# ROOTS OF EQUATIONS

- The bisection method or *interval-halving* is an extension of the direct-search method.
- It is used in cases where it is known that only one root occurs within a given interval of $x$.
- For the same level of precision, this method requires fewer calculations than the direct search method.

# Bisection Method

- Underlying Concept
  - The bisection (interval-halving) method is one of the simplest technique for determining roots of a function.
  - The basis for this method can be easily illustrated by considering the following function:

$$y = f(x)$$

# Bisection Method

- Underlying Concept
  - One objective is to find an $x$ value for which $y$ is zero.
  - Using this method, the function can be evaluated at two $x$ values, say $x_1$ and $x_2$ such that

$$f(x_1)f(x_2) < 0$$

## Bisection Method

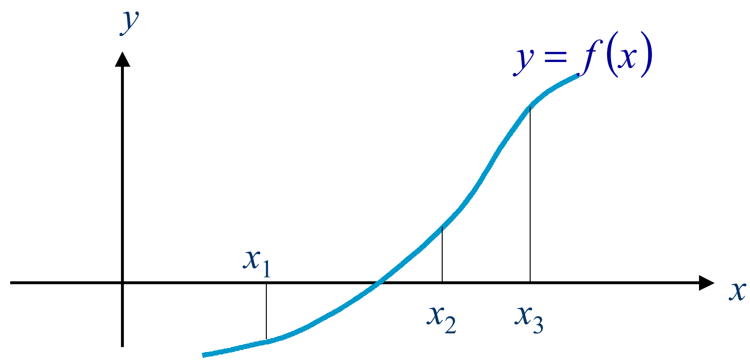
- Underlying Concept
  - The implication is that one of the values is negative and the other is positive.
  - Also, the function must be continuous for

$$x_1 \le x \le x_2$$

  - These conditions can be easily satisfied by sketching the function as shown in the following figure:

## Bisection Method

- Underlying Concept



## Bisection Method

- Underlying Concept
  - Looking at the figure, it is clear that the function is negative at $x_1$ and positive at $x_2$, and is continuous for $x_1 \le x \le x_2$.
  - Therefore, the root must between $x_1$ and $x_2$ and a new approximation to the root can be calculated as

$$x_3 = \frac{x_1 + x_2}{2}$$

## Bisection Method

- Underlying Concept
  - Clearly, $x_3$ and $x_1$ can be used to compute yet another value.
  - This process is continued until $f(x) \approx 0$ or the desired accuracy is achieved.
  - It is to be noted that at each iteration, the new $x$ value and one of the two previous values are used so that continuity and functional products are satisfied.

- Example 1: Bisection Method

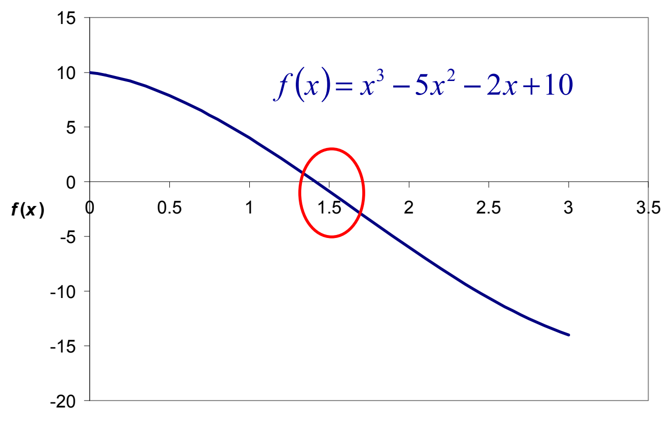
  Using graphical methods, the following function was found to have a real root between $x = 1$ and $x = 3$:

$$f(x) = x^3 - 5x^2 - 2x + 10$$

  Approximate the root.

# Bisection Method

- Example 1 (cont'd): Bisection Method



# Bisection Method

- Example 1 (cont'd): Bisection Method

  Evaluating the function at the initial values:

  $x_1 = 1: \qquad f(x_1) = 4$

  $x_2 = 3: \qquad f(x_1) = -14$

  Obviously, $f(1)f(3) = (4)(-14) < 0$ and the root has a value between 1 and 3. Therefore, a new value is approximated by

  $$x_3 = \frac{1+3}{2} = 2: \qquad f(x_3) = f(2) = -6$$

# Bisection Method

- Example 1 (cont'd): Bisection Method

  It is evident that the root is between $x_3$ and $x_1$, which must now be used to compute a new $x$ value. This is because $f(x_1)f(x_3) < 0$.

  Proceeding with the next five iterations, gives

$$x_4 = \frac{1+2}{2} = 1.5: \qquad f(x_4) = f(1.5) = -0.875$$

$$f(x_1)f(x_4) = f(1)f(1.5) = (4)(-0.875) < 0$$

# Bisection Method

- Example 1 (cont'd): Bisection Method

$$x_5 = \frac{x_1 + x_4}{2} = \frac{1+1.5}{2} = 1.25: \qquad f(x_5) = f(1.25) = 1.64063$$

$$f(x_1)f(x_5) = f(1)f(1.25) = (4)(1.875) > 0$$

$$f(x_4)f(x_5) = f(4)f(1.25) = (-0.875)(1.64063) < 0$$

$$x_6 = \frac{x_5 + x_4}{2} = \frac{1.25+1.5}{2} = 1.375: \qquad f(x_6) = f(1.375) = 0.39648$$

# Bisection Method

- Example 1 (cont'd): Bisection Method

$$f(x_5)f(x_6) = f(1.25)f(1.375) = (1.64063)(0.39648) > 0$$

$$f(x_6)f(x_4) = f(1.375)f(1.5) = (0.39648)(-0.875) < 0$$

$$x_7 = \frac{x_6 + x_4}{2} = \frac{1.375+1.5}{2} = 1.4375: \qquad f(x_7) = f(1.4375) = -0.23657$$

$$f(x_6)f(x_7) = f(1.375)f(1.4375) = (0.39648)(-0.23657) < 0$$

# Bisection Method

■ Example 1 (cont'd): Bisection Method

$$x_7 = \frac{x_6 + x_7}{2} = \frac{1.375 + 1.4375}{2} = 1.40625: \qquad f(x_7) = f(1.40625) = 0.08072$$

It is evident that the functional values are approaching zero as the number of iterations is increased.

After six iteration the approximated root of <u>1.40625</u> compares favorably with the exact value of $\sqrt{2}$

# Bisection Method

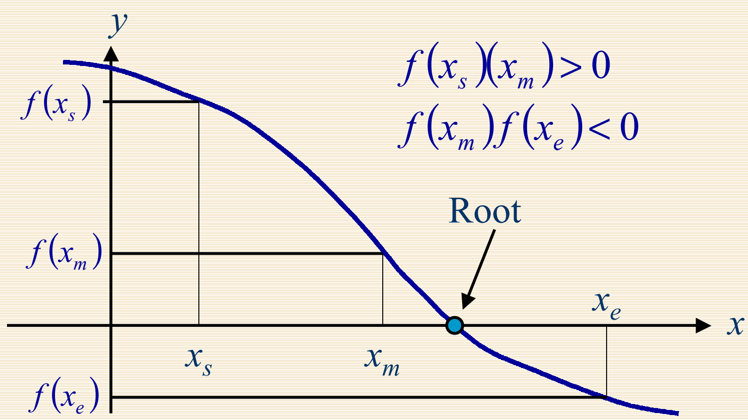
■ General Procedure

1. Sketch the function under consideration.
2. Establish the starting point $x_s$ and the end point $x_e$ of the interval such that $f(x_s)f(x_e) < 0$
3. From the starting point $x_s$ and the end point $x_e$, locate the midpoint $x_m$ at the center of the interval as follows:

$$x_m = \frac{x_s + x_e}{2}$$

# Bisection Method

■ General Procedure




# Bisection Method

■ General Procedure

4. At the starting point $x_s$, midpoint $x_m$, and the end point $x_e$, evaluate the function resulting from $f(x_s)$, $f(x_m)$, and $f(x_e)$, respectively.
5. Compute the product of the functions evaluated at the ends of the two intervals, that is, $f(x_s)f(x_m)$ and $f(x_m)f(x_e)$. The root lies in the interval for which the product is negative, and $x_m$ is used as an estimate.

## Bisection Method

- General Procedure
  5. Check for convergence as follows:
     a. If the convergence criterion (tolerance) is satisfied, then use $x_m$ as the final estimate of the root.
     b. If the tolerance has not been met, specify the ends of half-interval in which the root is located as the starting and ending points for a new interval and go to step 3.

## Bisection Method

- Error Analysis and Convergence Criterion
  - To ensure closure of the iteration loop, a convergence criterion is needed to terminate the iterative procedure for finding the root of a function.
  - The convergence criterion used in step 6 of the bisection method can be expressed in terms of either the absolute value of the difference percent relative error.

## Bisection Method

- Error Analysis and Convergence Criterion

$$\varepsilon_d = \left| x_{m,i+1} - x_{m,i} \right|$$

$$\varepsilon_r = \left| \frac{x_{m,i+1} - x_{m,i}}{x_{m,i+1}} \right| \times 100$$

Where $\varepsilon_d$ = absolute difference, $\varepsilon_r$ = percent relative error, $x_{m,i}$ = the midpoint in the previous root-search iteration, and $x_{m,i+1}$ is the midpoint in a new root-search iteration.

## Bisection Method

- Error Analysis and Convergence Criterion
  - The true accuracy of the solution at any iteration can be computed if the the true solution (root $x_t$) is known. The true error $\varepsilon_t$ in the $i^{th}$ iteration is given by

$$\varepsilon_t = \left| \frac{x_t - x_{m,i}}{x_t} \right|$$

# Bisection Method

- Example 2: Bisection Method

  The following polynomial has a root within the interval $3.75 \leq x \leq 5.00$:

$$f(x) = x^3 - x^2 - 10x - 8 = 0$$

  If a tolerance of 0.01 (1%) is required, find this root using bisection method.

# Bisection Method

- Example 2: Bisection Method

$x_s = 3.75, \qquad x_e = 5.00 \qquad \boxed{f(x) = x^3 - x^2 - 10x - 8 = 0}$

$i = 1$

$$x_m = \frac{x_s + x_e}{2} = \frac{3.75 + 5.00}{2} = 4.375$$

$$f(x_s) = f(3.75) = (3.75)^3 - (3.75)^2 - 10(3.75) - 8 = -6.828$$

$$f(x_m) = f(4.375) = (4.375)^3 - (4.375)^2 - 10(4.375) - 8 = 12.850$$

$$f(x_e) = f(5) = (5)^3 - (5)^2 - 10(5) - 8 = 42.000$$

$$f(x_s)f(x_m) < 0 \quad \text{(negative)}$$

$$f(x_m)f(x_e) > 0 \quad \text{(positive)}$$

# Bisection Method

- Example 2: Bisection Method

$$\boxed{f(x) = x^3 - x^2 - 10x - 8 = 0}$$

$x_s = 3.75 \qquad x_e = 4.375$

$i = 2$

$$x_m = \frac{x_s + x_e}{2} = \frac{3.75 + 4.375}{2} = 4.063$$

$f(x_s) = f(3.75) = -6.828$

$f(x_m) = f(4.063) = 1.918$

$f(x_s)f(x_m) < 0\ \text{(negative)}$

$f(x_e) = f(4.375) = 12.850$

# Bisection Method

- Example 2: Bisection Method

$$\text{error } \varepsilon_d = \left|x_{m,i+1} - x_{m,i}\right| = \left|4.063 - 4.375\right| = 0.312$$

$$\varepsilon_r = \left|\frac{x_{m,i+1} - x_{m,i}}{x_{m,i+1}}\right| \times 100 = \left|\frac{4.063 - 4.375}{4.063}\right| \times 100 = 7.68\%$$

$x_s = 3.75 \qquad x_e = 4.063$

$i = 3$

$$x_m = \frac{x_s + x_e}{2} = \frac{3.75 + 4.063}{2} = 3.907$$

$f(x_s) = f(3.75) = -6.828$

$f(x_m) = f(3.907) = -2.696$

$f(x_e) = f(4.063) = 1.934$

$f(x_m)f(x_e) < 0\ \text{(negative)}$

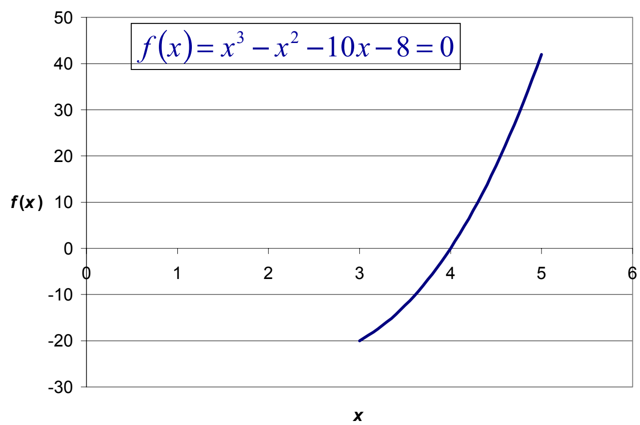
# Bisection Method

- Example 2: Bisection Method

| Iteration $i$ | $x_s$ | $x_m$ | $x_e$ | $f(x_s)$ | $f(x_m)$ | $f(x_e)$ | $f(x_s)f(x_m)$ | $f(x_m)f(x_e)$ | error $\varepsilon_d$ | error $\varepsilon_d$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3.7500 | 4.3750 | 5.0000 | -6.8281 | 12.8496 | 42.0000 | - | + | --- | --- |
| 2 | 3.7500 | 4.0625 | 4.3750 | -6.8281 | 1.9182 | 12.8496 | - | + | 0.31250 | 7.69 |
| 3 | 3.7500 | 3.9063 | 4.0625 | -6.8281 | -2.7166 | 1.9182 | + | - | 0.15625 | 4.00 |
| 4 | 3.9063 | 3.9844 | 4.0625 | -2.7166 | -0.4661 | 1.9182 | + | - | 0.07813 | 1.96 |
| 5 | 3.9844 | 4.0234 | 4.0625 | -0.4661 | 0.7092 | 1.9182 | - | + | 0.03906 | 0.97 |
| 6 | 3.9844 | 4.0039 | 4.0234 | -0.4661 | 0.1174 | 0.7092 | - | + | 0.01953 | 0.49 |
| 7 | 3.9844 | 3.9941 | 4.0039 | -0.4661 | -0.1754 | 0.1174 | + | - | 0.00977 | 0.24 |
| 8 | 3.9941 | 3.9990 | 4.0039 | -0.1754 | -0.0293 | 0.1174 | + | - | 0.00488 | 0.12 |
| 9 | 3.9990 | 4.0015 | 4.0039 | -0.0293 | 0.0440 | 0.1174 | - | + | 0.00244 | 0.06 |
| 10 | 3.9990 | 4.0002 | 4.0015 | -0.0293 | 0.0073 | 0.0440 | - | + | 0.00122 | 0.03 |
| 11 | 3.9990 | 3.9996 | 4.0002 | -0.0293 | -0.0110 | 0.0073 | + | - | 0.00061 | 0.02 |
| 12 | 3.9996 | 3.9999 | 4.0002 | -0.0110 | -0.0018 | 0.0073 | + | - | 0.00031 | 0.01 |
| 13 | 3.9999 | 4.0001 | 4.0002 | -0.0018 | 0.0027 | 0.0073 | - | + | 0.00015 | 0.00 |
| 14 | 3.9999 | 4.0000 | 4.0001 | -0.0018 | 0.0005 | 0.0027 | - | + | 0.00008 | 0.00 |
| 15 | 3.9999 | 4.0000 | 4.0000 | -0.0018 | -0.0007 | 0.0005 | + | - | 0.00004 | 0.00 |

# Bisection Method

- Example 2: Bisection Method



# Bisection Method

- Disadvantage of Bisection Method
  - Although the bisection (interval halving) method will always converge on the root, the rate of convergence is very slow.
  - A faster method for converging on a single root of a function is the Newton-Raphson iteration Method.