COURSE 3

ASYMPTOTIC THEORY OF NORMAL MODES AND SURFACE WAVES

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Contents

1. Introduction

The analysis of surface waves and normal modes plays an important role in the study of the internal structure of the earth and in particular of the uppermantle because of the way these waves sample this structure, making it possible to overcome to some extent the problems encountered in body wave analysis because of uneven distribution of seismic sources and receivers. Normal modes are, in principle, sensitive to the structure of the earth as a whole, surface waves, whose energy is stationary in the horizontal direction and decays exponentially with depth are particulary well suited for the investigation of large regions devoid of stations and have, for example, been the main source of information on lithospheric and uppermantle structure beneath the oceans.

For the past 20 years, studies of the lateral heterogeneity of the earth using surface waves have primarily been based on the analysis of their dispersive properties as expressed in the phase velocity curve $C(\omega)$ or the group velocity curve, $U(\omega)$. Two fundamental assumptions underlie such studies:

(1) surface waves propagate along the great circle γ defined by the epicenter and the station

(2) the phase is additive along the travel path ("minor arc theorem"):

$$\frac{X}{C(\omega)} = \int_S^R \frac{\mathrm{d}S}{C_{\mathrm{loc}}(\omega,\theta,\varphi)}, \tag{1.1}$$

where $C(\omega)$ is the phase velocity measured between the epicenter S and the receiver R, whose distance is X. C_{loc} is the local phase velocity at each point $M(\theta,\varphi)$ along travel path. It corresponds to an earth model whose elastic properties, as a function of depth, are those of the real earth immediately beneath point M.

Thus, the measured phase velocity depends only on the structure in the vertical plane defined by the great circle γ and represents an average of this structure for the portion contained between S and R.

That these assumptions are in general valid has been demonstrated by the consistent results obtained over the years by regional "pure path" studies, in particular when phase velocities can be measured along reversed

profiles (e.g. Romanowicz, 1982). They have led to the characterization of phase velocities curves in different tectonic regions (see Knopoff, 1972 for a review) and to the first global models of lateral heterogeneity in the uppermantle, based on a regionalization of the earth into several (4 to 7) main tectonic regions (Okal, 1977; Jordan, 1978; Lévêque, 1980). These studies were essentially based on the analysis of fundamental mode data.

The present global or regional tomographic models of the uppermantle also rely on these basic assumptions (cf Woodhouse and Dziewonski, 1984; Nataf et al., 1986; Montagner, 1986).

As more and more high quality digital data become available, evidence is accumulating in the observations for significant departures from the simple zeroth order geometrical optics assumptions presented above. In order to explain such effects, we need a more accurate description of the interaction of the waves with lateral heterogeneity such as that provided by first order perturbation theory. The computations involved in calculating the full interaction matrices are however very heavy and practically prohibitive, so that it is very useful to seek simplifying approximations.

In what follows we shall show how zeroth order asymptotic theory of normal modes can be used to first derive the "minor arc" theorem and the corresponding "great circle" theorem (for surface waves that have travelled around the complete great circle or alternatively normal mode eigenfrequencies). By carrying the asymptotic approximation to higher order we will show that it is possible to explain many of the observed features not taken into account in the zeroth order formulation, both in phase and amplitude. This can then be exploited to improve the quality of the tomographic models.

We first start by introducing asymptotic theory for the case of a spherical earth and then extend it to the case of a laterally heterogeneous earth.

2. Spherical earth

Let us consider a spherical nonrotating elastic and isotropic earth model (SNREI).

2.1. Notations

Let us start with the known expression for the displacement in a spherical earth, at time t and location $\boldsymbol{x}$, due to a point source, expressed by its

moment tensor M:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \mathrm{Re} \sum_K (-M : \epsilon^*)(\boldsymbol{x}_s) \boldsymbol{u}_K(\boldsymbol{x}) \exp(\mathrm{i}\omega_K t) \exp(-\alpha_K t) \tag{2.1}$$

where $\boldsymbol{x}_s$ is the location of the source, ϵ is the strain tensor evaluated at the source, and $\boldsymbol{u}_K$, ω_K and α_K are respectively the eigenvector, eigenfrequency and attenuation coefficient of multiplet K. The sum is extended over all multiplets K.

We recall that the eigenvector for displacement in a spherical earth can be written in the most general form as:

$$\begin{aligned} \boldsymbol{u}_K(\boldsymbol{x}) = {}_nU_\ell(r) Y_\ell^m(\theta, \varphi)\hat{r} + {}_nV_\ell(r)\nabla_1 Y_\ell^m(\theta, \varphi) \\ + {}_nW_\ell(r)\hat{r} \times \nabla_1 Y_\ell^m(\theta, \varphi) \end{aligned} \tag{2.2}$$

where (n, ℓ, m) are the radial, angular and azimuthal orders describing multiplet K, (r, θ, φ) are spherical coordinates of location $\boldsymbol{x}$ and

$$Y_\ell^m(\theta, \varphi) = X_\ell^m(\theta) e^{\mathrm{i}m\varphi} \tag{2.3}$$

where $X_\ell^m(\theta)$ are normalized Legendre functions such that Y_ℓ^m are fully normalized spherical harmonics

$$\int\int_\Omega (Y_\ell^m)^2(\theta, \varphi)\, \mathrm{d}\Omega = 1 \tag{2.4}$$

where Ω is the unit sphere.

${}_nU_\ell$, ${}_nV_\ell$, ${}_nW_\ell$ are the radial eigenfunctions for the earth model considered. The modes are normalized so that:

$$\begin{aligned} \int_0^{r_0} \rho(r)[U^2(r) + V^2(r)] r^2\, \mathrm{d}r = 1 \\ \int_0^{r_0} \rho(r) W^2(r) r^2\, \mathrm{d}r = 1 \end{aligned} \tag{2.5}$$

∇_1 is the surface gradient operator:

$$\nabla_1 Y_\ell^m = \frac{\partial Y_\ell^m}{\partial \theta}\hat{\theta} + \frac{1}{\sin\theta}\frac{\partial Y_\ell^m}{\partial \varphi}\hat{\varphi}. \tag{2.6}$$

Let us introduce the following operator formalism: we see from (2.2) that

$$\boldsymbol{u}_K(\boldsymbol{x}) = \boldsymbol{D}[Y_\ell^m(\theta, \varphi)] \tag{2.7}$$

where $\boldsymbol{D}$ is the displacement operator, defined by (2.2). We note that $\boldsymbol{D}$ does not depend on the azimuthal order m.

If we consider a particular component of observation $\boldsymbol{v}$ then

$$\boldsymbol{v}\cdot\boldsymbol{u}_K(\boldsymbol{x}) = (\boldsymbol{v}\cdot\boldsymbol{D})[Y_\ell^m(\theta_R,\varphi_R)] = R_K^m \tag{2.8}$$

where the operator $(\boldsymbol{v}\cdot\boldsymbol{D})$ acts on the receiver coordinates $(\theta_R,\phi_R)=(\theta,\phi)$.

On the other hand, it can be easily seen that the expression $(-M:\epsilon^*)(\boldsymbol{x}_S)$ can also be written in operator form:

$$-(M:\epsilon^*)(\boldsymbol{x}_S) = -(\boldsymbol{M}:\epsilon)\;Y_\ell^{m*}(\theta_S,\phi_S) = S_K^m. \tag{2.9}$$

To obtain this, we refer the reader to the expression of the strain tensor in spherical harmonics (Aki and Richards, 1980 p.353).

We note that the operator $(\boldsymbol{M}:\epsilon)$ also does not depend on m and acts on the coordinates of the source (θ_S,ϕ_S).

Expression (2.1) evaluated for a particular component of motion then becomes:

$$\boldsymbol{v}\cdot\boldsymbol{u}(\boldsymbol{x},t) = \mathrm{Re}\left\{\sum_K\left(\sum_{m=1}^{m=2l+1} R_K^m S_K^m\right)\exp \mathrm{i}\omega_K t\exp(-\alpha_K t)\right\} \tag{2.10}$$

where we have emphasized the fact that in a spherical earth multiplets are degenerate (the eigenfrequency does not depend on the azimuthal order m).

From (2.8) and (2.9) we obtain, since the operators do not depend on azimuthal order m:

$$\sum_m R_K^m S_K^m = (\boldsymbol{v}\cdot\boldsymbol{D})_R(\boldsymbol{M}:\epsilon)_S\left[\sum_m Y_\ell^m(\theta_R,\phi_R)Y_\ell^{m*}(\theta_S,\phi_S)\right]. \tag{2.11}$$

Using the addition theorem for spherical harmonics (Edmonds, 1960):

$$\sum_m Y_\ell^m(\theta_R,\phi_R)Y_\ell^{m*}(\theta_S,\phi_S) = f_\ell X_\ell^0(\cos\Delta), \tag{2.12}$$

where $f_l = \sqrt{(2l+1)/4\pi}$ and Δ is the epicentral distance between the source S and the receiver R.

Hence

$$\boldsymbol{v}\cdot\boldsymbol{u}(x,t) = \sum_K f_\ell(\boldsymbol{v}\cdot\boldsymbol{D})_R(\boldsymbol{M}:\epsilon)_S X_\ell^0(\Delta)\exp(\mathrm{i}\omega_K t)\exp(-\alpha_K t). \tag{2.13}$$

Expression (2.13) is most general. The simplest case is that of an isotropic source observed on the vertical component. Then

$$(\boldsymbol{v}\cdot\boldsymbol{D})_R(\boldsymbol{M}:\epsilon)_S X_\ell^0(\Delta) = a_\ell X_\ell^0(\Delta) \tag{2.14}$$

where

$$a_\ell = M_0\left(\partial_r U + (U - \tfrac{1}{2}\ell(\ell+1)V)\,r^{-1}\right). \tag{2.15}$$

M_0 is the seismic moment of the source and we have dropped indices n and ℓ.

For a more general source and spheroidal or toroidal modes observed on vertical or horizontal components:

$$(\boldsymbol{v}\cdot\boldsymbol{D})_R(\boldsymbol{M}:\epsilon)_S X_\ell^0(\Delta) = C_0X_\ell^0(\Delta) + C_1X_\ell^1(\Delta) + C_2X_\ell^2(\Delta) \tag{2.16}$$

where the coefficients C_0, C_1, C_2 depend on the moment tensor of the source (Appendix 1).

Now, the zeroth order asymptotic expression for the normalized Legendre function is as follows:

$$X_\ell^m(\Delta) = \frac{1}{\pi\sqrt{\sin\Delta}}\cos(k\Delta - \tfrac{1}{4}\pi + m\tfrac{1}{2}\pi) + \vartheta(\frac{1}{\ell}) \tag{2.17}$$

where $k=\ell+\frac{1}{2}$.

This expression is valid for large angular orders ($l \gg 1$) and for $\Delta \gg 0$ or $\Delta \ll \pi$, that is, away from the epicenter and its antipode.

When these conditions are satisfied, expression (2.13) becomes, to order zero in $1/\ell$:

$$\boldsymbol{v}\cdot\boldsymbol{u}(\boldsymbol{x},t) \simeq \mathrm{Re}\sum_K\left(\sum_\ell f_\ell a_\ell\frac{\cos(k\Delta-\frac{1}{4}\pi+\chi_\ell)}{\pi\sqrt{\sin\Delta}}\right. \\ \left.\exp(\mathrm{i}\omega_K t)\exp(-\alpha_K t) + \vartheta\left(\frac{1}{\ell}\right)\right) \tag{2.18}$$

where $\sum_K$ is now a sum over all mode branches,

$$a_\ell = \sqrt{(C_0-C_2)^2 + C_1^2}$$
$$\chi_l = \arctan\left(\frac{C_1}{C_0-C_2}\right) \tag{2.19}$$

a_ℓ and χ_ℓ are slowly varying functions of ℓ and from (2.18) we see in particular that, along a given dispersion branch, the amplitude of the mode, as a function of ℓ, has a periodicity which depends on the epicentral distance (periodicity of 2 in ℓ for example for $\Delta \simeq 90°$).

2.2. Correspondence with propagating waves

Let us consider a single mode branch n_0, and let $u(\boldsymbol{x}, t)$ be the corresponding displacement in a given component. Let $U(\boldsymbol{x}, \omega)$ be the Fourier transform of u_K. From (2.19) we obtain

$$U(\boldsymbol{x},\omega) = \sum_\ell \frac{f_\ell a_\ell}{2} \cos(k\Delta - \tfrac{1}{4}\pi + \chi_\ell) \times \left[\frac{1}{\mathrm{i}(\omega + \omega_K) - \alpha_K} + \frac{1}{\mathrm{i}(\omega - \omega_K) - \alpha_K}\right]. \tag{2.20}$$

We shall drop the first of the two terms in (2.20) which corresponds to negative frequencies ω. To the remaining term, which is of the form

$$\sum_{\ell=0}^{\infty} F_\ell$$

we shall apply Poisson's formula (Aki and Richards, 1980)

$$\sum_{\ell=0}^{\infty} F_\ell = \sum_{S=-\infty}^{S=+\infty} (-1)^S \int_0^\infty F(\nu)\,\mathrm{e}^{2\mathrm{i}\pi S\nu}\,\mathrm{d}\nu \tag{2.21}$$

with $\nu = \ell + \frac{1}{2}$.

Changing then variables to $k = \nu/r_0$ where r_0 is the radius of the earth, (2.20) becomes:

$$U(\boldsymbol{x},\omega) = \sum_{S=-\infty}^{S=+\infty} (-1)^S \int_0^\infty \frac{f_k a_k}{2\mathrm{i}} \times \cos(kX - \tfrac{1}{4}\pi + \chi_k) \frac{\mathrm{e}^{2\mathrm{i}\pi r_0 S k}}{(\omega - \omega_k + \mathrm{i}\alpha_k)} r_0\,\mathrm{d}k \tag{2.22}$$

where $X = a\Delta$ is distance in km.

We can evaluate the integral in (2.22) using the theorem of residues. For this we need to know the pole of the integrand, which is obtained by expanding ω_k in the vicinity of ω:

$$\omega_k - \omega \simeq (k - k(\omega))\frac{\partial\omega}{\partial k} \simeq (k - k(\omega))U(\omega)$$

where U is group velocity.

The pole is therefore

$$k = k(\omega) + \mathrm{i}\frac{\alpha(\omega)}{U}$$

and the application of the theorem of residues yields (Aki and Richards, 1980)

$$U(\boldsymbol{x},\omega) = \sum_{S=-\infty}^{S=+\infty} (-1)^S f(\omega)a(\omega)r_0\pi/U \times \cos\left[(k(\omega) + \mathrm{i}\frac{\alpha}{U})X - \tfrac{1}{4}\pi + \chi(\omega)\right] \exp\left(2\mathrm{i}\pi r_0 S(k + \mathrm{i}\frac{\alpha}{U})\right) \tag{2.23}$$

We now express the cosine in terms of exponential functions and obtain two families of waves propagating in opposite directions:

$$U^+(\boldsymbol{x},\omega) = \sum_{S=-\infty}^{S=+\infty} (-1)^S f(\omega)a(\omega)\frac{r_0\pi}{2U} \times \exp\left(\mathrm{i}(k(X + 2\pi r_0 S) - \tfrac{1}{4}\pi + \chi(\omega))\right) \exp\left(-\frac{\alpha}{U}(X + 2\pi r_0 S)\right)$$

$$U^-(\boldsymbol{x},\omega) = \sum_{S=-\infty}^{S=+\infty} (-1)^S f(\omega)a(\omega)\frac{r_0\pi}{2U} \times \exp\left[-(\mathrm{i}(k(2\pi r_0 S - X) + \tfrac{1}{4}\pi - \chi(\omega))\right] \times \exp\left[-\frac{\alpha}{U}(2\pi r_0 S - X)\right] \tag{2.24}$$

The first arriving R_1 and R_2 (or G_1 and G_2 for Love waves) wavetrains are obtained, for example by setting $S = 0$ in U_K^+ and $S = 1$ in U_K^- respectively. The first arriving train has the well-known form:

$$U^1(x,\omega) = +f(\omega)a(\omega)\frac{r_0\pi}{2U} \exp\left[\mathrm{i}(kX - \tfrac{1}{4}\pi + \chi(\omega))\right] \exp\left[\frac{-\alpha(\omega)X}{U}\right] \tag{2.25}$$

with $f(\omega) = \sqrt{2\pi/r_0 k}$.

We note from the preceding derivation that the correspondence between the phase velocity of surface waves and the eigenfrequency ω_ℓ of free oscillations is such that:

$$C(\omega_\ell) = \frac{\omega}{k} = \frac{\omega_\ell r_0}{\ell + \frac{1}{2}} \tag{2.26}$$

which is the well-known Jeans formula.

The spectrum of the surface wavetrain (2.25) is the product of two terms: a propagation term and a source term $A_S(\omega)e^{i\varphi_S(\omega)}$ where A_S and φ_S can be obtained from (2.19).

3. Aspherical earth

3.1. Introduction

For waves having travelled once around a complete great circle path, the minor arc theorem (1.1) becomes:

$$\frac{2\pi r_0}{C_G(\omega)} = \int_\gamma \frac{ds}{C_{\text{loc}}(\omega, \varphi)} \tag{3.1}$$

where $C_G$is the measured phase velocity along the great circle γ.

This naturally leads, remembering Jeans'formula, to the corresponding "great circle theorem" for measured eigenfrequencies of the earth : let Λ_K be the measured frequency shift for a given mode and a given source station geometry, with respect to a reference SNREI model of the earth. Then

$$\Lambda_K = \frac{1}{2\pi} \int_\gamma \delta\omega_{\text{loc}}^K(\theta, \varphi)\, dS \tag{3.2}$$

where $\delta\omega_{\text{loc}}^K$ represents the local frequency shift for points along the great circle γ defined by the epicenter and the station.

The frequency shifts and phase velocities are related by the relation:

$$\frac{\delta\omega}{\omega} = \frac{U}{C}\frac{\delta C}{C} \tag{3.3}$$

corresponding to a passage from a constant ω measurement to a constant k measurement.

The great circle theorem (3.2) was first derived from first order degenerate perturbation theory by Jordan (1978) and Dahlen (1979). Jordan calls Λ_K the "location parameter". That it is valid in general has been demonstrated by the remarkable result first obtained by Masters et al. (1982) when measuring frequency shifts of fundamental spheroidal modes for many great circle paths around the earth, leading to the discovery of the strong upper mantle degree 2 pattern.

The minor arc theorem (1.1) remained to be derived, and this was a problem for a while, since degenerate perturbation theory, which considers only isolated multiplets, permits to describe solely the even order lateral heterogeneity of the earth. Coupling between multiplets had to be introduced (Mochizuki, 1986; Park, 1987; Romanowicz, 1987).

We will show in what follows how all these results can be obtained by applying zeroth order asymptotics to quasi degenerate first order perturbation theory.

3.2. Zeroth order asymptotic theory

Let us start with the expression of Woodhouse (1983) for the displacement in a laterally heterogeneous earth and a given component, valid to first order in the model perturbations:

$$\begin{aligned} u(\boldsymbol{x}, t) = \mathrm{Re}\Bigg\{ \sum_K \Bigg(& \sum_m R_K^m S_K^m - \frac{2}{\omega_K} \sum_m R_K^m H_{KK}^{mm'} S_K^{m'} \\ & + it \sum_{mm'} R_K^m Z_{KK}^{mm'} S_K^{m'} \Bigg) \\ & + \sum_{K' \neq K} \frac{2\omega_K}{\omega_K^2 - \omega_{K'}^2} \Bigg(\sum_{mm'} R_K^m Z_{KK'}^{mm'} S_{K'}^{m'} + R_{K'}^{m'} Z_{K'K}^{m'm} S_K^m \Bigg) \\ & \times \exp(i\omega_K t) \exp(-\alpha_K t) \Bigg\} \end{aligned} \tag{3.4}$$

where ω_K, α_K, R_K^m S_K^m are as defined in I.

$H_{KK'}^{mm'}$ is the splitting matrix for interaction between multiplets K and K'

$$Z_{KK'}^{mm'} = H_{KK'}^{mm'} - \omega_K^2 P_{KK'}^{mm'}$$

where $P_{KK'}^{mm'}$ is the density interaction matrix.

3.3. Isolated multiplet

If we consider isolated multiplets K, expression (3.4) becomes

$$u(\boldsymbol{x},t) = \mathrm{Re}\left\{ \sum_K \left(\sum_m R_K^m S_K^m + \mathrm{i}t \sum_{mm'} R_K^m Z_{KK'}^{mm'} S_K^{m'} \right) \right.$$
$$\left. \times \exp(\mathrm{i}\omega_K t) \exp(-\alpha_K t) \right\} \tag{3.5}$$

where we have neglected the term

$$\frac{2}{\omega_K} \sum_m R_K^m H_{KK'}^{mm'} S_K^{m'}$$

Equation (3.5) can be written, away from the nodes of the radiation pattern of the event:

$$u(\boldsymbol{x},t) \simeq \mathrm{Re} \sum_K \left(\sum_m R_K^m S_K^m \right) \exp \mathrm{i}(\omega_K + \Lambda_K) \exp(-\alpha_K t) \tag{3.6}$$

with

$$\Lambda_K = \frac{\sum R_K^m Z_{KK'}^{mm'} S_{K'}^{m'}}{\sum R_K^m S_K^m} \tag{3.7}$$

where Λ_K is the location parameter of Jordan.

Equation (3.4) is a short time approximation, but we note that Woodhouse and Girnius (1982) have shown that the long time approximation:

$$\sum_K \left(R_K \mathrm{e}^{\mathrm{i}Z_K t} S_K \mathrm{e}^{\mathrm{i}\omega_K t} \mathrm{e}^{-\alpha_K t} \right) \tag{3.8}$$

is also valid, where R_K is the vector receiver $\{R_K^m\}$ S_K is the vector $\{S_K^m\}$ and Z_K the matrix $\left\{ Z_{KK}^{mm'} \right\}$.

We will now find an asymptotic expression for (3.7).

Woodhouse and Girnius (1982) have first shown, and Romanowicz (1987), Snieder and Romanowicz (1988) extended to more general situations, that the interaction matrix can be written in the form:

$$Z_{KK'}^{mm'} = \sum_{i=0}^{2} \int\!\!\int_\Omega \delta\omega_{KK'}(\theta,\varphi)(-\nabla^2)^i \left[Y_\ell^{m*}(\theta,\varphi) Y_{\ell'}^{m'}(\theta,\varphi) \right] \mathrm{d}\Omega \tag{3.9}$$

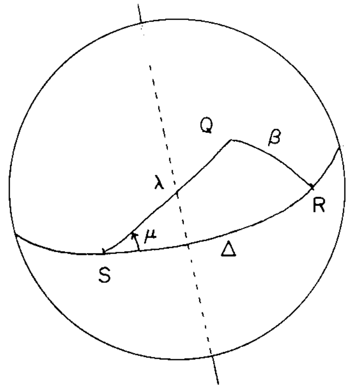

Fig. 1. Definition of angles λ, β and μ as used in text.

where the integration is over the unit sphere and $\delta\omega_{KK'}(\theta,\varphi)$ is a "local frequency" that can be expressed as an integral over depth of the local perturbations to the model parameters.

Using (3.9) and (2.8) (2.9) we can write:

$$\sum_{mm'} R_K^m Z_{KK}^{mm'} S_K^{m'} = \sum_{i=0}^{i=2} (\boldsymbol{v}\cdot\boldsymbol{D})(-\epsilon : \boldsymbol{M}) \int\!\!\int_\Omega (-\nabla^2)^i \sum_{mm'} Y_\ell^m(\theta_R,\phi_R)$$
$$\times Y_\ell^{m*}(\theta,\varphi) Y_\ell^{m'}(\theta,\varphi) Y_\ell^{m'*}(\theta_S,\phi_S)\,\mathrm{d}\Omega \tag{3.10}$$

Using the addition theorem for spherical harmonics, this becomes

$$\sum_{mm'} R_K^m Z_{KK}^{mm'} S_K^{m'} = \sum_{i=0}^{i=2} f_\ell^2 (\boldsymbol{v}\cdot\boldsymbol{D})(-\epsilon : \boldsymbol{M})$$
$$\times \int\!\!\int_\Omega \delta\omega_{KK}(\lambda,\mu)(-\nabla^2)^i X_\ell^0(\lambda) X_\ell^0(\beta)\,\mathrm{d}\Omega \tag{3.11}$$

where the angles λ, μ and β are defined in figure 1 and the operator $(-\nabla^2)^i$ acts on the running coordinates (θ,φ).

Using the zeroth order asymptotic approximation (2.11) we can easily see that the terms for $i \neq 0$ vanish and we can write:

$$A_K \simeq \int\!\!\int_\Omega X_\ell^0(\lambda) X_\ell^0(\beta) \delta\omega_{KK}(\lambda,\mu)\,\mathrm{d}\Omega$$
$$\simeq \int_0^\pi \frac{\sin\lambda \cos(k\lambda - \frac{1}{4}\pi)}{\pi\sqrt{\sin\lambda}}\,\mathrm{d}\lambda$$
$$\times \int_0^{2\pi} \delta\omega_{KK}(\lambda,\mu) \frac{\cos(k\beta - \frac{1}{4}\pi)}{\pi\sqrt{\sin\beta}}\,\mathrm{d}\mu \tag{3.12}$$

The second integral, $I(\lambda)$ can then be evaluated by the method of stationary phase, noting that $\partial\beta/\partial\mu = 0$ on the great circle γ and assuming that $\ell \gg s_{\max}$ where $s_{\max}$ is the maximum order of a spherical harmonics expansion of $\delta\omega_{KK}$. Then:

$$I_K(\lambda) \simeq \frac{1}{f_\ell}\sqrt{\frac{\sin\lambda\sin\beta_0}{\sin\Delta}}\cos k\beta_0\delta\omega_{KK}(\lambda, 0) \tag{3.13}$$

where $\beta_0 =\mid \lambda - \Delta \mid$, hence:

$$\begin{aligned} A_K &\simeq \frac{\delta\hat{\omega}_K}{f_\ell}\frac{\cos(k\Delta - \frac{1}{4}\pi)}{\pi\sqrt{\sin\Delta}} + \vartheta(1/\ell) \\ &\simeq \frac{\delta\hat{\omega}_K}{f_\ell}X_\ell^0(\Delta) + \vartheta(1/\ell) \end{aligned} \tag{3.14}$$

where

$$\delta\hat{\omega}_K = \frac{1}{2\pi}\int_\gamma \delta\omega_{KK}(\lambda, 0)\,\mathrm{d}\lambda$$

hence, to order 0 in $1/\ell$, (3.8) becomes, using the asymptotic expression for $\sum_m R_K^m S_K^m$ obtained in I:

$$\Lambda_K \simeq \frac{f_\ell(\boldsymbol{v}\cdot\boldsymbol{D})(-\epsilon:\boldsymbol{M})X_\ell^0(\Delta)\delta\hat{\omega}_K}{f_\ell(\boldsymbol{v}\cdot\boldsymbol{D})(-\epsilon:\boldsymbol{M})X_\ell^0(\Delta)}$$

i.e.,

$$\Lambda_K \simeq \delta\hat{\omega}_K \tag{3.15}$$

which establishes the great circle theorem. This can also be established in a similar manner from the long time approximation (3.8)

We note that degenerate perturbation theory enables us to retrieve only the even order part of lateral heterogeneity. In order to also include effects of the odd part of the earth's asphericity, we need to go one step further and consider the interaction between different multiplets.

3.4. Coupling terms included

In a similar way, we can obtain a zeroth order asymptotic expression for the coupling terms in expression (3.5). We will restrict ourselves to a single dispersion branch. Let:

$$S_K = \sum_{K'\neq K}\frac{2\omega_K}{\omega_K^2 - \omega_{K'}^2}\sum_{mm'}(R_K^m Z_{KK'}^{mm'}S_{K'}^{m'} + R_{K'}^{m'}Z_{K'K}^{m'm}S_K^m\ ) \tag{3.16}$$

where the sum is extended to all multiplets of the branch considered different from K (whose angular order is ℓ). We shall replace in what follows ω_K by ω_ℓ. Then

$$\begin{aligned} S_K = \sum_{n\neq 0}&\frac{2\omega_\ell f_\ell}{\omega_\ell^2 - \omega_{\ell+n}^2}(\epsilon:\boldsymbol{M})(\boldsymbol{v}\cdot\boldsymbol{D}) \\ &\times \int\int_\Omega \left[\delta\omega_{\ell,\ell+n}(\lambda,\mu)\cos(k\lambda - \tfrac{1}{4}\pi)\cos(k+n)(\beta - \tfrac{1}{4}\pi)\right. \\ &+ \left.\delta\omega_{\ell,\ell+n}(\lambda,\mu)\cos(k\beta - \tfrac{1}{4}\pi)\cos(k+n)\lambda - \tfrac{1}{4}\pi)\right] \\ &\times \frac{\pi^2\,\mathrm{d}\Omega}{\sqrt{\sin\lambda\sin\beta}} \end{aligned} \tag{3.17}$$

we note that:

$${\omega_\ell}^2 - {\omega_{\ell+n}}^2 = -2\omega_\ell\frac{U}{r_0}n \tag{3.18}$$

and apply, as in (3.1) the method of stationary phase to evaluate the integral in (3.17).

Thus, assuming that

$$\delta\omega_{\ell,\ell+n} \simeq \delta\omega_{\ell+n,\ell} \simeq \delta\omega_{\ell,\ell}$$

which we shall see is justified since the sum over n is in fact truncated to $n \leq s_{\max}$, where $s_{\max}$ is the maximum order of a spherical harmonics expansion of $\delta\omega_\ell$:

$$S_K = -\frac{r_0}{U}\frac{f_\ell}{\pi^2\sqrt{\sin\Delta}}\sum_{n\neq 0}\frac{1}{n}((\cos k\Delta - \tfrac{1}{4}\pi)I_K^n - J_K^n\sin(k\Delta - \tfrac{1}{4}\pi)) \tag{3.19}$$

where

$$I_K^n = \int_0^{2\pi}\delta\omega_{\ell\ell}(\lambda,0)(\cos n\lambda + \cos n(\Delta-\lambda))\,\mathrm{d}\lambda$$

and

$$J_K^n = \int_0^{2\pi}(\sin n(\lambda-\Delta) - \sin n\lambda)\delta\omega_{\ell\ell}(\lambda,0)\,\mathrm{d}\lambda.$$

We note that I_K^n is an even function of n, hence

$$\sum_{n\neq 0}\frac{I_K^n}{n}=0$$

whereas

$$\sum_{n\neq 0}\frac{J_K^n}{n}=\sum_{n\neq 0}\int_0^{2\pi}\sin n\lambda\delta\omega_\ell(\lambda,0)\,\mathrm{d}\lambda\left(-\int_0^{\Delta}\sin nx\,\mathrm{d}x\right)$$
$$-\sum_{n\neq 0}\int_0^{2\pi}\cos n\lambda\delta\omega_\ell(\lambda,0)\,\mathrm{d}\lambda\int_0^{\Delta}\cos nx\,\mathrm{d}x \tag{3.20}$$

We recognize the Fourier expansion of $\delta\omega_\ell(\lambda,0)$ and hence

$$\sum_{n\neq 0}\frac{J_K^n}{n}=-2\pi\left(\int_0^{\Delta}\delta\omega_\ell(\lambda,0)-\Delta\int_0^{2\pi}\delta\omega_\ell(\lambda,0)\,\mathrm{d}\lambda\right)$$
$$=+2\pi\Delta(\delta\hat{\omega}_K-\delta\tilde{\omega}_K) \tag{3.21}$$

where

$$\delta\hat{\omega}_K=\frac{1}{2\pi}\int_0^{2\pi}\delta\omega_\ell(\lambda,0)\,\mathrm{d}\lambda$$

is the great circle average and

$$\delta\tilde{\omega}_K=\frac{1}{\Delta}\int_0^{\Delta}\delta\omega_\ell(\lambda,0)\,\mathrm{d}\lambda$$

is the minor arc average.

We note from (3.21) that the sum over n needs only be extended to

$$n\leq s_{\max}.$$

Finally

$$S_K=+f_\ell\frac{r_0\Delta}{U}(\delta\tilde{\omega}_K-\delta\hat{\omega}_K)(\epsilon:\boldsymbol{M})(v.\boldsymbol{D})\frac{\sin(k\Delta-\frac{1}{4}\pi)}{\pi\sqrt{\sin\Delta}} \tag{3.22}$$

and the asymptotic expression for the displacement becomes

$$\begin{aligned}u(\boldsymbol{x},t)=&\mathrm{Re}\sum_\ell f_\ell(\epsilon:\boldsymbol{M})(\boldsymbol{v}\cdot\boldsymbol{D})\left[\frac{\cos(k\Delta-\frac{1}{4}\pi)}{\pi\sqrt{\sin\Delta}}\right.\\&\left.-\frac{r_0\Delta}{U}(\delta\hat{\omega}-\delta\tilde{\omega})\sin(k\Delta-\tfrac{1}{4}\pi)\right]\exp\left[\mathrm{i}(\omega_K+\Lambda_K)t\right]\exp\left(-\alpha_K t\right)\\&=\mathrm{Re}\sum_\ell f_\ell(\epsilon:\boldsymbol{M})(\boldsymbol{v}\cdot\boldsymbol{D})\left\{X_\ell^0(\Delta+\delta\Delta)\right\}\\&\times\exp\left[\mathrm{i}(\omega_K+\Lambda_K)t\right](\exp-\alpha_K t)\end{aligned} \tag{3.23}$$

where

$$\delta\Delta=\frac{r_0\Delta}{kU}(\delta\hat{\omega}_K-\delta\tilde{\omega}_K). \tag{3.24}$$

This expression is identical to that used by Woodhouse and Dziewonski (1984).

We see from (3.24) that the consideration of coupling terms has enabled us to introduce the effect of odd lateral heterogeneity.

If we now replace Δ by $\Delta+\delta\Delta$ in (2.2) we see that we obtain the correct expression for the propagation phase

$$\varphi_R=\frac{\omega X}{C}=\frac{\omega X}{C_0}-\omega\int_S^R\frac{\delta C\,\mathrm{d}S}{C_0^2} \tag{3.25}$$

with

$$\frac{U}{C_0}\frac{\delta C_{\mathrm{loc}}}{C_0}=\frac{\delta\omega_{\mathrm{loc}}}{\omega_0}$$

and C_0, is the phase velocity in the reference SNREI model.

We have thus established the great circle and minor arc theorems and showed that they are valid in the case of smooth lateral heterogeneity ($s_{\max}\ll\ell$). From (3.25) we also see that the amplitude of normal modes is sensitive to the odd part of lateral heterogeneity, a property being currently exploited in normal mode studies (Pollitz et al.; 1987).

3.5. Higher order asymptotics

While the zeroth order asymptotics appear to be verified in many instances, significant departures from this simple theory have been observed.

First, we expect from (3.3) the frequency shift to vary smoothly as a function of frequency for given dispersion branch. Second, from (3.24), we expect amplitudes to decay smoothly as a function of time or distance from the source.

In fact, there are many reports of amplitude anomalies (e.g. Lay and Kanamori, 1985; Romanowicz, 1987) and also systematic fluctuations of the eigenfrequency as a function of angular order (Jobert and Roult, 1977; Silver and Jordan, 1981; Roult et al., 1986).

It can be shown that such effects can be accounted for by applying higher order asymptotics to first order perturbation theory. Complete derivations can be found in Davis and Henson (1986), Romanowicz (1987) or Park (1987). We give here the results obtained.

The $(1/\ell\ )$ asymptotic expression for the Legendre function X_ℓ^m is

$$X_\ell^m(\theta,\varphi) = \frac{1}{\pi\sqrt{\sin\Delta}}\cos\left(k\Delta - \tfrac{1}{4}\pi + m\tfrac{1}{2}\pi + \frac{1}{\ell}(\tfrac{1}{2}m^2 - \tfrac{1}{8})\cot\Delta\right). \tag{3.26}$$

Introducing this expression in the derivations of (II.1) we obtain, to order $1/\ell$

$$\Lambda_K = \delta\hat{\omega}_K + \frac{1}{2k}\tan(k\Delta - \tfrac{1}{4}\pi + \lambda_K)(\alpha_K\hat{D}'_K + \beta_K\hat{D}''_K) \tag{3.27}$$

where α_K, β_K, λ_K depend on the source mechanism and

$$\begin{aligned}\hat{D}'_K =& \frac{1}{2\pi}\int_0^{2\pi}\partial_T(\delta\omega_K(s))\,\mathrm{d}s\\ \hat{D}''_K =& \frac{1}{2\pi}\int_0^{2\pi}\partial_T^2(\delta\omega_K(s))\,\mathrm{d}s\end{aligned} \tag{3.28}$$

where we have introduced the first and second transverse derivatives of $\delta\omega_K$ whose expressions are, in the epicentral reference frame

$$\begin{aligned}\partial_T(\delta\omega_K(\lambda,0)) =& (\cot\Delta - \cot\lambda)\frac{\partial\omega_K}{\partial\mu}(\lambda,0)\\ \partial_T^2(\delta\omega_K(\lambda,0)) =& (\cot\Delta - \cot\lambda)\frac{\partial^2\omega_K}{\partial\mu^2}(\lambda,0)\end{aligned} \tag{3.29}$$

We see that the frequency shift has fluctuations as a function of angular order ℓ with a periodicity depending on epicentral distance Δ.

In order to measure the great circle average we therefore have to extract the smoothly varying part of the frequency shift. This is what we have recently done in a global study of spheroidal mode eigenfrequency shifts (Romanowicz et al., 1987) yielding in particular improved constraints on the upper mantle degree two pattern. This of course required first filtering the data to extract a single mode branch (fundamental mode).

In the same way, it can be shown that to order $1/\ell$, the normal mode amplitudes are modified in such a way that

$$A_K(\Delta) = A_K^0(\Delta + \delta\Delta)(1 + \delta F_K(\Delta)) \tag{3.30}$$

where $\delta\Delta$ is as defined in (3.24) and

$$\delta F_K = \frac{r_0\Delta}{U}\left[\frac{\hat{D}_K - \tilde{D}_K}{2k} + \frac{\cot\Delta}{8k}(\delta\hat{\omega}_K - \delta\tilde{\omega}_K)\right] \tag{3.31}$$

where

$$\hat{D}_K = \alpha_K\hat{D}'_K + \beta_K\hat{D}''_K$$

and $\tilde{D}_K$ is the corresponding minor arc average. The term δF_K is a focussing term whose sign depends on the relative strength of great circle and minor arc averages of the local frequency and its transverse first and second derivatives.

By transforming to propagating waves (Romanowicz, 1987) we obtain an amplitude focussing term which is equivalent to that obtained by Woodhouse and Wong (1986) using ray theory.

By considering focussing terms (3.31) or their equivalent in propagating wave formalism we can hope to be able to separate to some extent effects of scattering and effects of intrinsic attenuation in amplitude measurements (Romanowicz, 1989).

Discussion. The higher order asymptotic theory described here is valid for large angular orders and smooth lateral variations of structure. It permits to go one step further in the interpretation of surface wave measurements. It breaks down however, when the scatterer is of small or comparable size to the wavelength of the waves, in which case a more complete theory has to be considered, including mode conversions, with the disadvantage of many times heavier calculations involved.

Appendix 1.

Excitation coefficients for a source represented by a moment tensor $\{M_{ij}\}$

$$(\epsilon : \boldsymbol{M})_S X_\ell^0(\lambda) = C_0 X_\ell^0(\lambda) + C_1 X_\ell'(\lambda) + C_2 X_\ell^2(\lambda)$$

where, for spheroidal modes:

$$\begin{aligned}
C_0 &= -\left[M_{rr}\partial r U + (M_{\theta\theta} + M_{\phi\phi})(U - \tfrac{1}{2}\ell(\ell+1)V)r^{-1}\right]\\
C_1 &= \sqrt{\ell(\ell+1)}(M_{r\theta}\cos\phi + M_{r\phi}\sin\phi)(\partial r V + \frac{U-V}{r})\\
C_2 &= -\ell(\ell+1)\frac{V}{r}\left[\tfrac{1}{2}(M_{\theta\theta} - M_{\phi\phi})\cos 2\phi_R + M_{\theta\phi}\sin 2\phi_R\right]
\end{aligned}$$

and for toroidal modes:

$$\begin{aligned}
C_0 &= 0\\
C_1 &= \sqrt{\ell(\ell+1)}(M_{r\phi}\cos\phi - M_{r\theta}\sin\phi)(\partial r W - \frac{1}{r}W)\\
C_2 &= -\ell(\ell+1)\frac{W}{r}\left[\tfrac{1}{2}(M_{\theta\theta} - M_{\phi\phi})\sin 2\phi_R - M_{\theta\phi}\cos 2\phi_R\right]
\end{aligned}$$

where the angle ϕ is the azimuth of point Q counted from South and $\widehat{SQ} = \lambda$ is the colatitude of point Q in the reference frame where the epicenter S is the pole.

References

Aki, K. and Richards (1980) Quantitative seismology: Theory and Methods, 2 vols., Freeman, San Francisco.

Dahlen, F.A. (1979) The spectra of unresolved split normal mode multiplets, Geophys. J. R. astr. Soc., 58, 1-34.

Davis P. and I. Henson (1986) Validity of the great circular average approximations for inversion of normal mode measurements, Geophys. J.R. astr. Soc., 85, 69-92.

Edmonds, A.R. (1960) Angular momentum in Quantum Mechanics, Princeton Univ. Press., New Jersey.

Jobert, N. and G. Roult (1976) Periods and damping of free oscillations observed in France after sixteen earthquakes, Geophys. J.R. Astr. Soc., 45, 155-176.

Jordan, T.H. (1978) A procedure for estimating lateral variations from low frequency eigenspectra data, Geophys. J.R. Astr. Soc., 52, 441-455.

Knopoff, L. (1972) Observation and inversion of surface wave dispersion, Tectonophys, 13, 497 519.

Lay, T. and H. Kanamori (1985) Geometric effects of global lateral heterogeneity on long period surface wave propagation, J. Geophys. Res. Lett., 90, 605-621.

Lévêque, J.J. (1980) Regional upper mantle S velocity models from phase velocities of great circle Rayleigh waves, Geophys. J.R. Astr. Soc., 63, 23-43.

Masters, G., J. Park and F. Gilbert (1982) Aspherical earth structure from fundamental spheroidal mode data, Nature, 298, 609-613.

Mochizuki, E. (1986) Free oscillations and surface waves on an aspherical earth, Geophys. Res. Lett., 13, 1478-1481.

Montagner, J.P. (1986) Regional three dimensional structures using long period surface waves, Ann. Geophys. 4, B3, 283-294.

Nataf, H.C., I. Nakanishi and D.L. Anderson (1986) Measurements of mantle wave velocities and inversion for lateral heterogeneities and anisotropy, 3. Inversion, J. Geophys. Res., 91, 7261-7307.

Okal, E. (1977) The effect of intrinsic ocean upper mantle heterogeneity on the regionalization of long-period Rayleigh wave phase velocities, Geophys. J.R. Astr. Soc., 49, 357-370.

Park, J. (1987) Asymptotic coupled-mode expressions for multiplet amplitude anomalies and frequency shifts on a laterally heterogeneous earth, Geophys. J.R. Astr. Soc., 90, 129-170.

Pollitz, F., J. Park, and F.A. Dahlen (1987) Observations of free oscillation amplitude anomalies, Geophys. Res. Lett., 14, 895-898.

Romanowicz, B. (1982) Constraints on the structure of the Tibet Plateau from pure path phase velocities of Love and Raleigh waves, J. Geophys. Res., 87, 6865-6883, 1982.

Romanowicz, B. (1987) Multiplet-multiplet coupling due to lateral heterogeneity: asymptotic effects on the amplitude and frequency of the earth's normal modes, Geophys. J.R. Astr. Soc., 90, 75-100.

Romanowicz, B. (1989) The upper mantle degree two : constraints and inferences from mantle wave attenuation measurements, submitted to J. Geophys. Res.

Romanowicz, B. G. Roult and T. Kohl (1987) The upper mantle degree two pattern: constraints form Geoscope fundamental spheroidal mode eigenfrequency and attenuation measurements, Geophys. Res. Lett., 14, 1219-1222.

Roult, G., B. Romanowicz and N. Jobert (1986) Observations of departures from classical approximations on very long period GEOSCOPE records, Ann. Geophys., 4, B, 3, 241-250.

Silver, P. and T. H. Jordan (1981) Fundamental spheroidal mode observations of aspherical heterogeneity, Geophys. J.R. Astr. Soc., 64, 605-634.

Snieder, R. and B. Romanowicz (1988) A new formalism for the effect of lateral heterogeneity on normal modes and surface waves, I: isotropic perturbations,

perturbations of interfaces and gravitational perturbations, Geophys. J., 92, 207-222.

Woodhouse, J.H. (1983) The joint inversion of seismic waveforms for lateral variations in earth structure and earthquake source parameters, in *Earthquakes: Observation, Theory and interpretation*, eds H. Kanamori and E. Boschi, Proc. Int. School of Phys. Enrico Fermi, 85, 366-397.

Woodhouse, J.H. and T.P. Girnius (1982) Surface waves and free oscillations in a regionalized earth model, Geophys. J. R. Astr. Soc., 68, 653-673.

Woodhouse, J.H. and A.M. Dziewonski (1984) Mapping the upper mantle: three dimensional modeling of earth structure by inversion of seismic waveforms, J. Geophys. Res., 89, 5953-5986.

Woodhouse, J.H. and Y.K. Wong (1986) Amplitude, phase and path anomies of mantle waves, Geophys. J.R. Astr. Soc., 87, 753-773.