta_i}$,where $x = (\xi_1, \xi_2, ..., \xi_n)$,

$y = (\eta_1, \eta_2, ..., \eta_n) \in \mathbb{C}^n$.

Since $\sum_{i=1}^{n} \xi_i \overline{\eta_i}$ is finite series, then it is convergent. Hence

$\langle x, y \rangle = \sum_{i=1}^{n} \xi_i \overline{\eta_i}$ is well defined

Now we show $\langle x, y \rangle = \sum_{i=1}^{n} \xi_i \overline{\eta_i} \ni x, y \in \mathbb{C}^n$ is an inner product .

1- $\langle x, x\rangle = \left(\sum_{i=1}^{n} |\xi_i|^2\right)^2 \geq 0$

2- $\langle x,x\rangle = 0 \Leftrightarrow \left(\sum_{i=1}^{n} |\xi_i|^2\right)^2 = 0 \Leftrightarrow \sum_{i=1}^{n} |\xi_i|^2 = 0$

$\Leftrightarrow |\xi_i|^2 = 0, \forall i = 1, 2, ..., n$

$\Leftrightarrow \xi_i = 0, \forall i = 1, 2, ..., n$

$\Leftrightarrow x = 0$

3-Let $z = (\beta_i)$ $\langle x + y, z\rangle = \sum_{i=1}^{n} (\xi_i + \eta_i)\overline{\beta_i}$

$= \sum_{i=1}^{n} \xi_i \overline{\beta_i} + \eta_i \overline{\beta_i} = \sum_{i=1}^{n} \xi_i \overline{\beta_i} + \sum_{i=1}^{n} \eta_i \overline{\beta_i}$

$= \langle x, z\rangle + \langle y, z\rangle$

4- $\langle \alpha x, y\rangle = \sum_{i=1}^{n} (\alpha\xi_i)\overline{\eta_i} = \sum_{i=1}^{n} \alpha\left(\xi_i \overline{\eta_i}\right) = \alpha\sum_{i=1}^{n} \xi_i \overline{\eta_i}$

$= \alpha\langle x, y\rangle$

5- $\overline{\langle y, x\rangle} = \overline{\sum_{i=1}^{n} \eta_i \overline{\xi_i}} = \sum_{i=1}^{n} \xi_i \overline{\eta_i} = \langle x, y\rangle$

3.1-5: The Space $\ell^2 = \left\{ x = (\xi_i)_{i=1}^{\infty}, \xi_i \in \mathbb{C}, \sum_{i=1}^{\infty} |\xi_i|^2 < \infty \right\}$ is

inner product space with an inner product

defined by $\langle , \rangle : \ell^2 \times \ell^2 \to \mathbb{C}$

$$x, y \in \ell^2, x = (\xi_i), y = (\eta_i) \ldots\ldots(*)\ \langle x, y \rangle = \sum_{i=1}^{\infty} \xi_i \overline{\eta_i}$$

Proof

Let $x, y \in \ell^2, x = (\xi_i), y = (\eta_i)$

By Cauchy-Schwarz inequality

$$\sum_{i=1}^{\infty} |\xi_i \overline{\eta_i}| \le \left(\sum_{i=1}^{\infty} |\xi_i|^2 \right)^{\frac{1}{2}} . \left(\sum_{i=1}^{\infty} |\overline{\eta_i}|^2 \right)^{\frac{1}{2}} < \infty$$

since $\sum_{i=1}^{\infty} |\xi_i|^2 < \infty, \sum_{i=1}^{\infty} |\overline{\eta_i}|^2 < \infty$ and $|\overline{\eta_i}| = |\eta_i|, \forall \eta_i \in \mathbb{C}$.

Then $\sum_{i=1}^{\infty} \xi_i \overline{\eta_i}$ is absolutely convergent series in $\mathbb{C}$ with usual metric since $\mathbb{C}$ is complet, every absolutely convergent series is convergent.

Hence, the map given by (*) wwell defined

Now we prove that (*) defines an inner product .

1- $\langle x, x\rangle = \left(\sum_{i=1}^{\infty} |\xi_i|^2\right)^2 \geq 0$

2- $\langle x, x\rangle = 0 \Leftrightarrow \left(\sum_{i=1}^{\infty} |\xi_i|^2\right)^2 = 0 \Leftrightarrow \sum_{i=1}^{\infty} |\xi_i|^2 = 0$

$\Leftrightarrow |\xi_i|^2 = 0, \forall i = 1, 2, ..., n$

$\Leftrightarrow \xi_i = 0, \forall i = 1, 2, ..., n$

$\Leftrightarrow x = 0$

3-Let $z = (\beta_i)$ $\langle x + y, z\rangle = \sum_{i=1}^{\infty} (\xi_i + \eta_i)\overline{\beta_i}$

$= \sum_{i=1}^{\infty} \xi_i \overline{\beta_i} + \eta_i \overline{\beta_i} = \sum_{i=1}^{\infty} \xi_i \overline{\beta_i} + \sum_{i=1}^{\infty} \eta_i \overline{\beta_i} = \langle x, z\rangle + \langle y, z\rangle$

4- $\langle \alpha x, y\rangle = \sum_{i=1}^{\infty} (\alpha \xi_i)\overline{\eta_i} = \sum_{i=1}^{\infty} \alpha\left(\xi_i \overline{\eta_i}\right) = \alpha \sum_{i=1}^{\infty} \xi_i \overline{\eta_i}$

$= \alpha\langle x, y\rangle$

5- $\overline{\langle y, x\rangle} = \sum_{i=1}^{\infty} \eta_i \overline{\xi_i} = \sum_{i=1}^{\infty} \xi_i \overline{\eta_i} = \langle x, y\rangle$

Note :

We can show by a simple straightforward calculation that a norm on an inner product space satisfies the important parallelogram equality.

$$\|x+y\|^2+\|x-y\|^2=2\left(\|x\|^2+\|y\|^2\right)^2$$

3.1-6: The $\left(C[a,b],\|.\|\right)$ with the norm defined by

$\|x\|=\max_{t\in[a,b]}|x(t)|$ is not an inner product space. We prove that by showing that the norm doesn't satisfy the important parallelogram equality.

$$\|x+y\|^2+\|x-y\|^2=2\left(\|x\|^2+\|y\|^2\right)^2$$

Let f,g $\in C[a,b]$,such that $f(t)=1, g(t)=\dfrac{t-a}{b-a}$

as $t\in[a,b]$,hear $\|f\|=\max_{t\in[a,b]}|f(t)|$,

Where f$\in C[a,b]$.

$\|f\| = 1$

$$= \max_{t\in[a,b]} \left|\frac{t-a}{b-a}\right| = \left|\frac{b-a}{b-a}\right| = 1 \quad \|g\| = \max_{t\in[a,b]} |g(t)|$$

$$\|f+g\| = \max_{t\in[a,b]} \left|1+\frac{t-a}{b-a}\right| = 1+\frac{b-a}{b-a} = 2$$

$$\|f-g\| = \max_{t\in[a,b]} \left|1-\frac{t-a}{b-a}\right| = 1+\frac{a-a}{b-a} = 1$$

$$\therefore \|f+g\|^2 + \|f-g\|^2 = 4+1 = 5$$

and $2\left(\|f\|^2 + \|g\|^2\right)^2 = 2(1+1) = 4 \neq 5$

$$\therefore \|f+g\|^2 + \|f-g\|^2 \neq 2\left(\|f\|^2 + \|g\|^2\right)^2$$

Hence $(C[a,b], \|.\|)$ is not an inner product space.

Applications

Application(1):Let X be a real product space, the condition $\|x\| = \|y\|$ implies $\langle x+y, x-y\rangle = 0$?

Proof:

$$\langle x+y, x-y\rangle = \langle x,x\rangle + \langle y,x\rangle - \langle x,y\rangle - \langle y,y\rangle$$
$$= \|x\|^2 + \langle y,x\rangle - \langle x,y\rangle - \|y\|^2$$

Since X is real $\Rightarrow \langle y,x\rangle = \langle x,y\rangle$ then;

$$\langle x+y, x-y\rangle = \langle x,x\rangle - \langle y,y\rangle = \|x\|^2 - \|y\|^2$$

Since $\|x\| = \|y\| \Rightarrow \|x\|^2 = \|y\|^2$ then $\langle x+y, x-y\rangle = 0$

Application(2):If an inner product space X, let $u, v \in X$. If $\langle x,u\rangle = \langle x,v\rangle$ for all $x \in X$ and ,then $u = v$.

Proof :

If $\langle x,u\rangle = \langle x,v\rangle$, $\forall x \in X$

$$\Rightarrow \langle x,u\rangle - \langle x,v\rangle = 0 \Rightarrow \langle x,u-v\rangle = 0$$

In particular when $x = u - v$.

$$\|u-v\|^2 = \langle u-v, u-v\rangle = 0 \Rightarrow u - v = 0 \Rightarrow u = v$$

3.2 Further Properties of Inner Product Space.

3.2-1:Lemma(Schwarz inequality, triangle inequality).

An inner product X and the corresponding norm satisfy the Schwarz inequality and triangle inequality as follows.

1- $|\langle x, y\rangle| \leq \|x\|\|y\|, \forall x, y \in X$ ..... (*) (Schwarz inequality)

Where the equality sign holds if and only if $\{x, y\}$ is a linearly dependent set.

2-The norm also satisfies $\|x+y\| \leq \|x\|+\|y\|$ (Tringle inequality), where the equality sign holds if and only if

$y=0$ or $x=cy$ $(c \in \mathbb{R}^{+})$

Proof:

Note that (*) holds if ether x or y is zero. So suppose that nether x or y is zero. Then for every scalar α we have,

$$0 \leq \|x-\alpha y\|^{2}=\langle x-\alpha y, x-\alpha y\rangle$$

$$=\langle x, x\rangle-\langle x, \alpha y\rangle-\langle\alpha y, x\rangle+\langle\alpha y, \alpha y\rangle$$

$$=\langle x,x\rangle-\overline{\alpha}\langle x,y\rangle-\alpha\langle y,x\rangle+\alpha\overline{\alpha}\langle y,y\rangle$$

$$=\langle x,x\rangle-\overline{\alpha}\langle x,y\rangle-\alpha\left[\langle y,x\rangle+\overline{\alpha}\langle y,y\rangle\right]$$

In particular, when $\overline{\alpha}=\frac{\overline{\langle x,y\rangle}}{\langle y,y\rangle}$,We have,

$$0\le\langle x,x\rangle-\frac{\overline{\langle x,y\rangle}}{\langle y,y\rangle}\langle x,y\rangle=\|x\|^2-\frac{|\langle x,y\rangle|^2}{\|y\|^2}$$

So, multiplying two sides of $0\le\|x\|^2-\frac{|\langle x,y\rangle|^2}{\|y\|^2}by\ \|y\|^2$,

then we have $0\le\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\Rightarrow|\langle x,y\rangle|^2\le\|x\|^2\|y\|^2$.

Hence $|\langle x,y\rangle|\le\|x\|\|y\|$

Now we show the equality in (*) holds if and only if x,y are linearly dependent.

If $y=\alpha x$ for some $\alpha\in\mathbb{F}$ then,

L.H.S $|\langle x,y\rangle|=|\overline{\alpha}\langle x,x\rangle|=|\alpha|\|x\|^2$

R.H.S $\|x\|\|y\|=\|x\|\|\alpha x\|=|\alpha|\|x\|^2$;

So $|\langle x,y\rangle|=\|x\|\|y\|$

Conversely, showing if $|\langle x,y\rangle| = \|x\|\|y\|$, then x, y are linearly dependent.

Suppose that $z = x - \frac{\langle x,y\rangle}{\|y\|^2} y$, for some $z \in X$

$$\langle z,z\rangle = \langle x - \frac{\langle x,y\rangle}{\|y\|^2} y, x - \frac{\langle x,y\rangle}{\|y\|^2} y\rangle$$

$$= \langle x,x\rangle - \frac{\overline{\langle x,y\rangle}}{\|y\|^2}\langle x,y\rangle - \frac{\langle x,y\rangle}{\|y\|^2}\langle y,x\rangle + \frac{|\langle x,y\rangle|^2}{\|y\|^4}\langle y,y\rangle$$

$$= \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2} - \frac{|\langle x,y\rangle|^2}{\|y\|^2} + \frac{|\langle x,y\rangle|^2}{\|y\|^4}\|y\|^2$$

$$= \|x\|^2 - \frac{\|x\|^2\|y\|^2}{\|y\|^2} - \frac{\|x\|^2\|y\|^2}{\|y\|^2} + \frac{\|x\|^2\|y\|^2}{\|y\|^2}\|y\|^2 = 0$$

Hence $\|z\|^2 = 0 \Rightarrow z = 0 \Rightarrow x - \frac{\langle x,y\rangle}{\|y\|^2} y = 0$

$$\Rightarrow x = \frac{\langle x,y\rangle}{\|y\|^2} y$$

We know $\frac{\langle x,y\rangle}{\|y\|^2} \in \mathbb{F}$ then x,y are linearly dependent.

3.2-2:Lemma: (continuity of inner product).

If in an inner product space, $x_n \to x$ and $y_n \to y$, then

$$\langle x_n, y_n \rangle \to \langle x, y \rangle$$

Proof:

We want to prove $\langle x_n, y_n \rangle \to \langle x, y \rangle$,since $x_n \to x$ and $y_n \to y$. This implies $x_n - x \to 0$ and $y_n - y \to 0$ .In inner product space that means, $\| x_n - x \| \to 0$ and $\| y_n - y \| \to 0$,as $n \to \infty$,so we have :

$$|\langle x_n, y_n \rangle - \langle x, y \rangle| = |\langle x_n, y_n \rangle - \langle x_n, y \rangle + \langle x_n, y \rangle - \langle x, y \rangle|$$

$$\leq |\langle x_n, y_n - y \rangle| + |\langle x_n - x, y \rangle|$$

(by tringle inequality)

$$\leq \| x_n \| \| y_n - y \| + \| x_n - x \| \| y \|$$

(by Schwarz inequality)

$$\to 0$$

$$\Rightarrow \langle x_n, y_n \rangle - \langle x, y \rangle \to 0 \Rightarrow \langle x_n, y_n \rangle \to \langle x, y \rangle$$

Applications

Application(1): Show that $y \perp x_n$ and $x_n \to x$ together imply $y \perp x$.

Let (x_n) be sequence in X such that (x_n) converges to an element $x \in X$. If $y \in X \ni: y \perp x_n, \forall n \in \mathbb{N}$ .Then $y \perp x$. Id

Since $y \perp x_n \Rightarrow \langle x_n, y \rangle = 0, \forall n \in \mathbb{N}$.

But (x_n) converges to x we have, by lemma 3.2-2,

$\Rightarrow \langle x, y \rangle = 0 \Rightarrow y \perp x$.

Application(2): For a sequence (x_n) in an inner product space the condition $\|x_n\| \to \|x\|$ and $\langle x_n, x \rangle \to \langle x, x \rangle$ imply convergence $x_n \to x$.

Proof :

Let (x_n) be sequence in Xsuch that if $\|x_n\| \to \|x\|$ and $\langle x_n, x \rangle \to \langle x, x \rangle$, then $\|x_n - x\| \to 0$.

Note that since $\langle x_n, x \rangle \to \langle x, x \rangle$,we have

$\langle x, x_n \rangle = \overline{\langle x_n, x \rangle} \to \overline{\langle x, x \rangle} = \langle x, x \rangle$

Therefore, $\|x_n - x\|^2 = \langle x_n - x, x_n - x\rangle$

$$= \|x_n\|^2 - \langle x_n, x\rangle - \langle x, x_n\rangle + \|x\|^2$$

$$\to \|x\|^2 - \langle x, x\rangle - \langle x, x\rangle + \|x\|^2 = 0$$

Hence $\|x_n - x\| \to 0$, i.e $x_n \to x$

Application(3): Let X be an inner product space ,and let $x, y \in X$. Then $x \perp y$ and only if we have $\|x + \alpha y\| = \|x - \alpha y\|$ for all scalar α .

Proof:

Let $x \perp y$, then

$\langle x, y\rangle = 0, \langle y, x\rangle = 0$. Therfore,

$\|x + \alpha y\|^2 = \langle x + \alpha y, x + \alpha y\rangle$

$$= \langle x, x\rangle + \alpha\langle y, x\rangle + \overline{\alpha}\langle x, y\rangle + |\alpha|^2 \langle y, y\rangle$$

$$= \|x\|^2 + |\alpha|^2 \|y\|^2 \quad \text{---------------(1)}$$

Also, $\|x - \alpha y\|^2 = \langle x - \alpha y, x - \alpha y\rangle$

$$=\langle x,x\rangle-\alpha\langle y,x\rangle-\overline{\alpha}\langle x,y\rangle+|\alpha|^2\langle y,y\rangle$$

$$=\|x\|^2+|\alpha|^2\|y\|^2 \quad \text{--------------}(2)$$

From (1) and (2) , we have

$$\|x-\alpha y\|^2=\|x\|^2+|\alpha|^2\|y\|^2=\|x+\alpha y\|^2$$

$$\Rightarrow \|x+\alpha y\|=\|x-\alpha y\|$$

Conversely, let $\|x+\alpha y\|=\|x-\alpha y\|$ for any scalar α , then

$$\Rightarrow \langle x+\alpha y,x+\alpha y\rangle^{\frac{1}{2}}=\langle x-\alpha y,x-\alpha y\rangle^{\frac{1}{2}}$$

$$\Rightarrow \langle x+\alpha y,x+\alpha y\rangle=\langle x-\alpha y,x-\alpha y\rangle \quad \text{for any scalar } \alpha .$$

$$\Rightarrow \|x\|^2+\alpha\langle y,x\rangle+\overline{\alpha}\langle x,y\rangle+|\alpha|^2\|y\|^2$$
$$=\|x\|^2-\alpha\langle y,x\rangle-\overline{\alpha}\langle x,y\rangle+|\alpha|^2\|y\|^2$$

$$\Rightarrow \alpha\langle y,x\rangle+\overline{\alpha}\langle x,y\rangle=0$$

In particular when $\alpha=i$, we have

$$\langle y,x\rangle+\langle x,y\rangle=0\Rightarrow\langle y,x\rangle=-\langle x,y\rangle$$

Also; when $\alpha=i$, we have

$i\langle y,x\rangle - i\langle x,y\rangle = 0$

Hence $i\langle y,x\rangle - i(-\langle y,x\rangle) = 0 \Rightarrow 2i\langle y,x\rangle = 0$

$\Rightarrow \langle y,x\rangle = 0 \Rightarrow x \perp y$.

3.3.Representation of Functional on Hilbert Spaces.

3.3-1 Theorem (Direct sum)

Let Y be any closed subspace of a Hilbert space H. Then

H=Y$\oplus$Z Z=$Y^{\perp}$.

3.3-2.Riesez Theorem (Functionals on Hilbert spaces).

Every bounded linear functional f on a Hilbert spaces H can be represented in terms of the inner product , namely,

$$f(x)= \langle x, z\rangle \qquad (1)$$

where z depends on f, is uniquely determined by f and has norm $\|z\| = \|f\|$ (2).

Proof:

We proof that

(a) f has representations (1),

(b) z in (1) is unique

(c) formula (2) holds

if f =0 then (1) and (2) hold, (a) Let f $\neq$ 0

since N(f) is a close subspace of H then

H=N(f) + $N(f)^{\perp}$ (by theorem 3.3-1)

Since f≠0 implies $N(f) \neq H$ so that $N(f)^{\perp} \neq \{0\}$

Hence $N(f)^{\perp}$ contains a $z_0 \neq 0$ and let x be any elemant in H

$v= f(x)\, z_0 - f(z_0)x$

applying f , we obtain

$f(v)= f(x)f(z_0)- f(z_0)f(x) =0$

This show that $v \in N(f)$ since $z_0 \perp N(f)$, we have

$0=\langle v, z_0\rangle = \langle f(x)z_0 - f(z_0)x, z_0\rangle$

$=f(x)\langle z_0, z_0\rangle + f(z_0)\langle x, z_0\rangle$

We solve for f(x).the result is

$$f(x)=\frac{\overline{f(z_0)}}{<z0,zo>} < x, z0 >$$

this can be written in the (1),where $z = z_0 \frac{\overline{f(z_0)}}{<zo,z0>}$

since x∈H was arbitrary, (1) is proved.

(b) To prove that z in (1) is unique,

Suppose that for all x∈H, $f(x)=\langle x, z_1\rangle = \langle x, z_2\rangle$

Then $\langle x, z_1 - z_2\rangle=0$ for all x.

Choosing the particular $x = z_1-z_2$,we have

$\langle x, z_1 - z_2\rangle = \langle z_1 - z_2, z_1 - z_2\rangle = \| z_1 - z_2 \|^2=0$

Hence $z_1-z_2=0$, so that $z_1=z_2$, the uniqueness.

(c)we finally prove (2).

From (1)with x=z and $|f(x)| \leq \|f\| \, \|x\|$ we obtain

$\|z\|^2 = \langle z, z\rangle = f(z) \leq \|f\| \, \|z\|$

$\|z\| \leq \|f\|$ (1) (since $\|z\| \neq 0$) $\Rightarrow$

Since $f(x) = \langle x, z\rangle$

$|f(x)| = |<x, z>| \leq \|x\| \, \|z\|$ (by Schwarz inequality) $\Rightarrow$

This implies $\|f\| = \sup_{\|x\|=1} < x, z > \leq \|z\|$ (2)

From (1)and (2) $\|f\| = \|z\|$

3.3-3lemma(Equlity).

if $\langle v_1, w\rangle = \langle v_2, w\rangle$ for all w in an inner product space X , then $v_1 = v_2$. In particular, $\langle v_1, w\rangle = 0$ for all $w \in X$ implies $v_1 = 0$

proof:

by assumption, for all w,

$\langle v_1 - v_2, w\rangle = \langle v_1, w\rangle - \langle v_2, w\rangle = 0$

For $w = v_1 - v_2$ this gives $\|v_1 - v_2\|^2 = 0$. Hence $v_1 - v_2 = 0$, so that $v_1 = v_2$

In particular, $\langle v_1, w\rangle = 0$ with $w = v_1$ gives $\|v_1\|^2 = 0$, so that $v_1 = 0$

3.3-4Definition(Sesquiliner form).

let X and Y be vector spaces over the same field K(=**R** or **C**). Then **a sesquilinear** form h on $X \times Y$

is mapping $h: X \times Y \rightarrow \mathbf{K}$ such that for all $x, x_1, x_2 \in Y$

and all scalars α, β

(a) $h(x_1 + x_2, y) = h(x_1, y) + h(x_2, y)$

(b) $h(x, y_1 + y_2) = h(x, y_1) + h(x, y_2)$

(c) h(αx,y)=αh(x,y)

(d) h(x,βy)=βh(x,y)

Hence h is linear in the first argument and conjugate linear in the second one. If X and Y are real then (d) is simply h(x,βy)=βh(x,y) ,$\forall x \in X, y \in Y, \beta \in \boldsymbol{R}$

h is called **bilinear** since it is linear in both argument .

If X and Y are normed spaces and if there is a real number c such that for all x, y

$$h(x, y)| \leq c \|x\| \|y\|$$

then h is said to be bounded , and the number

$$\|h\| = \sup_{\substack{x \in X-\{0\} \\ y \in Y-\{0\}}} \frac{|h(x,y)|}{\|x\| \|y\|} = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |h(x,y)| \qquad (\text{I})$$

Is called the norm of h.

3.3-5 Theorm (Riesz represntation).

Let H_1, H_2 be Hilbert spaces and $h: H_1 \times H_2 \rightarrow K$ a bounded sesquilinear form. Then h has a representation

$$h(x, y) = \langle Sx, y \rangle \qquad (1)$$

where $s: H_1 \rightarrow H_2$ is a bounded linear operator. S is uniquely determined by h and has norm

$$\|S\| = \|h\|$$

Proof: For each fixed x$\in H_1$ $define\ f_x\!: H_2 \to \boldsymbol{C}\ by$

$f_x(y) = \overline{h(x,y)}$. Then f_x is a linear in H_2, which is bounded since h is bounded. Then by the previous theorem , ∃unique element z$\in H_2$ $such\ that$

$$\overline{h(x,y)} = \langle y, z\rangle$$

Hence,

$$h(x, y) = \langle z, y\rangle \quad (*)$$

Define $S\!:H_1 \to H_2$ by $z = S\,x$

Substituting z = S x in (*), we have

$$h(x, y) = \langle Sx, y\rangle$$

S is linear. In fact, its domain is the vector space H_1, and from (1) $\langle S(\alpha x_1 + \beta x_2), y\rangle = h(\alpha x_1 + \beta x_2, y)$

$$= \alpha h(x_1, y) + \beta h(x_2, y)$$

$$= \alpha\langle Sx_1, y\rangle + \langle Sx_2, y\rangle$$

$$= \langle \alpha Sx_1 + \beta x_2, y\rangle$$

For all y in H_2, so that by Lemma 3.3-2,

$$S(\alpha x_1 + \beta x_2) = \alpha Sx_1 + \beta Sx_2$$

S is bounded. Indeed, leaving aside the trivial case S=0, we have from (I)and(*)

$$\|h\| = \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|\langle Sx, y\rangle|}{\|x\|\,\|y\|} \geq \sup_{\substack{x\neq 0\\ Sx0}} \frac{|\langle Sx, Sx\rangle|}{\|x\|\,\|Sx\|} = \sup_{x\neq 0} \frac{\|Sx\|}{\|x\|} = \|Sx\|$$

This proves boundednees. Moreover, $\|h\| \geq \|s\|$

Now, I want to prove $\|h\| \leq \|S\|$ by an application of the Schwarz inequality:

$$\|h\| = \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|\langle Sx,y\rangle|}{\|x\|\,\|y\|} \leq \sup_{x\neq 0} \frac{\|Sx\|\,\|y\|}{\|x\|\,\|y\|} = \|s\|$$

S is unique. In fact, assuming that there is a linear operator $T:H_1 \to H_2$ such that for all $x \in H_1$ and $y \in H_2$ we have :

$$h(x, y) = \langle Sx, y\rangle = \langle Tx, y\rangle$$

we see that S x=T x by lemma 3.3-2 for all $x \in H_1$. Hence S=T by definition.

3.3-6 Definition(Dual space X^*).

Let X be a normed space . Then the set of all bounded linear functional on X constitutes a normed space with norm defined by

$$\|f\| = \sup_{\substack{x\in X\\ x\neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x\in X\\ \|x\|=1}} |f(x)|$$

Which is called **the dual space of X** is denoted by X^*

3.3-7 Theorem:

The dual space X^* of a normed space X is a Banach space .

Applications

Application(1):if z any fixed element of an inner product space X, show that $f(x)=\langle x, z\rangle$ defines a bounded linear functional f on X,

of norm $\| z\|$.

proof:

To prove f is well defined, let $x_1=x_2$

$$\Rightarrow \langle x_1, z\rangle = \langle x_2, z\rangle$$

$\Rightarrow$ f(x_1)=f(x_2)

Now, we have to prove

f($\alpha x_1 + \beta x_2$)= $\alpha f(x_1) + \beta f(x_2)$ $\quad \forall x_1,\ x_2 \in X\ \alpha, \beta \in C$

f($\alpha x_1 + \beta x_2$)=$\langle \alpha x_1 + \beta x_2, z\rangle$=$\langle \alpha x_1, z\rangle$+$\langle \beta x_2, z\rangle$

$=\alpha\langle x_1, z\rangle + \beta\langle x_2, z\rangle$

$=\alpha f(x_1) + \beta f(x_2)$

Now, we prove f is bounded

$$|f(x)| = |\langle x, z\rangle| \leq \|x\| \, \|z\|$$

$$\|f\| = \sup_{\substack{x\in X \\ x\neq 0}} \frac{|f(x)|}{\|x\|} \leq \|z\| , \qquad (1)$$

$\Rightarrow$ f is bounded

$$\|f\| = \sup_{x\neq 0} \frac{|\langle x,z\rangle|}{\|x\|} \geq \frac{|\langle z,z\rangle|}{\|z\|} = \frac{\|z\|^2}{\|z\|} = \|z\| \qquad (2)$$

Then from (1)and (2) $\|f\| = \|z\|$

Application(2):show that the dual space $H^*$of a Hilbert space H, Then H^* is a Hilbert space with inner product$\langle .,.\rangle_1$ defined by

$$\langle f_z, f_v\rangle_1 = \overline{\langle z, v\rangle} = \langle v, z\rangle. \qquad (*)$$

Proof:

By the Riezs theorem $for\ each\ f \in H^*\ \exists\ unique\ z_f \equiv z \in H$

$$such\ that\ \ f(x) = \langle x, z\rangle\ \forall x \in H$$

Hence, for $f \in H^*$is of the form f=f_z for some unique element $z \in H$

$\Rightarrow$(*)is well-defined

Now, I want to prove (a) $\langle f, f\rangle \geq 0$

(b) $\langle f, f\rangle = 0 \Leftrightarrow f = 0$

(c) $\langle f, g\rangle = \overline{\langle g, f\rangle}$

(d) $\langle f + g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$

(a) $\langle f, f\rangle = \langle f_z, f_z\rangle = \overline{\langle z, z\rangle} = \|z\|^2 \geq 0$

(b)$0=\langle f, f\rangle = \langle f_z, f_z\rangle = \overline{\langle z, z\rangle} = \|z\|^2$

$\Leftrightarrow$ z=0 $\Leftrightarrow$ f_z=0 $\Leftrightarrow$ $f_z(x)=\langle x, 0\rangle$ =0 $\Leftrightarrow$ f=0

(c) $\langle f, g\rangle = \langle f_z, g_v\rangle = \overline{\langle z, v\rangle} = \langle g, v\rangle$

(d)$\langle f + g, h\rangle = \langle f_z + g_v, h_s\rangle = \overline{\langle z+v, s\rangle} = \langle s, z + v\rangle$

$=\langle s, z\rangle + \langle s, v\rangle = \overline{\langle z, s\rangle} + \overline{\langle v, s\rangle}$

$=\langle f, h\rangle + \langle g, h\rangle$

Application(3):Let M≠∅ be a subset of Hilbert space H, and let

$M^a=\{f\in H^*: f(x) = 0\ \forall x \in M\} \subseteq H^*$. let $M^{\perp}=\{y\in H: \langle y, x\rangle = 0\ \forall x \in M\} \subseteq H$

The relation between M^a and $M^{\perp}$ can be explained as a follows:

Let $f\in M^a \subseteq H^* \Rightarrow \exists$ *unique element* $z_f \in H \ni$:

$$\langle x, z_f\rangle = f(x), \forall x \in H$$

Hence $\forall x \in M$, $\langle x, z_f\rangle = f(x)$

$\Rightarrow z_f \perp M \Rightarrow z_f \in M^{\perp}$

Given any $f \in$
M^a *the uniqe element* z_f *exists by Riesz Theorem* belongs to $M^{\perp}$

Conversely, let $y_0 \in M^{\perp}$ $\exists$ *a bounded linear functional* $f_{y_0} \in H^*$ $\ni$:
$f_{y_0}(x) = \langle x, y_0 \rangle, \forall x \in H$

In particular, $\forall\ x \in M$, $f_{y_o}(x) = \langle x, y_0 \rangle = 0$

$\Rightarrow f_{y_0} \in M^a$

References

1-E. Kreyszig,"Introductory Functional Analysis with Application", John Wiley&sons, 1978

2-G.F. simmons "Topology and Modern Analysis",Mc Graw-Hill, Inc 1963