Chapter ①: Probability.

① Introduction to Probability:

1.1 Terminology:

* Random experiment:

it is the experiment with unknow outcome.

Examples: * Tossing a coin. (Head, Tail).

* Tossing a die (6 faces, 1, 2, ..., 6).
* Choosing randomly one book from a list of books.

* Probability space:

it is the set of all possible outcomes of a random experiment.

Examples: Tossing a coin: $S = \{H, T\}$.

Tossing a die: $S = \{1, 2, 3, 4, 5, 6\}$.

Tossing Two Coins: $S = \{HH, HT, TH, TT\}$.

The score of a student in a future Final exam:

$$S = [0, 40].$$

* We use also the name of "Sample space".

* Event:

it is a subset of a Sample space.

Examples: * $S = \{H, T\}$, $A = \{H\}$, $B = \{T\}$

S is the Certain event.

ϕ is the impossible event.

* $S = \{1, 2, \dots, 6\}$, $A = \{1\}$,

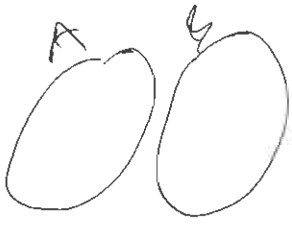
$B = \{2, 4, 6\}$, $C = \{$the number is odd$\}$
$= \{1, 3, 5\}$.

(1.2) Probability:

let S a sample space.

We call a Probability P on S, ~~to~~ any function from S to $\mathbb{R}$ (set of real number), Satisfying the following axioms:

* $P(A) \geq 0$, for any event A.
* $P(S) = 1$.
* let two disjoint events A and B:



$P(A \cup B) = P(A) + P(B)$.

* let a sequence of disjoint events (A_n):

$P(\bigcup_n A^n) = \sum_n P(A_n)$.

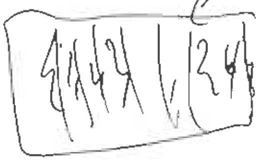
Example: $S = \{1, 2, \dots, 6\}$.

we define the Probability P by:

$$P(\{1\}) = P(\{2\}) = \dots = P(\{6\}) = \frac{1}{6}.$$

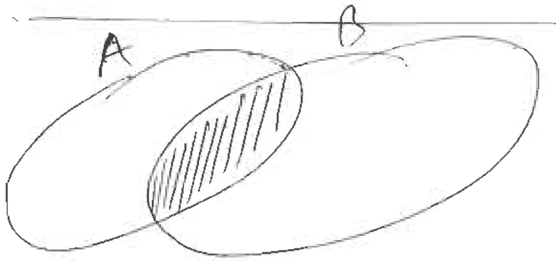
? $P(\{1,4,6\}) =$

$\{1,4,6\} = \{1\} \cup \{4\} \cup \{6\}$.



$$P(\{1,4,6\}) = P(\{1\} \cup \{4\} \cup \{6\})$$
$$= P(\{1\}) + P(\{4\}) + P(\{6\}).$$
$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 3/6 = \frac{1}{2}.$$

? $P(\{\text{the number} \geq 4\}) = P(\{4,5,6\})$
$$= P(\{4\}) + P(\{5\}) + P(\{6\}).$$
$$= 3/6 = 1/2.$$



$A \longrightarrow A', A^c, \bar{A}$

Consequences:

* $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
* $P(A') = 1 - P(A)$, A' is the Complementary event of A.
* $P(A) = P(A \cap B) + P(A \cap B')$.

Example: Suppose three events A, B, C:

$$P(A) = \frac{1}{4}, \quad P(B) = \frac{1}{2}, \quad P(C) = 1/6,$$

$$P(A \cap B) = \frac{1}{8}, \quad P(A \cup C) = 1/3.$$

Compute: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{1}{4} + \frac{1}{2} - \frac{1}{8} = 5/8.$$

$$P(A \cap C) = P(A) + P(C) - P(A \cup C)$$

$$= 1/4 + 1/6 - 1/3 = \frac{3+2-4}{12} = \frac{1}{12}.$$

$$P(A \cap B') = P(A) - P(A \cap B) = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}.$$

$$P(C') = 1 - P(C) = 1 - 1/6 = 5/6.$$

$$P(A \cup B') = P(A) + P(B') - \underline{P(A \cap B')}$$

$$= \frac{1}{4} + 1 - \frac{1}{2} - \frac{1}{8} = \frac{2+4-1}{8} = \frac{5}{8}.$$

<u>Particular Case:</u>

The Sample space is finite and the Probability of each elements are equal.

$|S|$ = Cardinal of the set S.

$S = \{a_1, \dots, a_n\}$. $P(\{a_i\}) = \frac{1}{n}$.

$A \subseteq S$: $\boxed{P(A) = \frac{|A|}{|S|}}$

In some cases, we use the words "at random" or "randomly" to signify that the elements have the same probability.

Example: From a group of 8 History books and 6 Maths books, one book is chosen at random.

Find the probability that:

a) this book is History.

b) ________ is Math.

$|S| = 8 + 6 = 14$.

a) $A = \{\text{the book is History}\} = \{H\}$

$|A| = 8$.

$P(A) = \frac{8}{14}$

b) $B = \{M\}$, $|B| = 6$, $P(B) = \frac{6}{14}$.

b) if the first letter is A or B.

2	26	26	26	10	9	8

$2 \times 26^3 \times 10 \times 9 \times 8$.

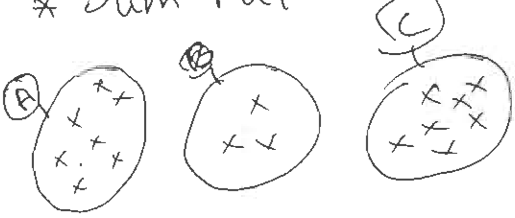
c) if the first two letters are identical.

26	1	26	26	10	9	8

$26^3 \times 10 \times 9 \times 8$.

1.2 Counting:

* Sum rule:



choose one element from A or B or C:

Number of ways $= |A| + |B| + |C|$.

* Product rule:

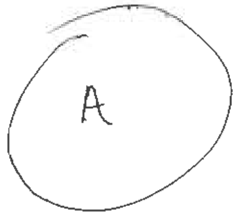
Choose one element, from each set:

Number of ways $= |A| \times |B| \times |C|$.

Example: In how many ways, one can choose one ball from Box1 (8 balls), and one ball from Box2 (10 balls).

$10 \times 8 = 80$.

* Combination:



Choose at time, r elements from A.

Number of ways $= \binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$

$A = \{a, b, c\}$

ab, ac, bc

~~ba~~, ~~ca~~

$\binom{3}{2} = \dfrac{3!}{2!\,1!} = 3$

Example : Box : 7 red + 8 white balls.

In how many ways, one can shoose :

a) Two balls,

$$\binom{15}{2} = \frac{15!}{2!\,13!} = \frac{14\times 15}{2} = 105.$$

b) Two red balls,

$$\binom{7}{2} = \frac{7!}{2!\,5!} = \frac{6\times 7}{2} = 21.$$

c) Two balls with the same Color.

(2R) OR (2W) : $\binom{7}{2} + \binom{8}{2} = 21 + \frac{8!}{2!\,6!} = 21 + \frac{7\times 8}{2} = 49.$

d) Three balls containing at most, one red.

$\boxed{(1R \text{ and } 2W)}$ OR $(0R \text{ and } 3W)$.

$$\binom{7}{1}\times\binom{8}{2} + \binom{8}{3} = 7\times 28 + 56 = 252.$$

* Arrangement and Repetition :

Example : A car plate is composed of 4 letters fellowed by 3 different digits.

How many car plates, one can obtain :

a) without restriction

26	26	26	26	10	9	8

$26^4 \times 10\times 9\times 8.$