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* final level

2. Time value of money

2.1 Time value of money

The **time value of money principle** states that a dollar today is worth more than a dollar tomorrow. This is due to the fact that the present is certain, but the future is not; present consumption is better than deferring gratification to the future. For this reason, an incentive is required to motivate us to defer gratification. This incentive is represented by the interest rate.

Interest rates are usually expressed on an annual basis, and express the remuneration that has to be paid from the borrower to the lender for the service of lending money. There are several types of interest rates.

2.1.1 Simple versus compound interest

Simple interest assumes that interest does not itself earn interest, and is calculated by the following formula:

$$\text{Simple interest} = (\text{initial value}) \cdot (\text{interest rate}) \cdot (\text{number of years})$$

The initial value is the principal amount on which interest is paid over a given period.

Example:
What is the simple interest earned on 1’000 CHF invested at 7% p.a. (per annum) after 10 years?

The answer is

$$1'000 \text{ CHF} \cdot 0.07 \cdot 10 = 700 \text{ CHF}$$

However in the real world interest payments received are reinvested to earn more interest in subsequent periods.

Compound interest assumes that interest is reinvested; so compound interest is simple interest plus interest earned on interest. The formula to calculate compound interest on a given initial value over a given period is:

$$\text{Compound interest} = (\text{initial amount}) \cdot [(1 + \text{interest rate})^{\text{number of years}} - 1]$$

Example:
What is the compounded interest earned on 1’000 CHF invested at R=7% p.a. (per annum) after 10 years?

The answer is

$$1'000 \text{ CHF} \cdot [(1+0.07)^{10} - 1] = 967.15 \text{ CHF}$$

Proof:

Year	Capital at begin of period [1]	Interest [2]=[1]·R	Capital at end of period [1]+[2]
1	1’000	70	1’070
2	1’070	74.9	1’144.9
3	1’144.9	80.14	1’225.04
…	…	…	…
10	1’838.46	128.69	1’967.15

So, we see that, after ten years, the compound interest is 967.15 CHF, as opposed to the simple interest of 700 CHF.

2.1.2 Present and future value

The process of determining the **present value** of a future payment (or receipt) or series of future payments (or receipts) is called **discounting**. The compound interest rate used for discounting cash flows is also called the **discount rate**. So the **present value** (or **actual value**) of a future income is given by:

$$\text{Present value} = \frac{\text{Future value}}{(1 + \text{Interest rate})^{\text{number of years}}}$$

The present value of a promised future cash flow is inversely related to both the length of the investment period and the level of interest rates.

Example:
A financial firm offers to pay you 100'000 CHF in 10 years, if you give the firm 60'000 CHF today. Using a 7% interest rate, the present value of 100'000 CHF in 10 years is:

$$\text{Present value} = \frac{100'000}{(1.07)^{10}} = 50'835\text{CHF}$$

Accepting this offer would imply paying 60'000 CHF for something that is worth 50'835 CHF. You should definitely refuse!

The process of finding the **future value** of the payment (or receipt) or series of payments (or receipts) using the concept of compound interest is known as **compounding**. The general formula for compounding is:

$$\text{Future value} = (\text{Present value}) \cdot (1 + \text{Interest rate})^{\text{number of years}}$$

Example:
100 CHF are deposited in a bank account with a 5% annual interest rate. What is the balance of the account at the end of the first and the second year respectively?

At the end of the first year, we have

$$100 \cdot (1 + 0.05) = 105 \text{ CHF}$$

At the end of the second year, we have

$$105 \cdot (1 + 0.05) = 110.25 \text{ CHF}$$

Why not just 110 CHF? Because we also have earned a 5% interest on the 5 CHF paid at the end of the first year ($5 \cdot 5\% = 0.25$). We could also have directly stated:

$$100 \cdot (1+0.05)^2 = 110.25 \text{ CHF}$$

A high interest rate environment and long investment period lead to greater accumulation of compound interest.

Example:
Suppose that 1'000 CHF were deposited in a saving account on 1st of January 1934. What is the balance on 31st of December 2000, if interest was paid at a rate of 10.5%?

There have been 67 years of compounding. The final balance is:

$$1'000 \cdot (1.105)^{67} = 804'030.69 \text{ CHF}$$

In the case of simple interest, we would have a balance of only

$$1'000 + (1'000 \cdot 0.105 \cdot 67) = 8'035 \text{ CHF}$$

2.1.3 Annuities

In the special case of an annuity, a fixed amount of money is paid each year for a specified number of years. The present value of this series of cash flows is given by the following formula:

$$\text{Present value} = \sum_{t=1}^{n} \frac{\mathrm{CF}}{(1+\mathrm{R})^{t}} = \frac{\mathrm{CF}}{\mathrm{R}} \cdot \left(1 - \frac{1}{(1+\mathrm{R})^{n}}\right)$$

When using this formula we suppose that the first payment is received in one year from now.

Example:
The same financial firm as above offers to pay you 10’000 CHF at the end of each year during 10 years, if you give the firm 70'000 CHF today. Using a 7% interest rate, the present value of this 10-year annuity is:

$$\text{Present value} = \frac{10'000}{0.07} \cdot \left(1 - \frac{1}{(1+0.07)^{10}}\right) = 70'236$$

Accepting this offer would imply paying 70'000 CHF for something that is worth 70’236 CHF. You should definitely accept it!

The future value of the same series of cash-flows assumes that all individual cash-flows are reinvested at the same interest rate R, and can be calculated with the following formula:

$$\text{Future value} = \sum_{t=0}^{n-1} \mathrm{CF} \cdot (1+\mathrm{R})^{t} = \mathrm{CF} \cdot \left(\frac{(1+\mathrm{R})^{n} - 1}{\mathrm{R}}\right)$$

Like for the present value, the formula above supposes that the first cash flow is received in one year from now.

Let us illustrate this with a simple example.

Example:
What is the future value of the series of coupons of a 7%, 10-year bond purchased at par (1'000 CHF), if we assume that all payments are reinvested at a 7% rate?

The answer is:

$$70 \cdot \left(\frac{(1+0.07)^{10} - 1}{0.07} \right) = 967.15$$

The simple interest in this case would have been:

$$1'000 \cdot 0.07 \cdot 10 = 700 \text{ CHF}$$

267.15 CHF (= 967.15 CHF – 700 CHF) is the additional amount earned by reinvesting the coupon payments and earning interest on the interest.

2.1.4 Continuous discounting and compounding

Compounding can take place not only with annual frequency, but also with higher frequency. It can be shown that if there are m compounds per year (i.e. interest is paid m times per year) then an initial amount N_0 invested at an annual interest rate R during n years becomes $N_0 \cdot (1+\frac{R}{m})^{m \cdot n}$. When m tends to infinity (i.e. interest is paid at every instant), it can be shown that this formula becomes $N_0 \cdot e^{R \cdot n}$, where e ≅ 2.718 is the Euler number. So we can use the following formula:

$$\text{Future value} = (\text{Actual value}) \cdot e^{\text{Time} \cdot \text{Instantaneous interest rate}}$$

Continuous compounding will lead to a higher future value. As interest is paid continuously, there is more interest on interest.

Example:
100 CHF are deposited in a bank account with a 5% continuous annual interest. What is the account balance at the end of the first year and second year respectively, using continuous compounding?

At the end of the first year, the balance is

$$100 \cdot e^{0.05 \cdot 1} = 105.13 \text{ CHF}$$

At the end of the second year, the balance is

$$100 \cdot e^{0.05 \cdot 2} = 110.52 \text{ CHF}$$

2.2 Bond yield measures

2.2.1 Current yield

The **current yield** of a bond is simply the annual coupon payment divided by the market price of the bond (excluding accrued interest).

$$\text{Current yield} = \frac{\text{Annual coupon in USD}}{\text{Price in USD}}$$

Since the coupon rate is generally fixed, the bond's current yield varies inversely with the bond's price. As the bond's price rises (declines), its current yield falls (increases) since its coupon return is now a lesser (larger) amount per USD of bond value. So, all other factors held constant, a bond with a higher current yield sells at a lower price!

> **Example:**
> The XYZ 2002-2012 3.25% bond is quoted at 98.1 on the 13.06. Its current yield is 3.25/98.1 = 3.31%.

The current yield only takes into account the annual coupon income of the bond, and therefore, it is not an adequate tool to compare two bonds. For example,

- the current yield of a zero-coupon bond is zero as it pays no coupon.
- the current yield of a bond priced under par will decrease as the bond approaches maturity (the coupon is fixed, but the price rises toward par).

One should note that some Japanese bond dealers still use a modified version of the current yield, also called **Japanese current yield**, defined as:

$$\text{Japanese current yield} = \frac{\text{Annual coupon - } \dfrac{\text{Price (\%)} - 100}{\text{Remaining life}}}{\text{Price}}$$

The Japanese current yield considers not only the coupon payment, but also the capital gains/losses made by the investor over the life of the bond. Thus, a bond purchased at premium will have a lower current yield due to capital losses suffered over its life-time, while bonds purchased at discount will exhibit higher current yield due to the capital gains component.

2.2.2 Yield to maturity

The **yield to maturity** (YTM) is the discount rate that equates the present value of the bond's future cash flows till maturity with the current market price of the bond. Just after the coupon payment, we have :

$$P = \sum_{t=1}^{T} \frac{CF_t}{(1+YTM)^t} = \frac{CF_1}{(1+YTM)^1} + \frac{CF_2}{(1+YTM)^2} + \ldots + \frac{CF_T}{(1+YTM)^T}$$

where:

P	price of the bond (market price)
CF_t	cash flow received at the end of period t (coupons or repayment)
T	remaining life of the bond (time to maturity).

It can also be defined as the **internal rate of return** (IRR) of the investment in the bond.

Example:
An investor can buy a bond for 116.00 with a 10% coupon, 1'000 CHF face value and 4 years until maturity. The coupon has just been paid. What is the bond's yield to maturity?

The yield to maturity YTM solves the following equation:

$$\frac{100}{(1+\text{YTM})}+\frac{100}{(1+\text{YTM})^2}+\frac{100}{(1+\text{YTM})^3}+\frac{1'100}{(1+\text{YTM})^4}=1'160$$

By iteration with a computer, we find the correct answer, which is YTM = 5.44%.

The yield to maturity assumes that the bond is held to maturity, and that all cash flows are received as scheduled through final maturity. As will be seen later (in paragraph 2.4.3), the yield to maturity should not be confused with the total return on the bond investment.

The case of six-monthly coupons is easy to handle. First, calculate a six-monthly yield to maturity YTM_S:

$$P=\sum_{t=1}^{T}\frac{CF_t}{(1+\text{YTM}_S)^t}=\frac{CF_1}{(1+\text{YTM}_S)^1}+\frac{CF_2}{(1+\text{YTM}_S)^2}+...+\frac{CF_T}{(1+\text{YTM}_S)^T}$$

where CF_t is the cash flow received at the end of period semester t (coupons or repayment), and T is the number of semesters in the remaining life of the bond (time to maturity).

Then, convert this six-monthly yield in an annual yield YTM_A using one of the following formulae:

- on a Euromarket:

$$\text{YTM}_A=(1+\text{YTM}_S)^2-1$$

- on the US or English market:

$$\text{YTM}_A^{US}=2\cdot\text{YTM}_S$$

  by convention, the annual yields are not compounded on these markets (which gives a lower yield than on the Euromarkets).

The following example will illustrate this concept.

Example:
A 10-year eurobond pays semi-annually an annual coupons of 6% and is quoted at 110.00. The coupon has just been paid. What is its yield to maturity? What would be its yield to maturity if the coupon was paid annually?

The yield to maturity solves:

$$110 = \frac{3}{(1+\text{YTM}_S)} + \frac{3}{(1+\text{YTM}_S)^2} + ... + \frac{103}{(1+\text{YTM}_S)^{20}}$$

The solution is $\text{YTM}_S = 2.37\%$. The annual corresponding yield is

$$\text{YTM}_A = (1+\text{YTM}_S)^2 - 1 = 4.79\%$$

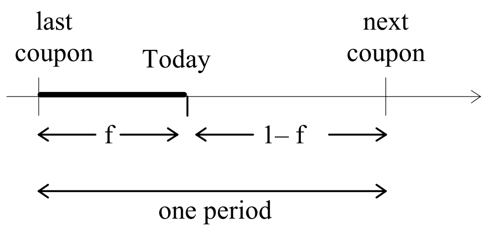
If the coupon was paid annually, we would have a yield of 4.72%, which is lower. This is predictable, as a semi-annual interest payment allows interest compounding.

Note that for a US bond, we would have $2 \cdot 2.37\% = 4.74\%$!

The same methodology can be used in the case of quarterly coupons.

2.2.2.1 Yield to maturity between two coupon payment dates

Between two coupon payment dates, the buyer of a bond must pay the accrued interest to the seller. The accrued interest is due for the fraction f of the total period between two coupon dates



Hence, the total price to be paid for the bond is (in the case of an annual coupon payment):

$$\text{Total price} = \text{Quoted price} + f \cdot \text{Coupon}$$

The formula to calculate the yield to maturity must be modified to include accrued interest in the total price:

$$\begin{aligned} P + f \cdot C &= \sum_{t=1}^{T} \frac{CF_t}{(1+\text{YTM})^{t-f}} = \frac{CF_1}{(1+\text{YTM})^{1-f}} + \frac{CF_2}{(1+\text{YTM})^{2-f}} + \ldots + \frac{CF_T}{(1+\text{YTM})^{T-f}} \\ &= (1+\text{YTM})^f \cdot \left[\frac{CF_1}{(1+\text{YTM})^1} + \frac{CF_2}{(1+\text{YTM})^2} + \ldots + \frac{CF_T}{(1+\text{YTM})^T} \right] \end{aligned}$$

Let us illustrate this.

Example:
A bond with an annual coupon of 6% is quoted at 108.00 on the market, 9 years and 3 months before its maturity. What is its yield?

We have:

$$f = 9 / 12 = 0.75 \text{ years}$$

The effective price paid is

$$108 + 0.75 \cdot 6 = 112.5$$

and the yield to maturity solves:

$$112.5 = (1+\text{YTM})^{0.75} \cdot \left[\frac{6}{(1+\text{YTM})} + \frac{6}{(1+\text{YTM})^2} + ... + \frac{106}{(1+\text{YTM})^{10}}\right]$$

The solution is YTM = 4.90% (by iteration).

While the methodology is general, one should always keep in mind that the way to calculate the number of days for the accrued interest varies from one country to another.

2.2.2.2 Influences on the yield to maturity: the coupon effect

The yield to maturity of two bonds having the same maturity but different cash flows is not necessarily the same. This is called the **coupon effect** or **coupon bias**[1].

Example:
Bond A is a two-year 10% coupon, while bond B is a two-year 5% coupon. The returns on money obtained for one year and two years are $R_{0,1} = 6\%$, and $R_{0,2} = 7\%$ respectively.

The bonds prices are

$$P_A = \frac{10}{1.06} + \frac{110}{1.07^2} = 105.512$$

$$P_B = \frac{5}{1.06} + \frac{105}{1.07^2} = 96.428$$

But the yields to maturity of the two bonds are different:

$$P_A = \frac{10}{(1+k_A)} + \frac{110}{(1+k_A)^2} = 105.512 \Rightarrow \text{YTM}_A = 6.953\%$$

$$P_B = \frac{5}{(1+k_B)} + \frac{105}{(1+k_B)^2} = 96.428 \Rightarrow \text{YTM}_B = 6.975\%$$

The differences in the bond yields arise because the yield to maturity is a complex average of the spot rates applied to one and two years cash investments. In our example, bond B has a greater fraction of its value tied to the higher two years interest rate.

1 Note that there is no coupon effect with zero-coupon bonds.

If one compares the price formula of a coupon bearing bond

$$P=\sum_{t=1}^{T}\frac{CF_t}{\left(1+R_{0,t}\right)^t}=\frac{CF_1}{\left(1+R_{0,1}\right)^1}+\frac{CF_2}{\left(1+R_{0,2}\right)^2}+...+\frac{CF_T}{\left(1+R_{0,T}\right)^T}$$

with the definition of the yield to maturity

$$P=\sum_{t=1}^{T}\frac{CF_t}{(1+YTM)^t}=\frac{CF_1}{(1+YTM)^1}+\frac{CF_2}{(1+YTM)^2}+...+\frac{CF_T}{(1+YTM)^T}$$

it is clear that **the yield to maturity is a complex average of the spot rates**. Hence, the reader should carefully distinguish between yield to maturity and the spot rate $R_{0,T}$.

In fact, if the series of rates $R_{0,t}$ are increasing, one can show that the yield to maturity (YTM) will underestimate the corresponding spot rate $R_{0,T}$.

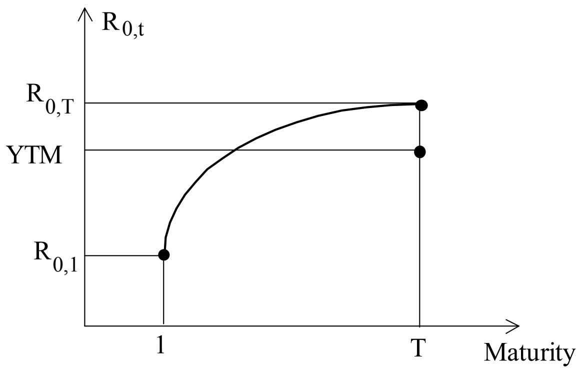


Figure 2-1: Yield to maturity versus spot rate

We can illustrate this by the following example[2]:

Example:
Let us consider the following increasing spot rates: $R_{0,1}$ = 2%, $R_{0,2}$ = 4%, $R_{0,3}$ = 5%, $R_{0,4}$ = 5.5%, $R_{0,5}$ = 6%. If we select three bonds A, B, and C differing only by their coupon rates, we can compute their prices and their returns to maturity.

	A	**B**	**C**
Maturity	5 years	5 years	5 years
Annual coupon rate	0%	3%	10%
Repayment	100%	100%	100%
Price	**74.73**	**87.69**	**117.96**
Yield to maturity	6%	5.91%	5.76%

Therefore, it is clear that using coupon-paying bonds, we will underestimate the effective spot rate for the considered maturity (6% for 5 years in our example). The bias will increase for larger coupon rates.

2 Source : DUMONT Pierre André, 1995, "Les obligations ordinaires: typologie, procédures d'émission et aspects boursiers", HEC-University of Geneva, Geneva.

If the series of rates $R_{0,t}$ are decreasing, one can show that the yield to maturity (YTM) will overestimate the corresponding spot rate ($R_{0,T}$).

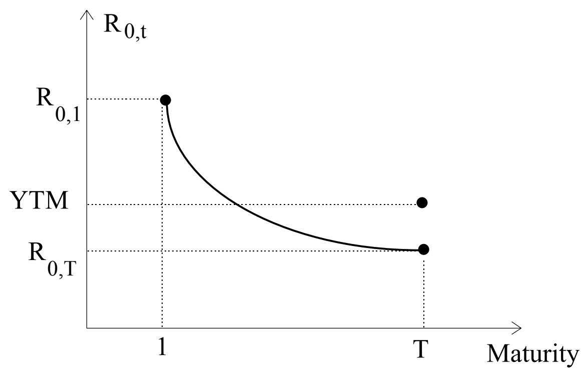


Figure 2-2: Yield to maturity versus spot rate

We can illustrate this by the following example:

Example:
Let us consider the following decreasing spot rates: $R_{0,1}$ = 6%, $R_{0,2}$ = 4%, $R_{0,3}$ = 3%, $R_{0,4}$ = 2.5%, $R_{0,5}$ = 2%. If we select three bonds A, B, and C differing only by their coupon rates, we can compute their prices and their yields to maturity.

	A	**B**	**C**
Maturity	5 years	5 years	5 years
Annual coupon rate	0%	3%	10%
Repayment	100%	100%	100%
Price	**90.57**	**104.35**	**136.52**
Yield to maturity	2%	2.08%	2.21%

Therefore, it is clear that using coupon-paying bonds, we will overestimate the effective spot rate for the considered maturity (2% for 5 years in our example). The bias will increase for larger coupon rates.

In the case of a coupon-bearing bond, the spot rate ($R_{0,T}$) and the yield to maturity (YTM) are usually unequal. But this issue will be dealt with later.

2.2.3 Yield to call

For callable bonds, the **yield to call** is the discount rate YTM_C that equates the present value of the bond's future cash flows received through the call date to the bond's current market price:

$$P = \sum_{t=1}^{TC} \frac{CF_t}{(1+YTM_C)^t} = \frac{CF_1}{(1+YTM_C)^1} + \frac{CF_2}{(1+YTM_C)^2} + \ldots + \frac{CF_{TC}}{(1+YTM_C)^{TC}}$$

where CF_t is the cash flow received at the end of period t (coupons or repayment), and TC is the remaining time until the call date. **It assumes that the bond will be called**, and that all cash flows are received as scheduled through the call date.

Example:
An investor can buy the same bond as in the previous example (10% coupon, 1'000 CHF face value, 4-year bond, quoted price of 116.00). The bond is callable in three years at 103. What is the bond's yield to call?

The yield to call YTM_C solves the following equation:

$$\frac{100}{\left(1+YTM_C\right)}+\frac{100}{\left(1+YTM_C\right)^2}+\frac{1030+100}{\left(1+YTM_C\right)^3}=1160$$

By successive approximation, we find the correct answer, which is YTM = 5.07%.

The yield to call differs from the yield to maturity as the discounting period is shorter (since the call date precedes the maturity date), and the final cash flow is generally higher (since the call price is generally above par-value).

2.2.4 *Other yields*

Yield to average life

The **yield to average life** YTM_{AL} is only used to compare bonds with a series of principal repayment (like sinking fund bonds, mortgage backed securities, ...) and bullet bonds that repay principal at maturity. For simplicity, the full principal repayment is supposed to occur on the **average life date**. The average life of a bond is the weighted average maturity of the principal repayment (note that the coupon rate plays no role in the average life, as it only considers principal repayments):

$$\text{Average life in years} = AL = \sum_{t=0}^{T}\frac{\text{Principal paid at time t}}{\text{Total principal to be repaid}}\cdot t$$

The yield to average life is simply the internal rate of return (IRR) to the average life date (as if the average life date was the final maturity date of the bond):

$$P=\sum_{t=1}^{AL}\frac{CF_t}{\left(1+YTM_{AL}\right)^t}=\frac{C_1}{\left(1+YTM_{AL}\right)^1}+\frac{C_2}{\left(1+YTM_{AL}\right)^2}+\ldots$$
$$+\frac{C_{AL}}{\left(1+YTM_{AL}\right)^{AL}}+\frac{\text{Principal}}{\left(1+YTM_{AL}\right)^{AL}}$$

where C_t is the coupon received at the end of period t (without repayment), and AL is the average life of the bond. All principal repayment are assumed to be made on the average date.

Example:
An investor can buy a 5% coupon, 1'000 CHF face value, 10-year bond, quoted at a price of 102. The bond has a 90% sinker[3], with sinking funds payments starting at the end of the first year, and repaying 10% of the bonds annually through the ninth year. All repayments are made at par value.

The average life is:

$$AL=\frac{100}{1000}\cdot 1+\frac{100}{1000}\cdot 2+\ldots+\frac{100}{1000}\cdot 10=5.50\text{ years}$$

3 The **sinker percentage** is the percentage of bonds retired before maturity.

The yield to average life solves the following equation:

$$\frac{50}{(1+\text{YTM}_{\text{AL}})}+\frac{50}{(1+\text{YTM}_{\text{AL}})^2}+\ldots+\frac{50}{(1+\text{YTM}_{\text{AL}})^5}+\frac{1025}{(1+\text{YTM}_{\text{AL}})^{5.5}}=1020$$

Note that the last payment is 1'025 CHF because we only earn interest (of 25 CHF) for half a year, as the average life is five years and a half.

By iteration, we find the correct answer, which is YTM_{AL} = 4.58%. **In essence, our bond is roughly comparable with a bullet bond giving a 4.58% yield to maturity, and maturing in 5.5 years.**

Call-adjusted yield

For a callable bond, the **call-adjusted yield** is simply the yield to maturity at the grossed-up, non callable equivalent bond price.

Example:
Assume XYZ Inc. can issue at par (= 100.00) a 8% coupon, 10-year bond, with an 8% yield to maturity. The bond is callable in 5 years at 108.00.

At the same time, a similar long term government bond yields 7%. At first, one may think that the investor receives 1% (= 100 basis points) additional yield for the credit risk of XYZ.

But if (for example) the call option is valued at 4.00 points, we have:

Price of the non callable bond = Price of the callable bond + Value of the call option

or

Price of the non callable bond = 100.00 + 4.00 = 104.00

The yield to maturity of our bond at par is 8% (quoted yield). But the yield to maturity of the same bond if the price was 104.00 is 7.41 (call-adjusted yield). Thus, our investor receives only 0.41% (41 basis points) for the credit risk of XYZ.

2.2.5 *Other basic concepts*

Three dates are essential while determining any rate of interest:

- the commitment date, which is the date at which the borrower and lender set the fixed rate on the loan.
- the lending date, at which the money is to be loaned.
- the repayment date, at which the money is to be repaid.

In the following discussion, we will use the concept of spot rates and forward rates which are defined below.

2.2.5.1 Spot rates

The **spot rate**, denoted by $R_{0,t}$, is defined as the annual interest rate received on a pure discount security[4] (zero-coupon bond) maturing at time t. It is, at time 0, the required rate of return to lend money from time 0 to time t, if only one final payment is made for both interest and principal.

4 Note that the exact definition of a pure discount bond is a bond that pays 1 CHF at maturity.

One should remember that:

- the commitment date and the lending date are the same.
- spot rates are interest rates on loans or bonds that pay only **one** cash flow to the investor.

Generally, spot rates are quoted as annual rates.

Example:
A bond involves an investment of 797.19 CHF and returns a principal of 1'000 CHF in exactly two years. What is the two years spot rate? What about the one year spot rate?

As the bond is a pure discount bond, its return will be the two-year spot rate.

$$797.19 = \frac{1'000}{(1+R_{0,2})^2} \quad \Rightarrow \quad R_{0,2} = 12\% \text{ p.a.}$$

The rate is expressed on an annualised basis. Nothing can be said about the one-year spot rate.

2.2.5.2 Forward rates

The **forward rate**, denoted by $F_{t,h}$, is the rate of interest on a bond where the commitment date (0) and the date the money is lent (t) are different. If a commitment is made today on a two-years loan to begin in one year (t), the annualised interest rate from year 1 to year 3 (from year t to year h) is a forward rate.

One should remember that:

- the commitment date is today, but the lending date differs.
- forward rates are interest rates on loans or bonds that pay only one cash flow to the investor.

Forward rates are also generally quoted as annual rates.

Example:
A commitment involves a loan of 841.68 CHF in one year and a principal and interest repayment of 1'000 CHF in three years. What is the two-year forward rate to begin in year one?

As the bond is a pure discount bond, its return will be the two-year forward rate.

$$841.68 = \frac{1'000}{(1+F_{1,3})^2} \quad \Rightarrow \quad F_{1,3} = 9\% \text{ p.a.}$$

The rate is expressed on an annualised basis.

2.2.5.3 Relation between spot rate and forward rate

Generally, the forward rate can be calculated as the ratio of the end-of-period wealth to the beginning-of-period wealth, or as the ratio of the corresponding spot rates:

$$F_{t1,t2} = \sqrt[t2-t1]{\frac{\left(1+R_{0,t2}\right)^{t2}}{\left(1+R_{0,t1}\right)^{t1}}} - 1$$

Example:
The spot rates are $R_{0,1}$ = 6%, $R_{0,2}$ = 7% and $R_{0,3}$ = 7.5%. What is the implicit one-year forward rate at the end of the first year? At the end of the second year?

The one-year forward rate at the end of the first year can be derived by using two consecutive spot rates. If the investor invests 100 CHF for two years at the current spot rate $R_{0,2}$, he will receive at the end of the second year:

$$100 \cdot (1 + 0.07) \cdot (1 + 0.07) = 114.49 \text{ CHF}$$

Another solution would be to invest 100 CHF for one year at the current spot rate $R_{0,1}$, and to reinvest the proceeds at the forward rate $F_{1,2}$. Note that all positions in this trading strategy are determined at time 0. At the end of the first year, the investor will receive

$$100 \cdot (1.06) = 106.00 \text{ CHF}$$

He will reinvest this amount at the forward rate, and he should end with the same amount as the two-year strategy. Hence:

$$106.00 \cdot (1 + F_{1,2}) = 114.49 \text{ CHF}$$

and we have:

$$F_{1,2} = \frac{\text{Wealth position (end of year 2)}}{\text{Wealth position (end of year 1)}} - 1 = \frac{114.49}{106.00} - 1 = 0.0801 = 8.01\%$$

Similarly, we can calculate the implicit one year forward rate at the end of the second year:

$$F_{2,3} = \frac{\text{Wealth position (end of year 3)}}{\text{Wealth position (end of year 2)}} - 1 = \frac{100 \cdot (1.075)^3}{114.49} - 1 = 0.0851 = 8.51\%$$

By construction, the **spot rate** may also be seen as the **geometric average** of implicit consecutive forward rates:

$$\left(1+R_{0,t}\right) = \left[\left(1+R_{0,1}\right)\cdot\left(1+F_{1,2}\right)\cdot\left(1+F_{2,3}\right)\ldots\left(1+F_{t-1,t}\right)\right]^{\frac{1}{t}}$$

2.3 Term structure of interest rates

2.3.1 Yield curves and shapes

The economics of interest rates deals with the pure **price of time** (time value of money). Awareness and appreciation of the interest rate-maturity relationship is essential in bond management.

The relationship between the yields on otherwise comparable bonds with different maturities is called the **term structure of interest rates**; its graphical depiction is known as the yield curve.

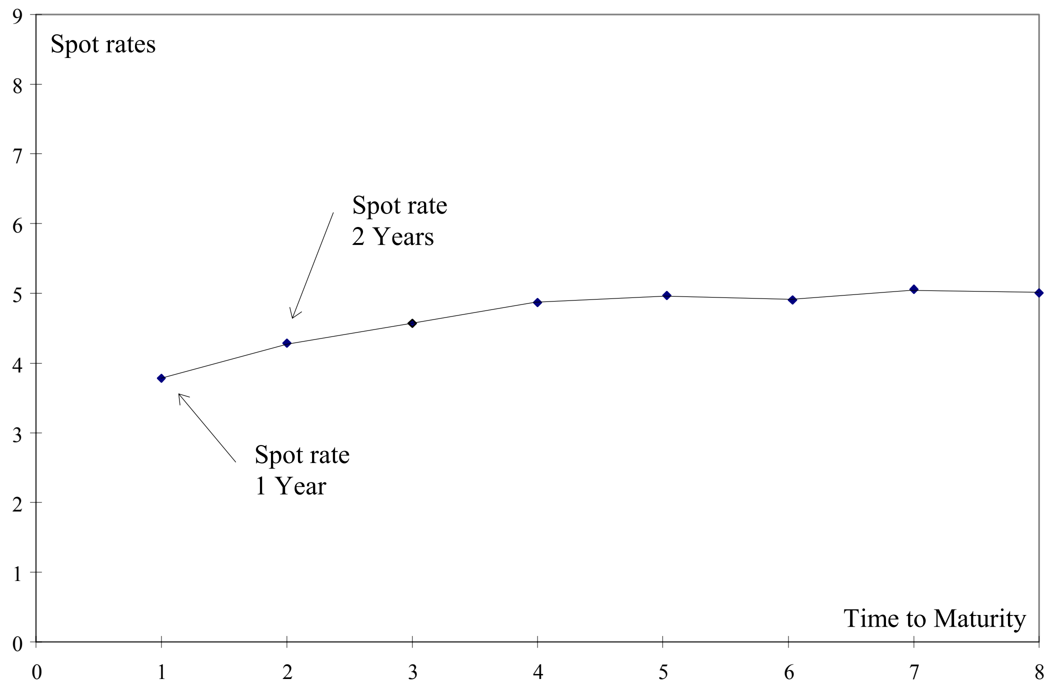


Figure 2-3: The term structure of interest rates

The problems in building the term structure of interest rates are that

- to avoid coupon effects and reinvestment risk, the term structure of interest rates should be built using only zero-coupon bonds.
- some rates are not available: one usually knows the 1, 2, 3, 5, and 10 year rates, but how about a 7.5 year rate?
- there are no spot rates published for non-government bonds, as there are very few corporate zero-coupon bonds.

Thus, most people will instead use the **yield curve**, which plots the yield to maturity of various bonds against their respective maturity, holding all other factors equal.

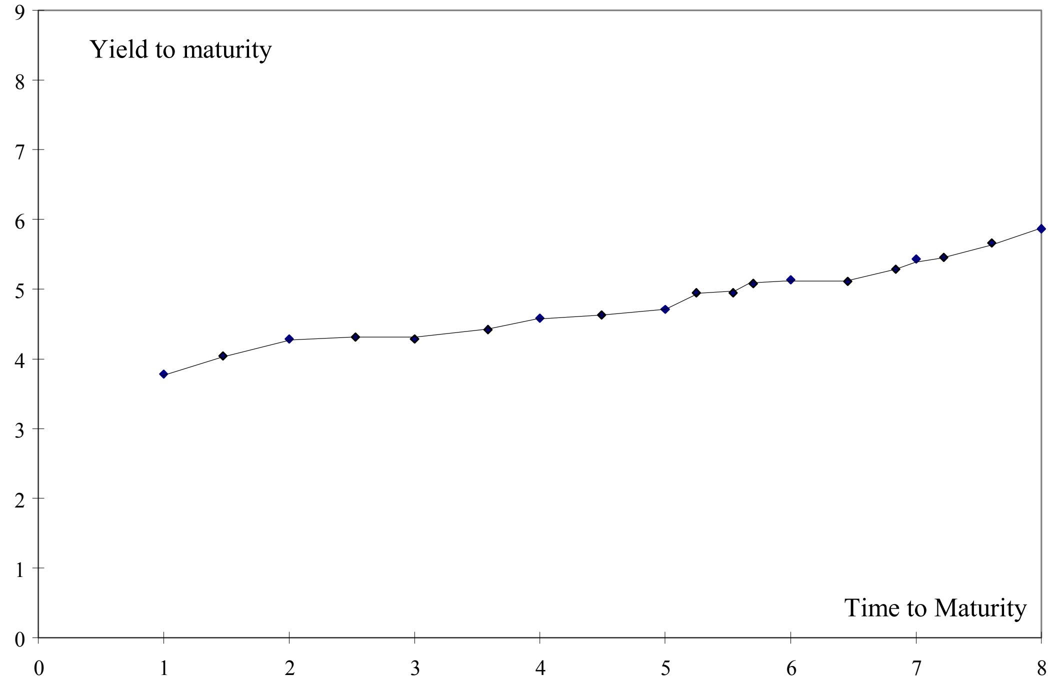


Figure 2-4: The yield curve

Formally, the term structure deals with the relationship between spot rates and time to maturity, whereas the yield curve deals with yield to maturity and time to maturity. Generally, both are very similar. But in the analysis of maturity-return relationship, it is better to work with spot rates rather than yields to maturity, as they are (among other things) not contaminated by the coupon effect.

A nominal interest rate can be decomposed into three basic components:

nominal rate = real interest rate + inflation premium + risk premium

The **real interest rate** is the compensation for the investor for deferring consumption to a future period (time value of money).

The **inflation premium** is intended to preserve the investor's purchasing power over time, and reflects the expected future inflation level over the life span of the investment.

The **risk premium** protects the investor against all other potential negatives, including default risk, redemption risk, market risk, etc.[5]

[5] Note that some of these risks can be diversified.

The yields level of all bonds will reflect these three components. Consequently, different issuer sectors will be plotted on different yield curves. Low quality sectors (lower ratings) will be traded at higher yields.

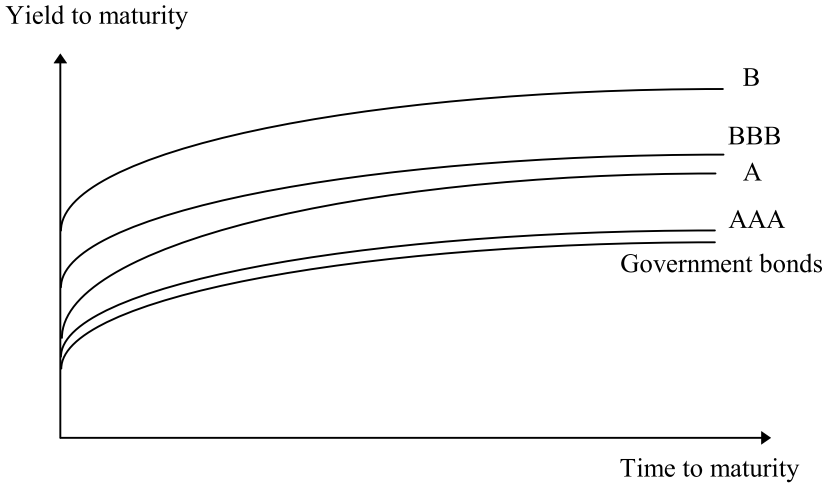


Figure 2-5: The yield/time to maturity relationship of various ratings

For the same reasons, callable bonds will be traded at higher yields than similar quality non-callable issues.

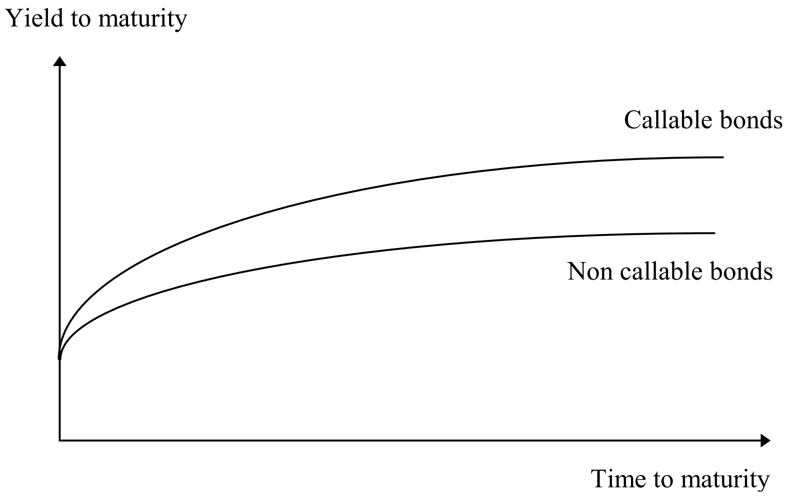


Figure 2-6: The yield/time to maturity relationship of callable/non callable bonds

Because of this, the liquidity risk, credit risk, call risk, coupon rate, and degree of premium/discount as well as any other risk should be sufficiently similar between the issues in order to build a useful yield curve.

The term structure of interest rates can exhibit four basic shapes: positively sloped, negatively sloped, flat, and humped. The following figures shows these four configurations for illustrative purposes only.

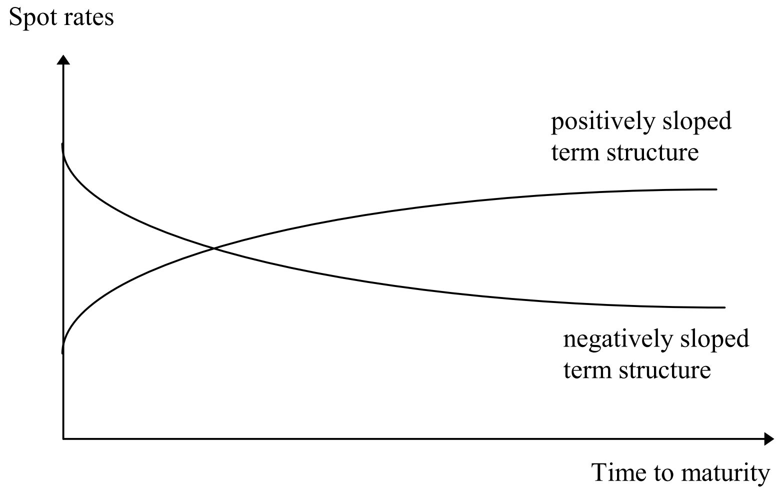


Figure 2-7: Basic shapes of the term structure: positively and negatively sloped

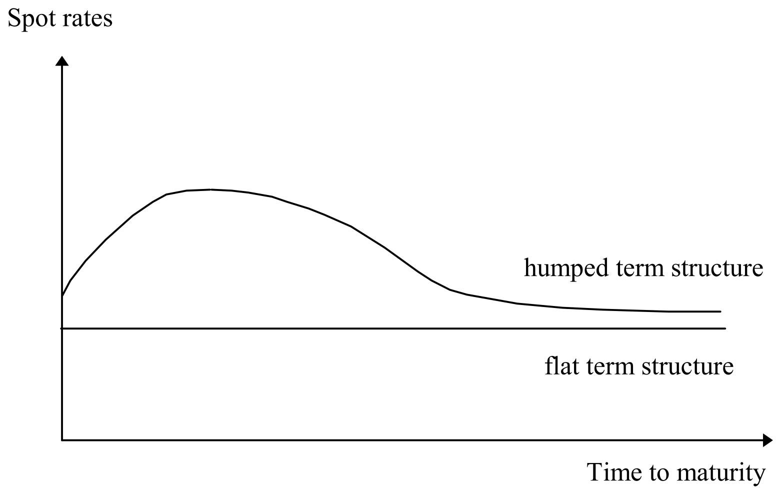


Figure 2-8: Basic shapes of the term structure: flat and humped

The short maturity section is mainly influenced by monetary policy, while the long maturity segment is more sensitive to inflationary expectations.

2.3.2 Theories of term structures

There are three primary theories that try to explain the shape of the term structure of interest rates: the expectations hypothesis, the liquidity preference, and the market segmentation theory.

2.3.2.1 Expectations hypothesis

The expectations theory contends that the shape of the term structure reflects the market consensus forecast on future interest rates levels.

Advocates of this theory believe **that the implicit forward rate is an unbiased estimate of the future spot rate**.

$$F_{t,h} = E\left(\tilde{R}_{t,h}\right)$$

A good way to understand this theory is to assume that investors are risk neutral, and that they will select the securities that give them the highest expected return (whatever their time horizon is). Consider the following example:

Example:
The yield to maturity on a one-year pure discount bond is 10%, and 12% for a two-year pure discount bond. Investors expect the one-year spot rate to be 16% in one year. What should investors do if they consider one or two years investment horizon?

The two years investor can invest 1 CHF in a two-year bond, with a final value of

$$1 \cdot (1.12) \cdot (1.12) = 1.254 \text{ CHF}$$

or hold two one-year bonds, with an expected final value of

$$1 \cdot (1.10) \cdot (1.16) = 1.276 \text{ CHF}.$$

All two-years investors will want to hold two one-year bonds. The one year investor can invest 1 CHF in a one-year bond, with a final value of

$$1 \cdot (1.10) = 1.10 \text{ CHF}$$

or hold a two-year bond, that they will sell in one year, with expected final value of

$$1 \cdot (1.12) \cdot \frac{(1.12)}{(1.16)} = 1.081 \text{ CHF}$$

So, all one year investors will also want to hold one-year bonds.

Given this universal preference, all investors will choose a **rollover strategy** and hold only the one-year bonds; thus, prices (i.e. interest rates) should adjust until the expected return from holding a two-years bond is exactly the same as the expected return from holding two one-year bonds.

Example:
The yield to maturity on a one-year discount bond is 10%, and 12.96% for a two-year discount bond. Investors expect the one year spot rate to be 16% in one year. What should investors do if they consider one or two years investment horizon?

The two years investor can invest 1 CHF in a two-year bond, with a final value of

$$1 \cdot (1.1296) \cdot (1.1296) = 1.276 \text{ CHF}$$

or hold two one-year bonds, with an expected final value of

$$1 \cdot (1.10) \cdot (1.16) = 1.276 \text{ CHF}$$

Two-year investors will be indifferent between both bonds. The one-year investor can invest one franc in a one-year bond, with a final value (after one year) of

$$1 \cdot (1.10) = 1.10 \text{ CHF}$$

or hold a two-year bond, that they will sell in one year, with expected final value (after one year) of

$$1 \cdot (1.1296) \cdot \frac{(1.1296)}{(1.16)} = 1.10 \text{ CHF}$$

So, all one year investors will also be indifferent between both bonds.

If we accept the fact that the implicit forward rate is an unbiased estimate of the future spot rate, future spot rates can be derived from spot rates, as they are also the marginal yield between two spot rates; and it implies that, if there are no transaction costs, **each bond is a perfect substitute for any other bond, whatever its maturity**. The expected return will be the same, whatever the bond combination selected by the investor. Indeed, an investor who has a three year investment horizon could do any of the following:

- buy a pure zero-coupon bond that matures at the end of the investment period, and hold it to maturity ("**buy and hold strategy**").
- buy a short-term maturity bond, and reinvest regularly the proceeds ("**rollover strategy**").
- buy a long-term bond, and sell it with a loss or a gain prior to maturity. The loss or the gain is predictable, using the forward rates.

The investor would expect the same return as long as one accepts the idea that

$$F_{t1,t2} = E(\widetilde{R}_{t1,t2})$$

The expectations theory can explain the four different shapes of the term structure of interest rates: a positively sloped (respectively negatively sloped) term structure implies that interest rates are expected to rise (respectively to decrease) in the future, while a flat term structure represents a market consensus for stable yields. Finally, a humped term structure shows that market participants expect a rising rate environment over the intermediate times to maturity, followed by a long-term decline in yield levels.

The statement that implicit forward rates are unbiased estimates of future spot rates is based on the following assumptions:

- investors have homogenous expectations.
- investors choose between short or long-term bonds in order to maximise their final expected wealth for a given investment period.
- there are no transaction costs.
- bond markets are efficient, and new information is instantaneously reflected in bond prices.

In reality, all of these assumptions are subject to criticism.

Expectations theory or Expectations theories?

One should note that in fact, there are several different versions of the expectations theory. We will briefly present them:

- The **naive expectations hypothesis** (or **globally equal expected holding period return**) version states that expected returns from any strategy for any holding period are equal. We will show later that this hypothesis is internally inconsistent.

- The **local expectations** version states that the expected total returns from a long-term bond over a short-term investment horizon is the same as today's interest rate over this horizon; more generally if we denote by P(t,T) the price at time t of a zero-coupon bond paying 1 CHF at time T, independently of the bond's maturity date, we have

$$E\left[\frac{dP}{P(t,T)}\right] = r(t)\cdot dt \rightarrow P(t,T) = E\left[\exp\left(-\int_t^T r(t)\cdot ds\right)\middle| r(t)\right]$$

  where r is the instantaneous rate of return at time t. The above equation states the common sense result that if there is no term premium, then, the discount rate to be used at each instant is the prevailing spot rate. Thus, the local expectations version is less comprehensive than the naive expectations hypothesis, as **it refers only to total returns over a (short) period beginning at the present**.

- The **unbiased expectations** (or **Malkiel's hypothesis**) version states that the forward rates are equal to the future expected spot rates, that is,

$$F_{t1,t2} = E(\tilde{R}_{t1,t2})$$

  In that case, we have:

$$\frac{1}{P(t,T)} = \left(1 + R_{t,t+1}\right)\cdot E\left(1+\tilde{R}_{t+1,t+2}\right)\cdot E\left(1+\tilde{R}_{t+2,t+3}\right)\cdot \ldots \cdot E\left(1+\tilde{R}_{T-1,T}\right)$$

- The **return to maturity expectations** (or **Lutz's hypothesis**) version states that the expected return of holding any discount bond up to maturity has to be equal to the expected return we would obtain by rolling over a sequence of single-period bonds over the same horizon, i.e.:

$$\frac{1}{P(t,T)} = E\left[\left(1 + R_{t,t+1}\right)\cdot\left(1+\tilde{R}_{t+1,t+2}\right)\cdot\left(1+\tilde{R}_{t+2,t+3}\right)\cdot\ldots\cdot\left(1+\tilde{R}_{T-1,T}\right)\right]$$

  This is incompatible with the unbiased expectations, unless the level of future interest rates are mutually uncorrelated. When interest rates are positively correlated over time, because of uncertain cross products terms, bonds with maturities in excess of two periods will have higher prices under the unbiased expectations hypothesis than under the return to maturity expectations hypothesis.

- The **yield to maturity version** states that the periodic rate of return (or holding period yield, such as an annual return) from holding a bond to maturity is equal to the expected holding period yield from rolling over a sequence of short term bonds over the same horizon. This version deals with periodic returns, while the return to maturity version is concerned with total returns over the investment horizon.

- In a remarkable paper[6], Cox, Ingersoll, and Ross have shown that

  ◊ The naive expectations hypothesis cannot be literally valid if there is uncertainty about future interest rates.
  ◊ The remaining four versions are not equivalent or even consistent with each other with uncertain interest rates.
  ◊ **Only the local expectations hypothesis is consistent with an equilibrium**. All other versions would imply that some strategies can earn excessive returns, and create some arbitrage profits.

Inconsistency of the naive expectations hypothesis

Why is the naive expectations hypothesis inconsistent? Let's imagine we have a two-period economy, and two pure discount bonds maturing at the end of period t_1 and at the end of period t_2 respectively. We will denote by $P(t_0,t_1)$ the price at time t_0 of a zero-coupon bond paying 1 CHF at time t_1.

- For the **one year holding-period**, we can

  → buy the one year zero coupon for $P(t_0,t_1)$, and receive 1 CHF at t_1. Our return is certain:

$$R = \frac{1}{P(t_{0,}t_1)}$$

  → buy the two year zero-coupon, and sell it at t_1. Our expected return is

$$E(\tilde{R}) = E\left[\frac{\tilde{P}(t_1,t_2)}{P(t_0,t_2)}\right]$$

According to the naive expectations hypothesis, we must have

$$\frac{1}{P(t_0,t_1)} = E\left[\frac{\tilde{P}(t_1,t_2)}{P(t_0,t_2)}\right]$$

that we can rewrite as

$$\frac{P(t_0,t_1)}{P(t_0,t_2)} = \frac{1}{E\left[\tilde{P}(t_1,t_2)\right]} \qquad \text{(relation A)}$$

[6] COX John C., INGERSOLL Jonathan and ROSS Stephen A., 1981, "A Re-Examination of Traditional Hypothesis about the Term Structure of Interest Rates", Journal of Finance, pp. 769-99.

- For the **two years holding period**, we can

  → buy the two years zero-coupon bond for $P(t_0,t_2)$, and receive 1 CHF at t_2. Our return is certain:

$$R = \frac{1}{P(t_{0,}t_2)}$$

  → buy the one year zero-coupon bond for $P(t_0,t_1)$, and receive 1 CHF at t_1; then, reinvest in buying bond which matures at t_2 (rollover strategy). Our expected return is equal to the expected payoff divided by the original price:

$$E(\tilde{R}) = \frac{1}{P(t_0,t_1)} \cdot E\left[\frac{1}{\tilde{P}(t_1,t_2)}\right]$$

According to the naive expectations hypothesis, we must have

$$\frac{1}{P(t_0,t_2)} = \frac{1}{P(t_0,t_1)} \cdot E\left[\frac{1}{\tilde{P}(t_1,t_2)}\right]$$

that we can rewrite as

$$\frac{P(t_0,t_1)}{P(t_0,t_2)} = E\left[\frac{1}{\tilde{P}(t_1,t_2)}\right] \qquad \text{(relation B)}$$

But in fact, relation A and B are contradictory.

$$\frac{P(t_0,t_1)}{P(t_0,t_2)} = \frac{1}{E\left[\tilde{P}(t_1,t_2)\right]}$$

is inconsistent with

$$\frac{P(t_0,t_1)}{P(t_0,t_2)} = E\left[\frac{1}{\tilde{P}(t_1,t_2)}\right]$$

since, by Jensen's inequality[7], we know that

$$\frac{1}{E\left[\tilde{P}(t_1,t_2)\right]} \neq E\left[\frac{1}{\tilde{P}(t_1,t_2)}\right]$$

unless $\tilde{P}(t_1,t_2)$ is deterministic.

Thus, the naive expectations hypothesis is internally inconsistent.

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7 Jensen's inequality states that E(f(x)) is not equal to f(E(x)).

2.3.2.2 *Liquidity preferences*

In the expectations theory, we assumed that investors do not have any maturity preference; in fact, there is no obvious reason to have a maturity preference in a world without uncertainty, since any combination of bonds would provide the same return for the same investment period. Once the certainty assumption is relaxed, we have just seen that only the local expectations hypothesis is sustainable.

The **liquidity preference theory** asserts that investors prefer to hold liquid securities, liquidity being defined as the ability to convert rapidly a bond into cash, while minimising the principal loss. As the fluctuation risk in long-term securities is higher than for short-term ones, investors will prefer short-term securities[8]. Thus, there is a shortage of longer term investors.

> **Example**:
> Bond A is a one year bond with a 6% coupon. Bond B is a two-year bond with a 6% coupon. The one year and two-year spot rates are equal to 6%. What happens if there is a sudden unexpected rise of all spot rates to 7%?
>
> Both bonds were priced at par (= 100.00). After the rise, the two-year security will drop to 98.19 in price, while the one year bond will drop to 99.07. So, for one percentage point increase in yield, the two-years security has a more important price decrease than the one year security. Hence, for a risk-averse investor, the two-year security seems riskier.

On the other hand, the borrowers (governments, firms, ...) prefer to issue long-term securities, to avoid interest rates fluctuations consequences on their expenses. In order to induce investors to invest in long-term securities, they will offer them an **extra risk premium** (a **liquidity premium** or **term premium**).

So, there are in fact two factors in the observed term structure of interest rates:

- future expected short-term spot rates (as predicted in the expectations hypothesis theory)
- a positive liquidity premium

8 Another simple reason to justify this is the unexpected inflation risk: if there is an unexpected rise in the inflation rate, the nominal interest rates should also rise; investors holding short term bonds will be able to reinvest their money at a higher rate, while investors holding long term bonds will have to wait for the final reimbursement before to take profit of the higher interest rates.

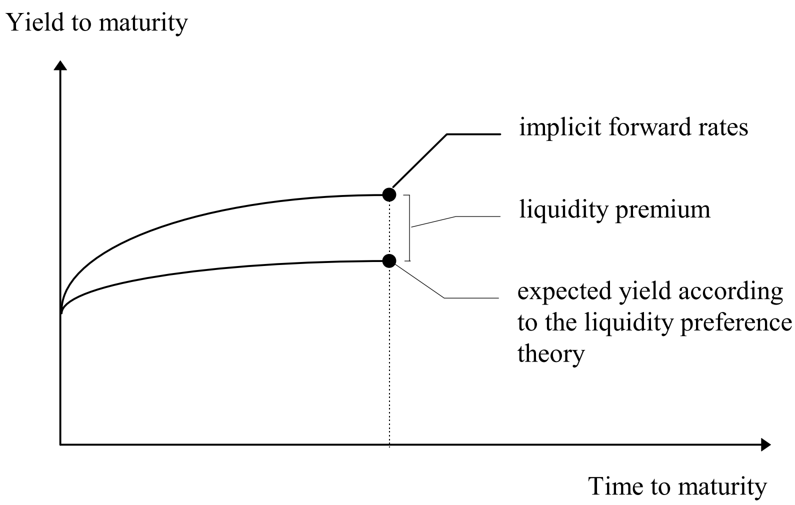

Figure 2-9: The liquidity premium concept

We should have a positive difference between implicit forward rates and expected spot rates:

$$F_{t,t+1} = E\left(\tilde{R}_{t,t+1}\right) + L_{t,t+1} \quad (\text{for } t > 0)$$

and

$$L_{t,t+1} > 0$$

As this theory suggests a higher yield for longer maturity issues caused by their lower degree of "liquidity", the expected return on a buy and hold strategy has to be higher than the expected return on a rollover strategy.

$$\left(1+R_{0,t}\right)^t = \left(1+R_{0,1}\right)\cdot\left(1+R_{1,2}+L_{1,2}\right)\cdot\left(1+R_{2,3}+L_{2,3}\right)\cdot\ldots\cdot\left(1+R_{t-1,t}+L_{t-1,t}\right)$$

Furthermore, as risk increases with time, we should observe:

$$L_{1,2} < L_{2,3} < L_{3,4} < \ldots < L_{n-1,n}$$

Hence, the term structure of interest rates should be (mainly) upward sloping because of the preference of investors for liquidity.

2.3.2.3 Market segmentation and preferred habitat theories

The **market segmentation theory** (or **preferred habitat theory**) views the bond market as a series of distinct markets that differ by their maturity. Each issuer or investor will have a preferred maturity, and he will be sufficiently risk-averse to operate only in his desired maturity spectrum. Within a given maturity range, the relative supply and demand for funds determines the appropriate clearing price (i.e. the appropriate interest rate).

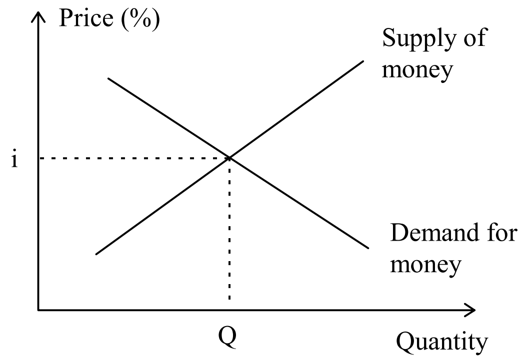

Figure 2-10: Supply and demand of money for a given maturity

Thus, contrary to the liquidity preference theory, the market segmentation theory leads to the result that the risk premium attached to bonds can be either positive, negative, or zero.

$$F_{t,t+1} = E\left(\tilde{R}_{t,t+1}\right) + \Pi_{t,t+1}$$

and nothing can be said ex ante about the sign of $\Pi_{t,t+1}$.

In the market segmentation theory, the money is considered as a commodity, its market clearing price is the interest rate, and the supply and demand of each individual segment connect to create the overall composite term structure of interest rates. By examining flows of funds into the market segments, one could - in theory - predict changes in the term structure of interest rates.

The market segmentation theory explains all four basic term structure of interest rates shapes:

- In the case of a positively sloped term structure of interest rates, the investors (buyers) have a preference for the short-term segments of the market, thus prices of bonds with small maturities are high and their yields are low; the reverse is true for the long-term segment.
- In the case of a negatively sloped term structure of interest rates, we have the reverse case of the above scenario.
- If the term structure of interest rates is flat, investors have similar preferences for all segments of the market.
- finally, the humped term structure of interest rates is due to different preferences, which depend on the maturity segments.

The major criticism of this theory is that even if investors have a strong maturity preference, the effect of segmentation on interest rates should be offset as soon as some investors start considering relative yields and allocate their funds to another segment which offers a (sufficiently) higher yield. Any investor will try to reduce risks by staying in its preferred habitat; but he will leave it as soon as he is given a risk premium high enough to cover the assumed risks and the cost of leaving its preferred habitat.

2.3.2.4 Other theories

The expectation hypothesis, liquidity preference and market segmentation theory are three non exclusive ways of thinking about interest rates.

But as far as bond pricing is concerned, the most promising theories are the **stochastic process no-arbitrage approaches**. They rely on the following assumptions:

- the term structure and the bond prices are related to some stochastic factors
- these factors evolve over time according to a particular hypothesised stochastic process (i.e. a process with some uncertainty)
- there should be no arbitrage opportunity

Various models have been developed, using single or multiple factors. For example, the Ogden model (1987) assumes that the term structure of interest rates is driven only by the short term interest rate fluctuations, and uses the following process to describe the short term interest rate variations:

$$dr = \underbrace{\beta \cdot (u - r) \cdot dt}_{\text{predictable component}} + \underbrace{\sigma \cdot r \cdot dZ(t)}_{\text{unpredictable component}}$$

where dr is the instantaneous change in the rate, β is a speed-of-adjustment component, u is the average level of the rate, dt is the passage of time, dZ(t) is a stochastic process, and $\sigma \cdot r$ is the standard deviation of the process.

In words, such an equation says that the change in the short term rate has two components: one is predictable (the extent to which the current rate differs from its long-term value, multiplied by a coefficient that measures its rate of adjustment to its long-term value), and one unpredictable (the product of the standard deviation of the rate, of the initial level, and of some stochastic process, that acts as a random generator).

Using this specification of the short-term rate, and by solving a partial differential equation, it is possible to find an analytical solution (or a numerical solution) for the bond prices, and therefore for the term structure of interest rates.

Of course, other specifications of the process followed by the short-term rates would lead to another term structure of interest rates. It is also possible to use other factors (such as the long-term rate, the spread between short-term and long-term interest rates, ...), or more than one factors. But each factor stochastic process has to be carefully specified, and the addition of factors complicates the solving of the partial differential equation[9].

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9 For some interesting specifications, see: BRENNAN Michael and SCHWARTZ Eduardo S., 1982, "Bond Pricing and Market Efficiency", Financial Analysts Journal, pp. 49-56.

2.4 Bond price analysis

2.4.1 Yield spread analysis

Fixed income instruments differ in a variety of ways (marketability, tax status, credit risk..). It is thus possible to examine the impact of the various differences in the characteristics of bonds on their yield to maturity. It is often helpful to consider one difference at a time: for example, when the only difference is given by the maturity, attention is paid to the term structure of yields; when the bonds are equal but their credit risk, the attention is focused on the risk structure of yields and so on. In any case, the differential in the yields of two or more bonds is called **yield spread** and the analysis of the causes and consequences of those spreads is called **yield spread analysis**. The yield spread of a given instrument is usually measured against the yield of a Treasury security having comparable maturity; as a matter of fact, Treasury securities are the instruments of highest quality in the fixed income market in terms of marketability, credit risk and often of tax status. The yield spread between a given security and the corresponding Treasury security has the nature of risk premium because it reflects the risks (lower liquidity, higher credit risk ...) that an investor has to face when he invests in non-Treasury securities.

Normally yield spreads are measured in terms of **basis points**, where a basis point, as previously mentioned, is equal to 0.01%: so, for example, a given bond A has a yield to maturity equal to 7% and another to 7.50%, the yield spread is 0.50%, that is 50 basis points. Moreover, one can also measure yield spread in terms of the yield level and calculate the **relative yield spread** defined as

$$\text{relative yield spread} = \frac{\text{yield bond B - yield bond A}}{\text{yield bond A}}$$

or the **yield ratio**

$$\text{yield ratio} = \frac{\text{yield bond B}}{\text{yield bond A}}$$

In our example, the relative yield spread is 0.0714 (0.50/7) and the yield ratio 1.0714.

2.4.1.1 Types of spreads

The bond market can be subdivided into sectors by type of issuer (Treasury, corporate, financial institutions), credit quality (summarized by the rating grade), maturity (short, medium and long term) and level of coupon. Each sector of the market is characterized by a yield spread versus other sectors of the bond market.

2.4.1.2 Determinants of yield spreads

Many factors may affect the yield spread; in principle, any difference in any characteristics between two bonds should be reflected in a yield differential. For simplicity, the determinants of the yield spread are usually classified in:

- maturity of the instrument;
- creditworthiness of the issuer;
- embedded options;
- tax status of the instrument;
- liquidity of the security.

As far as the first item is concerned the reader is referred to section 2.3.

The creditworthiness (probability of default) of the issuer clearly affects the yield: as long as there is a possibility of default, the **expected yield** (which takes into account the likelihood of default) is lower than the **promised yield** (which is calculated on the promised cash flows taken at their face value). The greater the likelihood of default (default probability) and the greater the amount lost in case of default (the loss given the event of default), the wider the spread between promised and expected yield (yield spread).

An example of the relationship between credit risk and expected yield is depicted in Figure Figure 2-11 (**risk structure of interest rates**): higher risk bonds command higher yields to maturity.

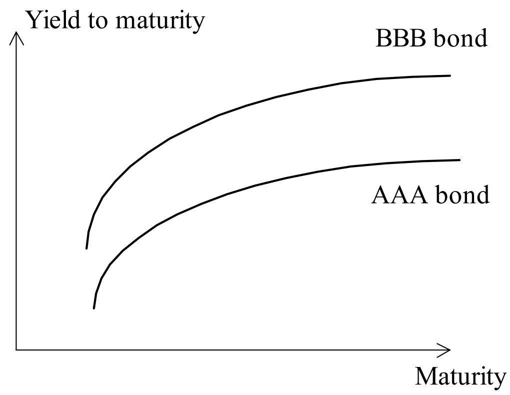


Figure 2-11: Term structure of different rating grade bonds

Yield spreads due to creditworthiness tend to widen when the economy is likely to face a recession and narrow in boom phases; investors in fact see a higher probability of default and a greater severity of losses when the economy is not in good health and consequently they assign an increasing probability of default to lower quality borrowers and require a higher risk premium.

When the credit risk is taken into account, the yield earned by the investor must be high enough to account for default probability and for the losses incurred in case of default. The following relationship holds, where r_f is the risk free rate, r_p is the risky rate and (1-p) is the default probability (so that p is the probability of exact fulfillment of the payment obligations). Let's assume at first that, in case of default, the bondholders do not get any payment. In that case:

$$p \cdot (1 + r_p) + (1 - p) \cdot 0 = 1 + r_f$$

hence

$$r_p = \frac{1+r_f}{p} - 1$$

and the premium for the credit risk is

$$\text{premium} = r_p - r_f = \frac{(1+r_f)}{p} - (1+r_f)$$

If the bondholder, as it is generally true, can get a fraction γ of the amount owed by the issuer, then

$$p \cdot (1+r_p) + \gamma \cdot (1-p) \cdot (1+r_p) = 1 + r_f$$

and the credit risk premium is

$$r_p - r_f = \frac{\frac{(1+r_f)}{(\gamma + p - \gamma \cdot p)}}{(1+r_f)}$$

As far as **embedded options** are concerned, many bond issues include provisions granting either the bondholder or the issuer, or both, some options to take some actions in his self interest (like repaying the bond before the maturity). Clearly, the option benefits the party who can choose to exercise it (the holder of the option) it and "damages" the party who has written it. The advantage of holding an option has to be paid for by a **yield differential**. For example, the call provision allows the issuer to shorten at his exclusive choice the maturity of the bond; that option will be exercised only when is advantageous for the issuer himself who is able to replace a higher yield instrument with a lower yield one. Since each option has to be paid for, a bond with an embedded call provision should command a higher yield (the issuer has to pay the option) than a similar bond without the call provision itself. Conversely, a putable bond (a bond that can be resold to the issuer at a given price by the investor at his exclusive choice) should carry a lower yield (the bondholder has to pay for the option he holds) than a bond similar but for the put provision.

The effect of the **tax status** is quite clear: anyone looks at the net income (from labour, from investments and so on). Given that, a taxable bond (like a corporate bond, for example) has to pay a higher gross yield in order to compete with an exempt bond (like a Government or municipal bond, for example).

The after tax (net) yield on a taxable fixed income security is defined as:

$$\text{after-tax (net) yield} = \text{pretax (gross) yield} \cdot (1\text{- tax rate})$$

Example:
If the yield on a given taxable bond is 10% and the relevant tax rate is 30%, the after tax yield is

$$10\% \cdot (1 - 30\%) = 7\%$$

Likewise, it is possible to calculate the equivalent taxable yield of a tax exempt security with the following formula:

$$\text{equivalent taxable yield} = \frac{\text{tax exempt yield}}{1\text{ - tax rate}}$$

As far as the **liquidity** is concerned, the greater the expected liquidity, the lower the yield required by the investors. The reason is straightforward: when an issue is illiquid, the investor might experience some problems should he decide to sell the bond before the maturity. The liquidity of bonds has three dimensions:

- marketability, that is the existence of a broad and deep market on a given instrument;
- time to maturity, because at maturity (unlike stocks for example) the bond will be paid back and so "transformed" in cash;
- financiability, to the extent that a given issue can be utilized as a collateral in order to borrow funds.

2.4.2 Bond valuation

Bond valuation is based on the discounted cash flow method and on the equilibrium concept.

2.4.2.1 Valuation of a zero-coupon bond

The simplest bond to consider is a zero-coupon bond which pays a single cash flow CF_t at the end of period t. The price of such a bond, denoted by $B_{0,t}$, is equal to the present value of its final (and only) cash flow:

$$B_{0,t} = \frac{CF_t}{(1+k)^t}$$

where CF_t is the cash flow received at the end of period t, and k is the appropriate discount rate.

Example:
What is the price today of a zero-coupon bond that will pay 1'000 CHF in exactly 5 years, assuming a discount rate of 7%? How about a 7-year bond, still assuming a discount rate of 7%?

$$B_{0,5} = \frac{1'000}{(1.07)^5} = 712.99 \text{ CHF}$$

and

$$B_{0,7} = \frac{1'000}{(1.07)^7} = 622.75 \text{ CHF}$$

In the previous example, the discount rate was assumed to be the same regardless of the maturity of the bond. But generally the discount rate varies from maturity to maturity. If we denote the annual rate of return demanded by a lender to lend money from time 0 to time t (also called the **spot rate**) by $R_{0,t}$ and if only one final payment CF is made towards both interest and principal, the price of such a zero-coupon bond will be calculated as:

$$B_{0,t} = \frac{CF_t}{(1+R_{0,t})^t}$$

The above formula allows us to use different discount rates for different maturities.

Example:
What is the price today of a zero-coupon bond that will pay 1'000 CHF in exactly 5 years, assuming a 5-year spot rate $R_{0,5}$ = 5%? How about a 7-year bond, assuming a 7-year spot rate $R_{0,7}$ = 6%?

$$B_{0,5} = \frac{1'000}{(1.05)^5} = 783.53 \text{ CHF}$$

and

$$B_{0,7} = \frac{1'000}{(1.06)^7} = 665.06 \text{ CHF}$$

Given the spot rates, we are now able to price zero-coupon bonds of different maturities. This principle can be extended to calculate the value of coupon-bearing bonds, as follows.

2.4.2.2 Static arbitrage and valuation of coupon bonds

A coupon-bearing bond can be visualised as a stream of future cash flows. Such a stream can be replicated by **a portfolio of zero-coupon bonds**.

Example:
Investor A holds one straight bond with a 4-year maturity, 1'000 CHF face value, 6% coupon rate. Investor B wants to replicate the same cash flows as investor A's bond, but he can only use zero-coupon bonds. What should he buy?

Investor B should buy:

- a zero-coupon bond that matures in exactly one year and pays 60 CHF.
- a zero-coupon bond that matures in exactly two years and pays 60 CHF.
- a zero-coupon bond that matures in exactly three years and pays 60 CHF.
- a zero-coupon bond that matures in exactly four years and pays 1'060 CHF.

With such a portfolio, he will receive exactly the same cash flows as investor A.

What should be the price of such a portfolio? By definition, it should be exactly the same as the price of the replicated bond, as both yield same cash flows. Otherwise, there would be scope for arbitrage (buy the cheaper of the two bonds, sell short the other bond, match the cash flows, and keep the price difference as risk-free profit). So, we can calculate the price of bond using following:

Bond price = Price of replicating zero-coupon bond portfolio

And as the portfolio price is the sum of all the zero-coupon bond prices, the price of any coupon-bearing bond is the sum of the present value of all its individual cash payments:

$$P_0 = \sum_{t=1}^{T} \frac{CF_t}{\left(1+R_{0,t}\right)^t} = \frac{CF_1}{\left(1+R_{0,1}\right)^1} + \frac{CF_2}{\left(1+R_{0,2}\right)^2} + \ldots + \frac{CF_T}{\left(1+R_{0,T}\right)^T}$$

where CF_t is the cash flow received at the end of period t (coupons or repayment), and T is the number of years remaining until maturity (time to maturity).

Example:
What is the price of a straight bond with a 4-year maturity, 1'000 CHF face value, 6% coupon rate? The spot rates are $R_{0,1}$ = 7%, $R_{0,2}$ = 8% $R_{0,3}$ = 8.5% and $R_{0,4}$ = 9%.

The following table represents the stream of cash flows:

time	0	1	2	3	4	4
Cash Flows in CHF		60	60	60	60	1'000
Discount rate	0	0.07	0.08	0.085	0.09	0.09

Therefore the bond price is:

$$P = \frac{60}{(1+0.07)} + \frac{60}{(1+0.08)^2} + \frac{60}{(1.+0.085)^3} + \frac{(1000+60)}{(1+0.09)^4} = 905.42 \text{ CHF}$$

2.4.2.2.1 Special cases

Note that even if the **final repayment is not made at par**, but with a premium or a discount, then methodology remains the same, only the final cash flow is modified.

Example:
What is the price of a bond with a maturity in 4 years, 1'000 CHF face value, 6% coupon rate? The spot rates are $R_{0,1}$ = 7%, $R_{0,2}$ = 8%, $R_{0,3}$ = 8.5% and $R_{0,4}$ = 9%. The bond will be repaid with a 2% premium (total 102%).

The following table represents the stream of cash flows:

time	0	1	2	3	4	4
Cash Flows in CHF		60	60	60	60	1'020
Discount rate	0	0.07	0.08	0.085	0.09	0.09

Therefore the bond price is:

$$P = \frac{60}{(1+0.07)} + \frac{60}{(1+0.08)^2} + \frac{60}{(1+0.085)^3} + \frac{(1020+60)}{(1+0.09)^4} = 919.59 \text{ CHF}$$

If the bond pays a **semi-annual coupon**, we can still use the same formula

$$P_0 = \sum_{t=1}^{T} \frac{CF_t}{\left(1+R_{0,t}\right)^t} = \frac{CF_1}{\left(1+R_{0,1}\right)^1} + \frac{CF_2}{\left(1+R_{0,2}\right)^2} + \ldots + \frac{CF_T}{\left(1+R_{0,T}\right)^T}$$

where CF_t is the cash flow received at the end of the semi-annual period t (coupons or repayment), T is the number of semi-annual periods remaining until final maturity, and $R_{0,t}$ the required rate of return to lend money from time 0 to the end of the **semi-annual** period t.

In the case of **floating rate bonds**, the quoted price should tend toward par as we get closer to the next coupon payment date. Generally, the coupon rate for the next period is set up at the previous-coupon payment date, and will be equal to the market spot rate for the next period. Thus, just after the coupon payment, the bond should be quoted at par.

Between two coupon dates, the floating rate bond can be considered as a short term zero-coupon bond that will pay interest and principal (as it is possible to sell the bond at par at just after the coupon payment date) in at the most six months if the coupon is paid semi-annually, or in one year in the case of an annual coupon.

Thus, the price of a floating rate bond should theoretically be given by

$$P_{cum} = P_{ex} + f \cdot C_1 = \frac{C_1}{(1+R_{0,1})^{1-f}} + \frac{100}{(1+R_{0,1})^{1-f}}$$

where P_{cum} is the price of the bond with coupon, P_{ex} is the price of the bond without coupon, f is the fraction of that is time elapsed since the last coupon payment date, and C_1 is the next coupon.

Example:
Company XYZ has issued a CHF floating rate bond, 2002-2007. Coupons are paid semi-annually on the 31st of March and the 30th of September. The coupon rate for the following period is set at the same dates equal to the 6-month CHF LIBOR rate.

On the 6th of September 2002, the next coupon to be paid was set to 1.75% (annual rate). The six-month risk-free interest rate is 0.75% (annual rate). What is the price of such a bond?

Since the last coupon payment, the time elapsed is f = 156 / 180 = 0.866 semester. The accrued interest is equal to $0.866 \cdot 0.875 \cong 0.76$. The six-month rate is $(1.0075^{0.5}-1) = 0.374\%$.

Thus, the bond price is

$$P_{cum} = P_{ex} + f \cdot C = \frac{0.875 + 100.0}{1.00374^{1-0.866}} = 100.82$$

that is, a quoted price of:

$$P_{ex} = 100.82 - 0.76 = 100.06$$

But in practice, things are more complicated as the coupon rate is not always equal to the market spot rate for the next period, and default risk may exist. Also, in some cases, the announcement of the next coupon rate is not made exactly on the previous-coupon payment date (and thus, anticipation biases may occur)[10].

2.4.2.2.2 Influences on the bond price

The impact of the coupon rate

The bond price depends on the promised payments (or the expected cash flows), which appear in the numerators of the summation; hence, it is directly related to the bond's coupon rate.

$$P_0 = \sum_{t=1}^{T} \frac{CF_t}{\left(1+R_{0,t}\right)^t} = \frac{CF_1}{\left(1+R_{0,1}\right)^1} + \frac{CF_2}{\left(1+R_{0,2}\right)^2} + \ldots + \frac{CF_T}{\left(1+R_{0,T}\right)^T}$$

---

10 Note that another common practice to value floating rate notes is to consider that the current coupon rate will remain the same in the future, and to use the traditional cash flows discounting model. Using such a model, one should remember that it totally ignores the fact that the coupons **will not** remain constant.

A higher coupon bond will be worth more than a lower coupon issue with the same maturity, since the expected cash flows are higher.

Example:
Consider two bonds with 4-year maturities, 1'000 CHF face value and a 6% and 7% coupon rate. What are their prices, if the spot rates are $R_{0,1}$ = 7%, $R_{0,2}$ = 8%, $R_{0,3}$ = 8.5% and $R_{0,4}$ = 9%?

The following table represents the cash flows for both bonds:

Time	**0**	**1**	**2**	**3**	**4**	**4**
Cash flows for bond 1 in CHF		60	60	60	60	1'000
Cash flows for bond 2 in CHF		70	70	70	70	1'000
Discount rate	0	0.07	0.08	0.085	0.09	0.09

Hence, the prices of the bonds are:

$$P_{\text{bond 1}} = \frac{60}{(1+0.07)} + \frac{60}{(1+0.08)^2} + \frac{60}{(1+0.085)^3} + \frac{(1000+60)}{(1+0.09)^4} = 905.42 \text{ CHF}$$
$$P_{\text{bond 2}} = \frac{70}{(1+0.07)} + \frac{70}{(1+0.08)^2} + \frac{70}{(1+0.085)^3} + \frac{(1000+70)}{(1+0.09)^4} = 938.25 \text{ CHF}$$

As expected, the price of the second bond is higher.

The impact of the discount rate of interest

The bond price also depends on the discount rates, which appear in the denominators in the summation; hence, the bond price is inversely related to the discount rates of interest. (the bond's value rises if any of the discount rates is reduced, and falls if any of the discount rates is increased).

Example:
Consider a bond with a 4-year maturity, 1'000 CHF face value, 6% nominal interest rate. What is its price, if the spot rates are $R_{0,1}$ = 7%, $R_{0,2}$ = 8%, $R_{0,3}$ = 8.5% and $R_{0,4}$ = 9%? What happens if the 2-year spot rate increases by 0.5%?

The following table represents the stream of cash flows of the bond:

Time	**0**	**1**	**2**	**3**	**4**	**4**
Cash flows on CHF		60	60	60	60	1'000
Discount rate	0	0.07	0.08	0.085	0.09	0.09

Price of the bond is:

$$P = \frac{60}{(1+0.07)} + \frac{60}{(1+0.08)^2} + \frac{60}{(1+0.085)^3} + \frac{(1000+60)}{(1+0.09)^4} = 905.42 \text{ CHF}$$

If there is an increase in the 2-year spot rate, we have $R_{0,2}$ = 8.5%, and the new price for our bond is:

$$P = \frac{60}{(1+0.07)} + \frac{60}{(1+0.085)^2} + \frac{60}{(1+0.085)^3} + \frac{(1000+60)}{(1+0.09)^4} = 904.95 \text{ CHF}$$

As expected, the new price is lower.

2.4.2.3 Strips markets

The acronym STRIPS stands for Separately Traded Registered Interest and Principal Securities; they are zero coupon notes and bonds created by separating (stripping) bond coupon and principal payments from Treasury securities so that they can be separately traded.

When a Treasury bond is stripped, each interest payment (coupons) and the principal payment (due at maturity) become separate zero-coupon securities that can be held and traded separately. For example, a Treasury note with 10 years remaining to maturity consists of a single principal payment at maturity and 20 coupons (interest payments), one every six months for 10 years. If the note is converted into a STRIPS, each of the 20 coupon payments and the principal payment becomes a separate security (zero-coupon). For example, if the bond has a nominal value of USD 100 million and the coupon rate is 8%, the security can be stripped in 20 zero coupon bonds of USD 4 million (8% divided by two times USD 100 million) with maturities ranging form 6 months to 10 years and a zero coupon of USD 100 million 10 years from now.

The Treasury does not issue or sell STRIPS directly to investors. They are "created" by financial institutions. Primary intermediaries (like Merrill Lynch and Salomon Brothers) create the synthetic security by purchasing Treasury Bonds, depositing them in a bank custody account and then issuing receipts representing the separate right to receive each coupon payment and a security entitling the holder to the payment of the principal (also called *corpus*) at maturity.

The STRIPS market is attractive for investors who want to receive a prespecified sum of money at a prespecified date in the future and are not interested in periodic payments (as in the case of coupons carried by ordinary bonds and notes). The credit risk of STRIPS securities is the same as the original security (i.e. very low given that the issuer of the underlying security is the Treasury and the underlying bond is deposited in a bank custody account) while the interest rate risk may be higher for longer maturities given that they have only one single payment at maturity (and so their duration is by definition equal to the time to maturity). Of course, if the STRIPS security is held until maturity there is no interest rate risk, as with all the zero coupon bonds.

2.4.3 Price / yield relationship

The total return realized from holding a bond during a given time-period can be broken down into three components

Total return = Price return + Coupon return + Reinvestment return

The price component is the variation in the quoted price, the coupon component is the periodic payment from the bond, the reinvestment return is the income generated by reinvesting all previously received cash flows (“interest on interest”).

The price return itself can be decomposed in two components:

Price return = Price return due to yield change + Amortisation of premium/discount

The price return due to the yield change comes from the change of at least one of the spot rates (used in the discounting process). The amortisation of the premium or discount is the fact that the bond price will tend to converge toward the final payment value.

As we have seen already, the bond's current yield varies inversely with the bond price, and there is a one-to-one correspondence between the price and the current yield (except, of course, for zero-coupon bonds); but the current yield is not a good estimate of the total return, as it focuses solely on the coupon component of the return.

There is also a one-to-one correspondence between the price and the yield to maturity.

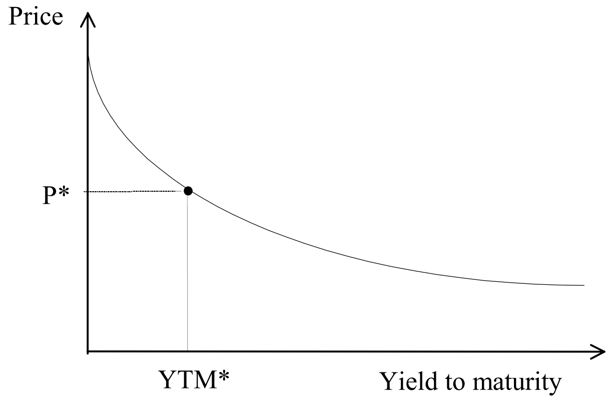


Figure 2-12: Yield to maturity and price relationship

We have seen that the yield to maturity is the internal rate of return of the bond. Despite this, **it can only serve as a proxy for the total return under a series of very restrictive assumptions**:

- **All coupon reinvestments are made at the yield to maturity rate**. This assumption is very open to criticism, as in reality, each coupon is reinvested at the market rate prevailing on the date of cash receipt. If this rate differs from the yield to maturity, the total return will differ from the yield to maturity. Higher (lower) reinvestment rate lead to higher (lower) total return compared to the yield to maturity.

- The **reinvestment rate risk** is particularly important with long term bonds, and high coupons rates, for which the realised return may differ significantly from the one inferred by the yield to maturity.

- If a bond has to be sold before maturity, **interest rates must be stable, and the bond has to be sold at a yield level exactly equal to the yield to maturity on the date of purchase.**

- **For bonds paying semi-annual coupon, an annual basis-point add-on is required to determine the equivalent annual return**. This conversion is necessary due to the reporting of total returns on an annualised basis.

In most cases, using the yield to maturity as an expected return proxy implies that whatever the maturity, there is only one prevailing rate to lend and borrow money, and that this rate will never change in the future.

The important thing to remember is that the equivalence of yield to maturity and total return, although naively accepted by many investors in practice, is not supported by theory.

Example:
Assume that a bond with a 4-year maturity, 6% annual coupon has been bought at par in year 0; hence with a yield to maturity of 6%. Assume further that in the following years interest rates follow a downtrend, so that the coupon received in year 1 is invested for three years at 4.5%; the coupon received in year 2 is reinvested for two years at 3%, and the coupon received in year 3 is reinvested for 1 year at 2%. What's the total realized return in year 4?

The final capital in year 4 (for a 100 initial investment) is: $6 \cdot 1.045^3 + 6 \cdot 1.03^2 + 6 \cdot 1.02 + 106 = 125.33$.

This means that the realized return is $\sqrt[4]{\frac{125.33}{100}} - 1 = 5.81\%$ p.a., lower than 6%!

2.5 Risk measurement

The return from holding a bond for a given period can be decomposed in two components: the change in the market value of the security (selling price minus purchase price), and the cash flows received from the security plus any additional income from reinvesting those cash flows. Several environmental factors impact one or both of these parts of the return.

Hence, we will define the risk of a bond as a **measure of the impact of the market factors on the return characteristics of the bond**.

Hereafter, we will examine the external factors that can affect bond prices.

We have seen that using the yield to maturity concept (denoted YTM), a bond price can be defined as:

$$P = \sum_{t=1}^{T} \frac{CF_t}{(1+YTM)^t} = \frac{CF_1}{(1+YTM)^1} + \frac{CF_2}{(1+YTM)^2} + \ldots + \frac{CF_T}{(1+YTM)^T}$$

where CF_t is the cash flow received at the end of period t (coupons or repayment), and T is the remaining life of the bond (time to maturity).

Hence, it should be clear that the price of a typical fixed income security moves in the opposite direction of the change in interest rates: as interest rates rise (fall), the price of a fixed income security will fall (rise)[11]. A bond's **systematic risk** is defined as the volatility in the total return from a bond due to any instantaneous interest rate fluctuation.

In the past, bonds were considered as safe investments. Interest rates were stable, and investing in bonds was a rather conservative strategy. But the increasing interest rates volatility has transformed bonds into an exciting as well as risky investment vehicle.

- Price risk:
  For an investor who plans to hold a bond to maturity, the change in the price prior to maturity is of no concern. But if the investor plans to sell the bond prior to the maturity date, an increase in the interest rate will result in a capital loss. This is referred to as the **price risk**, which is by far the major risk faced by an investor in the fixed income market.

- Reinvestment risk:
  The reinvestment risk is defined as the variability of the reinvestment income from a given strategy due to changes in interest rates. For example, if interest rates fall, interim cash flows will be reinvested at a lower rate.

It should be noted that price risk and reinvestment risk act in opposite directions. If interest rates rise, the market price of a bond decreases. But, at the same time, the income received by reinvesting the coupons increases. **A strategy based on equalising and thereby nullifying these two offsetting risks is called "immunization"** and will be examined later.

11 There are some exceptions, with price changes in the same direction as interest rates, like some putable bonds.

What happens if there is an instantaneous change in the bond's yield? Empirical investigations show that:

- **Long maturity bonds are more price sensitive than short maturity bonds:** thus, an investor holding a short-term bond can more readily profit from a market rate increase: he will wait until his bond matures (at par), and then reinvest all proceeds at the higher interest rate.

Example:
The following table lists various bonds differing only by maturity. All bonds have the same face value of 1'000 CHF. If the market yield changes from 5% to 5.5%, the bond prices adjust to reflect the new yield. Long-term bonds are clearly more volatile than short-term bonds, and have a larger depreciation.

	Bond 1	**Bond 2**	**Bond 3**	**Bond 4**	**Bond 5**
Maturity (in years)	20	10	5	3	1
Market yield	5.00%	5.00%	5.00%	5.00%	5.00%
Coupon	7.00%	7.00%	7.00%	7.00%	7.00%
Market price (CHF)	1'249.24	1'154.43	1'086.59	1'054.46	1'019.05
New market yield	5.50%	5.50%	5.50%	5.50%	5.50%
New market price (CHF)	1'179.26	1'113.06	1'064.05	1'040.47	1'014.22
ΔP (CHF)	–69.99	–41.37	–22.54	–14.00	–4.83
ΔP / P	**–5.60%**	**–3.58%**	**–2.07%**	**–1.33%**	**–0.47%**

We should note that the relationship is the same in the case of an interest rate decrease. In the case of a decrease in the interest rate, long-term bonds will have the largest price appreciation.

Example:
The following table lists various bonds differing only by maturity. All bonds have the same face value of 1'000 CHF. If the market yield changes from 5% to 4.5%, the bond prices adjust to reflect the new yield. Long-term bonds are clearly more volatile than short-term bonds, and have a larger appreciation.

	Bond 1	**Bond 2**	**Bond 3**	**Bond 4**	**Bond 5**
Maturity (in years)	20	10	5	3	1
Market yield	5.00%	5.00%	5.00%	5.00%	5.00%
Coupon	7.00%	7.00%	7.00%	7.00%	7.00%
Market price (CHF)	1'249.24	1'154.43	1'086.59	1'054.46	1'019.05
New market yield	4.50%	4.50%	4.50%	4.50%	4.50%
New market price (CHF)	1'325.20	1'197.82	1'109.75	1'068.72	1'023.92
ΔP (CHF)	+75.95	+43.38	+23.16	+14.26	+4.88
ΔP /P	**+6.08%**	**+3.76%**	**+2.13%**	**+1.35%**	**+0.48%**

Thus, the price/yield curve is steeper for longer maturity issues than for shorter maturity issues. Therefore, a small change in the yield will have a greater impact on the price of a long term bond compared to a short term bond.

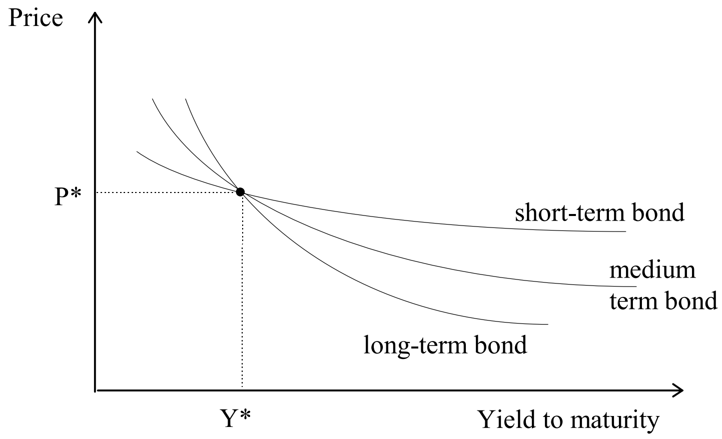


Figure 2-13: Price/yield relationship for various maturities

- For a given maturity**, low coupon bonds are more volatile than high coupon bonds**. Clearly, zero-coupon bonds have the greatest volatility.

Example:
The following table lists various bonds differing only by coupon. All bonds have the same face value of 1'000 CHF. If the market yield changes from 5% to 5.5%, the bond prices adjust to reflect the new yield. Low coupons bonds have a greater volatility.

	Bond 1	**Bond 2**	**Bond 3**	**Bond 4**	**Bond 5**
Maturity (in years)	10	10	10	10	10
Market yield	5.00%	5.00%	5.00%	5.00%	5.00%
Coupon	10.00%	7.00%	5.00%	3.00%	0.00%
Market price (CHF)	1'386.09	1'154.43	1'000.00	845.57	613.91
New market yield	5.50%	5.50%	5.50%	5.50%	5.50%
New market price (CHF)	1'339.19	1'113.06	962.31	811.56	585.43
ΔP (CHF)	–46.89	–41.37	–37.69	–34.01	–28.48
ΔP / P	**–3.38%**	**–3.58%**	**–3.77%**	**–4.02%**	**–4.64%**

- For a given maturity**, low yield bonds are more price volatile than high yield bonds**. Thus, the price volatility should be greater in a low interest rate environment.

Example:
The following table lists the effect of a market yield increase on a bond price, with different initial yield levels. All bonds have the same face value of 1'000 CHF. The higher the initial yield, the lower the price decrease. Hence, the bond price is much more volatile when the yield is low, for a same variation of the market yield.

	Bond 1	**Bond 2**	**Bond 3**	**Bond 4**	**Bond 5**
Maturity (in years)	10	10	10	10	10
Market yield	10.00%	7.00%	6.00%	5.00%	3.00%
Coupon	6.00%	6.00%	6.00%	6.00%	6.00%
Market price (CHF)	754.22	929.76	1'000.00	1'077.22	1'255.91
New market yield	10.50%	7.50%	6.50%	5.50%	3.50%
New market price (CHF)	729.34	897.04	964.06	1'037.69	1'207.92
ΔP (CHF)	–24.88	–32.73	–35.94	–39.53	–47.99
ΔP / P	**–3.30%**	**–3.52%**	**–3.59%**	**–3.67%**	**–3.82%**

- **A bond with a sinking fund provision is less volatile** than a similar maturity bullet bond. If there is an interest rate increase, a bondholder may be able to reinvest the proceeds of the sinking fund immediately at a higher rate, while the bondholder of a non-sinking counterpart will have to wait until final maturity. Alternatively, the existence of a sinking fund provision reduces the effective time to maturity of the bond issue and therefore such bonds are less volatile.

- **Callable bonds exhibit less price volatility** than their similar maturity non-callable counterparts. This can be deduced from the maturity effect: a callable bond has an effective time to maturity that is shorter (or at most equal) than its non-callable counterpart, as there is the call risk.

- **Price volatility is not a symmetric phenomenon**; at a given price, a decrease in the market yield does not have the same effect on the bond price as an identical increase in the market yield.

Example:
Bond A is a 6%, 10 years bond with a face value of 1'000 CHF. The current market yield is 6%. What happens if this market yield falls/increases by 50 basis points? By 100 basis points?

The following table lists the various results of the yield modification:

New market yield	**5.00%**	**5.50%**	**6.50%**	**7.00%**
New market price (CHF)	1'077.22	1'037.69	**964.06**	**929.76**
ΔP (CHF)	+77.22	+37.69	**–35.94**	**–70.24**
ΔP / P	**+7.72%**	**+3.77%**	**–3.59%**	**–7.02%**

The price of the bond has a lower variation (in absolute terms) if the yield increases than if the yield decreases.

But all these observations are not sufficient for us to derive an adequate bond risk measure. In particular, we cannot compare the riskiness of two bonds differing in coupon as well as maturity.

Example:
Bonds A and B are described in the following table.

	Bond A	**Bond B**
Coupon	10%	2%
Time to maturity	12 years	8 years
Market rate	8%	8%
Actual market price	115.07	65.52

What happens if the new market rate is 8.5%? Which bond will be more volatile?

Without any calculation, it is impossible to say anything: bond A has a longer maturity and should have a greater volatility. But bond B's coupon is lower, which should give her the larger volatility.

A quick calculation would lead to the following results:

	Bond A	**Bond B**
New market rate	8.5%	8.5%
New market price	111.02	63.34
ΔP	–4.06	–2.18

ΔP / P	**–3.52%**	**–3.32%**

Bond A is more volatile. But the results are very close.

Given the various factors affecting bond price volatility, one wonders whether it is possible to derive an adequate bond risk measure to capture the volatility characteristics of a particular issue.

2.5.1 Risk measurement tools

The most basic bond risk proxies are time to maturity, weighted average maturity, and weighted average cash flow.

The **time to maturity** is the number of years remaining until the bond's final maturity date. It assesses a bond's risk from a final maturity date perspective. Long maturity bonds are supposed to be riskier than short maturity issues, as the investors have to wait longer to recover the principal, and also because long term bonds are more sensitive to interest rate fluctuations.

But it is a weak proxy for a bond's inherent risk, because:

- it does not consider the cash flows received prior to final maturity, which leads to errors in the risk assessment process.

Example:
Consider a 10% bond and a zero-coupon bond maturing in 10 years. Both should have the same risk, because they have the same maturity.

But after 5 years, the owner of the coupon bond will have recover half of its initial investment (which can be reinvested at a higher rate in the case of an interest rate increase), while the owner of the zero-coupon bond will have cashed nothing (all of its proceeds are in terms of principal appreciation).

- it assumes implicitly that there is a linear relationship between time to maturity and price volatility. A 30-year bond would be three times as risky as a 10-year bond.

Example:
The following table lists the effect of a market yield increase on bonds with different maturities. All bonds have a face value of 1'000 CHF.

	Bond 1	**Bond 2**	**Bond 3**	**Bond 4**
Maturity (in years)	5	10	20	40
Market yield	6.00%	6.00%	6.00%	6.00%
Coupon	6.00%	6.00%	6.00%	6.00%
Market price (CHF)	1'000.00	1'000.00	1'000.00	1'000.00
New market yield	6.50%	6.50%	6.50%	6.50%
New market price (CHF)	979.22	964.06	944.91	929.27
ΔP (CHF)	–20.78	–35.94	–55.09	–70.73
ΔP / P	**–2.08%**	**–3.59%**	**–5.51%**	**–7.07%**

It is clear that doubling the time to maturity does not result in twice the initial volatility.

The **weighted average maturity**, or **average life**, is the weighted average maturity of the principal repayment (note that the coupon rate plays no role in the average life, as it only considers principal repayments):

$$\text{Average life} = \sum_{t=1}^{T} \frac{\text{Principal paid at time t}}{\text{Total principal to be repaid}} \cdot t$$

It is identical to the time to maturity for bullet bonds, but for sinking requirement and mortgage backed securities, it offers some improvements.

Example:
What is the weighted average maturity of a 6% coupon, 10 year sinking fund debenture priced at par (1'000 USD) to yield 6% (discounted cash flow)? The sinking fund retires 20% of the bonds annually, commencing at the end of the sixth year. The interest are paid six-monthly.

The principal cash flows are the following (we are not concerned with the interest payments):

Time	Principal Cash Flows
6	200 USD
7	200 USD
8	200 USD
9	200 USD
10	200 USD

Thus, the weighted average maturity of our bond is:

$$\text{WAM} = \frac{200 \cdot 6}{1000} + \frac{200 \cdot 7}{1000} + \frac{200 \cdot 8}{1000} + \frac{200 \cdot 9}{1000} + \frac{200 \cdot 10}{1000} = 8 \text{ years}$$

to be compared with the 10 years time to maturity.

But weighted average maturity is still a weak proxy for a bond's inherent risk. It is better than the term to maturity, as it considers the principal repayment cash flows. But it does not consider the full impact of the distribution of cash flows on the bond's risk, as it ignores the coupons. Thus, the weighted average maturity is insensitive to the coupon differentials. For example, an 8% and a 2% sinking-fund debentures could have the same average life, and hence, the same risk.

The **weighted average cash flow** is calculated similar to the weighted average maturity, except that it considers all the cash flows from a bond:

$$\text{Weighted average cash flow} = \sum_{t=1}^{T} \frac{\text{Cash flow paid at time t}}{\text{Total cash flows to be repaid}} \cdot t$$

It assesses a bond's risk by finding the average maturity of a bond's cash flows, considering coupons as well as principal repayments.

Example:
A bond has a face value of 1'000 CHF, expires in 4 years, and offers a 6% coupon rate. What is its weighted average cash flow?

The bond's weighted average cash flow is

$$\text{Weighted average cash flow} = \frac{60 \cdot 1}{1240} + \frac{60 \cdot 2}{1240} + \frac{60 \cdot 3}{1240} + \frac{1060 \cdot 4}{1240} = 3.71 \text{ years}$$

to be compared with the 4 years time to maturity.

The main drawback of the weighted average cash flow is that repayments are considered on a nominal basis rather than on a present value basis; and we all know that a CHF tomorrow does not have the same value as a CHF today.

Hence, these three average maturities do not provide an adequate bond price volatility measure[12].

2.5.2 Duration and modified duration

The duration as a measure of bond risk was initially proposed by Frederick R. Macaulay in 1938.

2.5.2.1 Definition

The concept of **duration** can be interpreted as an advanced version of the weighted average cash flow. The duration of a series of cash flows is equal to the average time at which the cash flows occur; the weight of each cash flow is calculated using the *present value* of the cash flow (instead of using the nominal value). The formula for Duration is:

$$\text{Duration} = D = \sum_{t=1}^{T} \frac{PV(CF_t)}{P} \cdot t = \sum_{t=1}^{T} w_t \cdot t$$

If we discount all cash flows at the bond's yield to maturity k (as we did to calculate the price), the weight of each cash flow is $w_t = \frac{CF_t/(1+k)^t}{P}$ and the complete formula for duration is:

$$\begin{aligned}\text{Duration} = D &= \sum_{t=1}^{T} \frac{PV(CF_t)}{P} \cdot t \\ &= \frac{1}{P} \cdot \sum_{t=1}^{T} \frac{CF_t}{(1+k)^t} \cdot t \\ &= \frac{1}{P} \cdot \left[\frac{CF_1}{(1+k)^1} \cdot 1 + \frac{CF_2}{(1+k)^2} \cdot 2 + \frac{CF_3}{(1+k)^3} \cdot 3 + \ldots + \frac{CF_T}{(1+k)^T} \cdot T \right]\end{aligned}$$

12 Note that the relationships between the three basis measures are the following:

Bond type	Relationship
Coupon bearing bullet bond	WACF < WAM = TTM
Sinking fund bond	WACF < WAM < TTM
Zero-coupon bond	WACF = WAM = TTM

where:

CF_t	amount of the cash flow (coupon or principal) received at date t
P	the market price of the bond[13] (or present value of all future payments)
T	time to maturity
k	discount rate (market yield)

Similar to the weighted average cash flows, **duration is measured in years**. This formulation of duration is often called **Macaulay's Duration**.

Example:
A bond with a 10-year maturity pays a 8% annual coupon. Its yield to maturity is k=10%. What is its Macaulay's Duration?

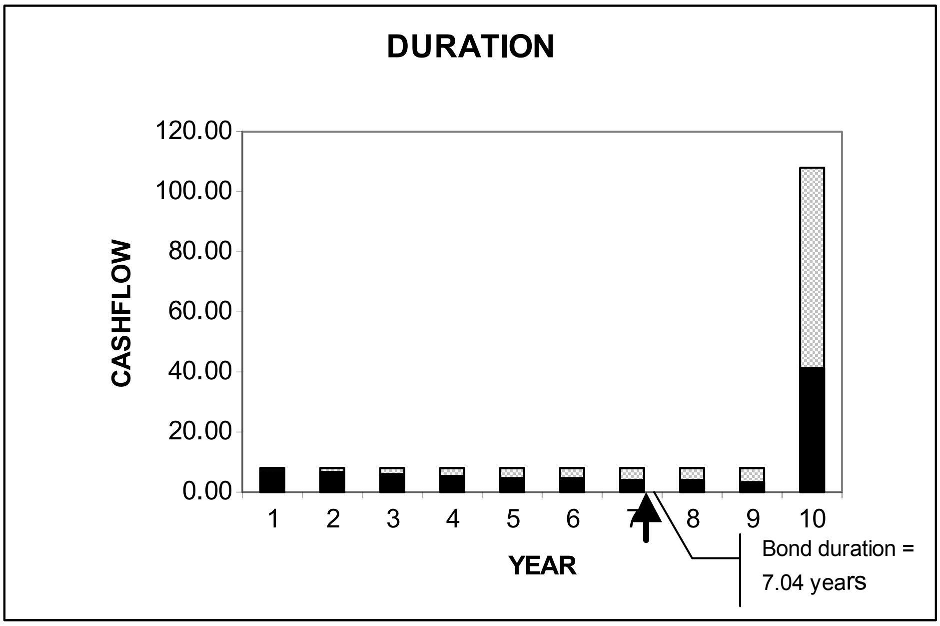
t (Years) [1]	Cash flow CF [2]	PV (CF) $[3]=[2]/(1+k)^t$	CF weight [4] = [3] / Price	Time weighted by CF weight [5] = [1] · [4]
1	8	7.27	0.0829	0.083
2	8	6.61	0.0754	0.151
3	8	6.01	0.0685	0.206
4	8	5.46	0.0623	0.249
5	8	4.97	0.0566	0.283
6	8	4.52	0.0515	0.309
7	8	4.11	0.0468	0.328
8	8	3.73	0.0425	0.340
9	8	3.39	0.0387	0.348
10	108	41.64	0.4747	4.747
	Price:	**87.71**	**Duration:**	**7.04**

The bond's Duration is 7.04 years.

We can graphically represent the duration by plotting the cash flows as a function of time. The height of each bar is the cash flow received [column 2 in the above table]; the lower portion of each bar (in black) is the present value of the cash flow [column 3]. If we think of these values as physical weights placed on a horizontal bar, the duration (marked with an arrow) is the fulcrum point of these weights.

13 Note that if the price of the bond is not known, we can use the following formula.

$$\text{Duration} = \frac{\sum_{t=1}^{T} \frac{t \cdot CF_t}{(1+k)^t}}{P} = \frac{\sum_{t=1}^{T} \frac{t \cdot CF_t}{(1+k)^t}}{\sum_{t=1}^{T} \frac{CF_t}{(1+k)^t}}$$



In the case of a zero-coupon bond, as there are no intermediate payments, duration is simply the present value of the final cash flow multiplied by the maturity, divided by the price. But as the price is itself the present value of the final cash flow, **the duration of a zero-coupon bond is equal to its maturity**.

Example:
A 10-year zero-coupon bond with a face value of 1'000 CHF is traded at 558.39 CHF. Its yield to maturity is 6%. What is its duration?

The bond's duration is:

$$\text{Duration} = \frac{\frac{10 \cdot 1000}{1.06^{10}}}{558.39} = \frac{10 \cdot 558.39}{558.39} = 10 \text{ years}$$

which is exactly its time to maturity.

Even if the bond has features which modify its cash flows, such as six-monthly coupon payments or sinking fund requirements, the methodology remains the same.

Example:
What is the duration of a 6% coupon, 10 year sinking fund debenture priced at par (1'000 USD) with a current market yield of 6%? The sinking fund retires 20% of the bonds annually, commencing at the end of the sixth year. The interest is paid six-monthly.

The following table lists the various steps that are necessary to calculate the duration.

T (Years) [1]	total cash flow [2]	PV factor [3]	PV (CF) [4] = [2] · [3]	CF weight [5] = [4] / Price	PV weighted by time t [6] = [1] · [5]
0.5	30.00	0.9709	29.13	0.0291	0.01
1	30.00	0.9426	28.28	0.0283	0.03
1.5	30.00	0.9151	27.45	0.0275	0.04
2	30.00	0.8885	26.65	0.0267	0.05
2.5	30.00	0.8626	25.88	0.0259	0.06
3	30.00	0.8375	25.12	0.0251	0.08
3.5	30.00	0.8131	24.39	0.0244	0.09
4	30.00	0.7894	23.68	0.0237	0.09
4.5	30.00	0.7664	22.99	0.0230	0.10
5	30.00	0.7441	22.32	0.0223	0.11
5.5	30.00	0.7224	21.67	0.0217	0.12
6	230.00	0.7014	161.32	0.1613	0.97
6.5	24.00	0.6810	16.34	0.0163	0.11
7	224.00	0.6611	148.09	0.1481	1.04
7.5	18.00	0.6419	11.55	0.0116	0.09
8	218.00	0.6232	135.85	0.1359	1.09
8.5	12.00	0.6050	7.26	0.0073	0.06
9	212.00	0.5874	124.53	0.1245	1.12
9.5	6.00	0.5703	3.42	0.0034	0.03
10	206.00	0.5537	114.06	0.1141	1.14
		Price	**1'000.00**	**Duration**	**6.43**

The duration of our bond is 6.43 years.

2.5.2.2 Interpretations and implicit assumptions

Duration takes into account all of the following variables affecting the bond price volatility:

- all the cash flows
- the yield to maturity
- the current market price of the bond

But what does duration really mean? In fact, it is more than just a sophisticated average maturity, and there is one basic property that helps in understanding the concept of duration:

- **In interest-rate risk terms, an investor is indifferent between a coupon-bearing bond investment and a zero-coupon instrument maturing on the duration date of the coupon bearing issue.**

Using Macaulay's duration, we implicitly assume that all cash flows are discounted (or reinvested) at the same discount rate k, equal to the bond's yield to maturity. But in fact, each cash flow should be discounted at the appropriate rate $R_{0,t}$, and we do not have one single yield, but a part of the term structure of interest rates.

Thus, the assumption made using the Macaulay's duration is that **the term structure of interest rates is flat** (that is, **the yields for all maturities are equal** to a single value, called the **market yield**).

Note that if the term structure of interest rates (or the yield curve) is not flat, the implied spot rate curve supplies a series of discount rates (rather than a single one) applicable to the bond's future cash flows, generating a duration that differs from Macaulay's duration. For example, the Fisher and Weil's duration is defined as:

$$\text{Duration} = D_{FW} = \sum_{t=1}^{T} \frac{PV(CF_t)}{P} \cdot t = \frac{1}{P} \cdot \sum_{t=1}^{T} \frac{t \cdot CF_t}{(1+R_{0,t})^t}$$

$$= \frac{1}{P} \cdot \left[\frac{1 \cdot CF_1}{(1+R_{0,1})^1} + \frac{2 \cdot CF_2}{(1+R_{0,2})^2} + \frac{3 \cdot CF_3}{(1+R_{0,3})^3} + \ldots + \frac{T \cdot CF_T}{(1+R_{0,T})^T} \right]$$

2.5.2.3 An example to illustrate the calculation of duration

Consider a 10-year bond with a face value of 100 CHF and a 10% coupon. The current market yield (for all maturities) is 8%.

Let us calculate Macaulay's duration for the bond

$$\text{Duration} = \frac{\dfrac{1 \cdot 10}{1.08^1} + \dfrac{2 \cdot 10}{1.08^2} + \dfrac{3 \cdot 10}{1.08^3} + \ldots + \dfrac{10 \cdot 110}{1.08^{10}}}{\dfrac{10}{1.08^1} + \dfrac{10}{1.08^2} + \dfrac{10}{1.08^3} + \ldots + \dfrac{110}{1.08^{10}}} = 6.97 \text{ years}$$

and the current bond price

$$P_{k=8\%} = \frac{10}{1.08^1} + \frac{10}{1.08^2} + \frac{10}{1.08^3} + \ldots + \frac{110}{1.08^{10}} = 113.42$$

Now let us determine what happens if immediately after we bought this bond for 113.42 CHF, the market yield decreases from 8% to 4%. The new price of this bond is then

$$P_{k=4\%} = \frac{10}{1.04^1} + \frac{10}{1.04^2} + \frac{10}{1.04^3} + \ldots + \frac{110}{1.04^{10}} = 148.67$$

We can see that with this (extreme) interest rate variation the bondholder has a capital gain of 35.25 CHF. To illustrate the use of duration, we calculate the bond's price for each year remaining until the bond's maturity for a market yield of 8% (before) and a market rate of 4% (after). We also calculate the present value for each year and each market yield of the reinvested coupons C.

$$\text{Future value of reinvested coupons in year t} = \sum_{i=1}^{t} C_i \cdot (1+k)^{t-i}$$

The total value of this bond (calculated as the sum of the bond price and the value of the reinvested coupons) is also given in the following table:

Year	YTM = 8%			YTM = 4%		
	Bond price	**Value of reinvested coupons**	**Total value**	**Bond price**	**Value of reinvested coupons**	**Total value**
	[1]	[2]	[1] + [2]	[3]	[4]	[3] + [4]
0	113.42	0.00	**113.42**	148.67	0.00	**148.67**
1	112.49	10.00	**122.49**	144.61	10.00	**154.61**
2	111.49	20.80	**132.29**	140.40	20.40	**160.80**
3	110.41	32.46	**142.88**	136.01	31.22	**167.23**
4	109.25	45.06	**154.31**	131.45	42.46	**173.92**
5	107.99	58.67	**166.65**	126.71	54.16	**180.87**
6	106.62	73.36	**179.98**	121.78	66.33	**188.11**
7	***105.15***	***89.23***	***194.38***	***116.65***	***78.98***	***195.63***
8	103.57	106.37	**209.93**	111.32	92.14	**203.46**
9	101.85	124.88	**226.73**	105.77	105.83	**211.60**
10	100.00	144.87	**244.87**	100.00	120.06	**220.06**

Table 2-1: Time to maturity and value of a bond

The following figure illustrates the results obtained in the table above:

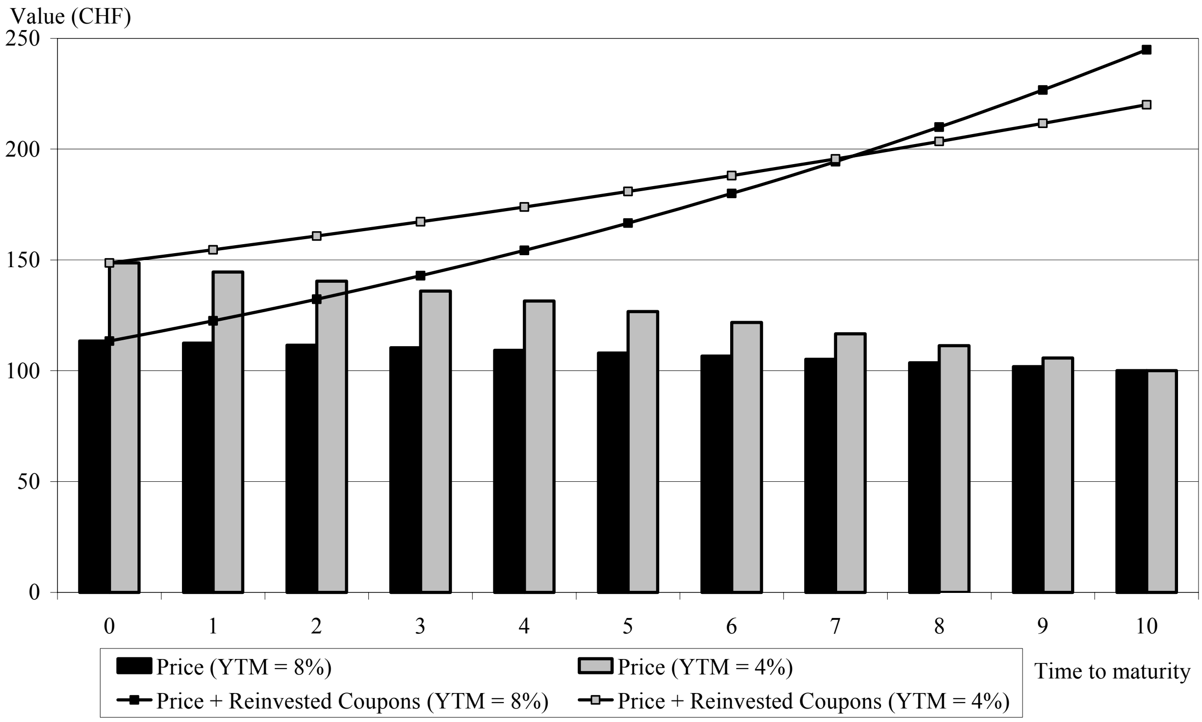


Figure 2-14: Time to maturity and value of a bond

It’s interesting to see that in year 7, the total value (= price + reinvested coupons) is almost equal for the two scenarios (194.38 CHF versus 195.63 CHF). From above we know that the duration of this bond is 6.97 years. This is the interpretation of the duration. Duration is equal to the time (in years) at which the total value of the bond is not sensitive to interest rate variations. The total value is the amount we get from this investment.

Another way to show that the total value of the bond is insensitive to interest rate variations in year 7 is to calculate the holding period returns. Remember that we bought the bond in year 0 for 113.42 CHF at a market yield of 8%. What are the holding period returns if the market yield has decreased to 4%? The holding period return is defined as

$$\mathrm{HPR}_{0,t} = \sqrt[t]{\frac{\text{Total value in t}}{\text{Total value in 0}}} - 1$$

Year	Holding period return
1	36.31%
2	19.07%
3	13.82%
4	11.28%
5	9.78%
6	8.78%
7	***8.10%***
8	7.6%
9	7.17%
10	6.85%

We see that in year 7 (= duration), the holding period return is almost equal to 8% which was the current market yield at which we bought the bond. If you sell a bond at the time of its duration, you have a holding period return equal to the current market yield.

In this example, we calculated the effect on the bond of an extreme interest rate variation (for the purpose of illustration). Now we calculate the total value of this bond in year 7 for other interest rate variations:

market yield	Change in market yield	Bond price [1]	Value of reinvested coupons [2]	Total value [1] + [2]
12%	+4%	95.20	100.89	**196.09**
11%	+3%	97.56	97.83	**195.39**
10%	+2%	100.00	94.87	**194.87**
9%	+1%	102.53	92.00	**194.54**
8%	***0%***	***105.15***	***89.23***	***194.38***
7%	–1%	107.87	86.54	**194.41**
6%	–2%	110.69	83.94	**194.63**
5%	–3%	113.62	81.42	**195.04**
4%	–4%	116.65	78.98	**195.63**

Table 2-2: Value of a bond in its duration year for different interest rate changes

Again we see that the total value of the bond is in year 7 (= duration) almost insensitive to interest rate variations. We also see that with bigger interest rate variations (positive and

negative) the variation in the total value is bigger. We will come back to this effect in section 2.5.5. which discusses convexity.

2.5.2.4 Determinants of duration

Duration of a bond is a function of the bond's time to maturity, coupon rate, accrued interest, market yield, its sinking fund and call features, if any.

Duration is generally positively related to a bond's **time to maturity**: longer maturity bonds have longer durations[14]. But as maturity increases, duration increases at a decreasing rate: thus, duration cannot grow infinitely, whereas maturity can, and the maximum value of duration is (with an annual coupon payment[15]):

$$\text{Duration of a perpetual bond} = \frac{1}{\text{Bond's yield}} + 1$$

We should also note that a zero-coupon bond has a duration that exactly matches its time to maturity, while other bonds have durations shorter than their time to maturity (because of the coupon effect).

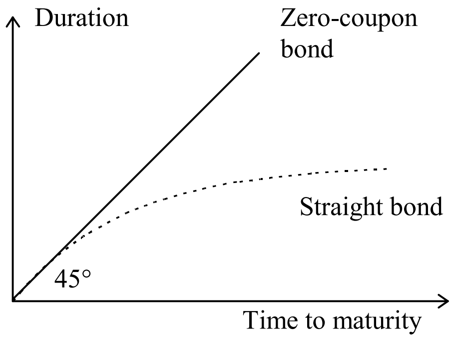


Figure 2-15: Relationship between duration and time to maturity

Duration is inversely related to the **coupon rate of interest**. Lower coupon bonds have longer durations than higher coupon bonds of similar maturity (compare with a zero-coupon bond). Progressively higher coupons lead to a decline in duration, but at a diminishing rate.

14 In fact, if the bond is sold at par or over the par duration always increases with maturity. If the bond is sold under par (with a discount), duration also increases with maturity, but starts decreasing at a certain level. It can be shown that the duration of a bond paying an annual coupon C, with yield to maturity y and time to maturity T years is given by: $D = \frac{(1+y)}{y} - \frac{(1+y) + T \cdot (C-y)}{C \cdot \left[(1+y)^T - 1\right] + y}$. We can note that, when the coupon C is smaller than the yield y, for large enough T the expression (1+y)+T·(C-y) becomes negative. This means that such a bond has a duration which is higher than the one of a perpetuity!

15 For a semi-annual payment, we have:

$$\text{Maximum Duration} = \frac{1}{\text{Bond's yield}} + 0.5$$

Example:
The following table lists the Duration of various bonds differing only by their coupon rate. Higher coupons rates lead to a decline in duration.

	Bond 1	Bond 2	Bond 3	Bond 4	Bond 5
Maturity (years)	10	10	10	10	10
Market yield	6.00%	6.00%	6.00%	6.00%	6.00%
Coupon	0.00%	3.00%	6.00%	9.00%	12.00%
Market price (CHF)	558.39	779.20	1'000.00	1'220.80	1'441.61
Duration (years)	**10.00**	**8.59**	**7.80**	**7.30**	**6.95**

Coupon changes have more impact on duration the lower the initial coupon rate, and the longer the time to maturity.

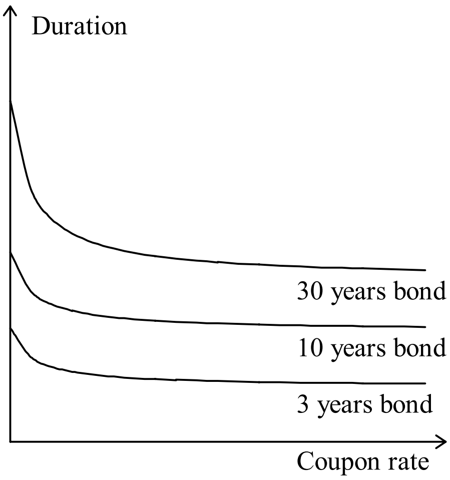


Figure 2-16: Relationship between duration and coupon rate (various maturities)

Duration is, of course, inversely related to the **buildup of accrued interest**. A bond's duration naturally increases on coupon payment date, as the accrued interest drops off. These effects are especially pronounced for high coupon issues and for long maturity bonds

We should also note that duration is inversely related to the **general level of interest rates** (**yield level**). As the concept of duration is based on the discounting process, a higher discount rate will lead to lower duration.

Example:
The following table lists the duration of the same bond using various yields levels. Lower yields lead to a duration increase.

	Case 1	Case 2	Case 3	Case 4	Case 5
Maturity (years)	10	10	10	10	10
Market yield	8.00%	7.00%	6.00%	5.00%	4.00%
Coupon	6.00%	6.00%	6.00%	6.00%	6.00%
Market price (CHF)	865.80	929.76	1'000.00	1'077.22	1'162.22
Duration (years)	**7.62**	**7.71**	**7.80**	**7.89**	**7.98**

2.5.2.5 *Using duration to approximate price changes*

From the mathematical derivation of the formula[16] for duration, we know that for small changes of the market yield:

$$\frac{\Delta P}{P} = -\frac{D}{(1+k)} \cdot \Delta k$$

This very important formula says that the percentage change in the price of a bond due to an interest rate change is, in first approximation, proportional to its duration.[17]
Note that the previous equation is often expressed as:

$$\frac{\Delta P}{P} = -D^{mod} \cdot \Delta k$$

where $D^{mod} = \frac{D}{1+k}$ is called the **modified duration** (or **sensitivity**) of the bond, or as

$$\Delta P = -D^{p} \cdot \Delta k$$

where $D^{p} = \frac{D}{1+k} \cdot P = -\frac{\Delta P}{\Delta k}$ is called the **price duration of the bond**[18].

Using the modified duration or the price duration of a bond, one can approximate the percentage price change for a given change in the required yield.

Example:
A bond has a face value of 1'000 CHF, expires in 4 years and offers a 6% coupon rate. The market yield is 7%. The bond's duration is 3.67 years. What happens if the yield changes by plus 50 basis points (+0.5%, and goes to 7.5%)? How about a 200 basis points change (+2%, to 9%)?

Using duration, we can write:

$$\frac{\Delta P}{P} = -D \cdot \frac{\Delta k}{1+k} = -3.67 \cdot \frac{+0.005}{(1+0.07)} = -1.71\%$$

The duration approach predicts a decrease of 1.71% of the bond price. As the price at a 7% market rate was:

$$P = \frac{60}{1.07^1} + \frac{60}{1.07^2} + \frac{60}{1.07^3} + \frac{1060}{1.07^4} = 966.13 \text{ CHF}$$

the new price should be 966.13 · (1 – 0.0171) = 949.61 CHF.

The effective price with a 7.5% yield is:

$$P = \frac{60}{1.075^1} + \frac{60}{1.075^2} + \frac{60}{1.075^3} + \frac{1060}{1.075^4} = 949.76 \text{ CHF}$$

which is very close from 949.61 CHF.

---

16 See for the mathematical derivation of the duration Appendix A.1 of this module.

17 This formula can be used to *define* duration. In fact the definition of duration as the weighted average life of a bond does not work with particular instruments such as some classes of CMO's (Collateralised Mortgage Obligations) and Inverse Floaters.

18 Or **dollar duration** of the bond in the United States.

The same computation for $\Delta k = +2\%$ would predict a decrease of 6.85% of the price, i.e. a price of 899.95 CHF. The effective price with a 9% yield is 902.81 CHF.

From the above example, it seems that the duration approach does a good job in estimating the change in the price of a bond **for a small change in the yield**, but not for a large change. We will explain the reason behind this in the next section.

*2.5.3 Convexity**

How accurately does duration allow us to calculate approximate bond price changes?

Example:
Let us start from the following situation: a 6% 10-year bond is priced at par. Therefore, the market yield is 6%. Its duration is 7.8 years. What are the differences between the effective market price and the price estimated with duration, if the market yield increases?

New market yield	New market price	Estimated price	ΔP / P	Estimated ΔP / P	Difference
0.0625	981.82	981.60	–1.82%	–1.84%	**0.02%**
0.0650	964.06	963.20	–3.59%	–3.68%	**0.09%**
0.0675	946.71	944.80	–5.33%	–5.52%	**0.19%**
0.0700	929.76	926.40	–7.02%	–7.36%	**0.34%**
0.0725	913.21	908.00	–8.68%	–9.20%	**0.52%**
0.0750	897.04	889.60	–10.30%	–11.04%	**0.74%**
0.0775	881.24	871.20	–11.88%	–12.88%	**1.00%**
0.0800	865.80	852.80	–13.42%	–14.72%	**1.30%**
0.0825	850.71	834.40	–14.93%	–16.56%	**1.63%**
0.0850	835.97	816.00	–16.40%	–18.40%	**2.00%**
0.0875	821.56	797.60	–17.84%	–20.24%	**2.40%**
0.0900	807.47	779.20	–19.25%	–22.08%	**2.83%**
0.0925	793.70	760.80	–20.63%	–23.92%	**3.29%**
0.0950	780.24	742.40	–21.98%	–25.76%	**3.78%**
0.0975	767.08	724.00	–23.29%	–27.60%	**4.31%**
0.1000	754.22	705.60	–24.58%	–29.44%	**4.86%**
0.1025	741.64	687.20	–25.84%	–31.28%	**5.44%**
0.1050	729.34	668.80	–27.07%	–33.12%	**6.05%**
0.1075	717.30	650.40	–28.27%	–34.96%	**6.69%**
0.1100	705.54	632.00	–29.45%	–36.80%	**7.35%**
0.1125	694.03	613.60	–30.60%	–38.64%	**8.04%**
0.1150	682.77	595.20	–31.72%	–40.48%	**8.76%**
0.1175	671.76	576.79	–32.82%	–42.32%	**9.50%**

The approximation is accurate for small changes in the market yield, but the error increases for large changes.

The following figure represents the price/yield relationship. We can draw a tangent line at yield k^*, which shows the rate of change of price with respect to a change in interest rate at that point (yield level). The slope of this line is the **price duration**. Mathematically speaking, the price duration is the first derivative of the curvilinear price/yield function.

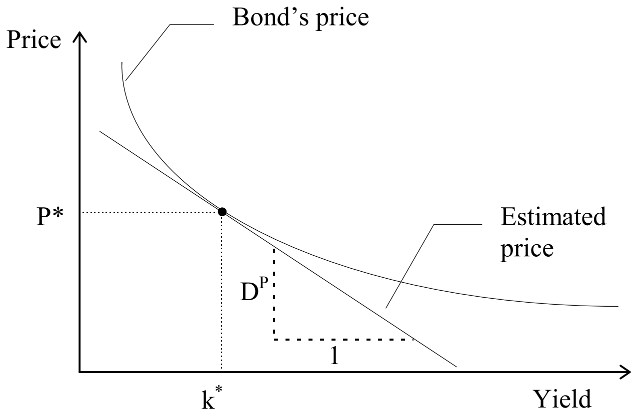


Figure 2-17: Bond's price and market yield

Modified and price duration consider a bond's price/yield relationship as a linear function. In reality, the price/yield function is a convex curve. Thus, duration attempts to estimate a convex relationship with a straight line. Consequently, error terms become large as prices and yields move away from current levels. The further away the new yield is from the initial yield k*, the greater the errors.

Hence,

- Duration is an instantaneous value that is continuously modified: even **time has an effect on duration**.
- Duration will not exhibit the **asymmetry in price volatility**.
- it should be clear that the approximation will **always underestimate the new price**.
- the **accuracy** of the approximation **depends of the convexity** of the price/yield relationship for the bond.
- we should **not use duration** to approximate a price change if there is a **large variation** in the market yield.

We can better estimate (bigger) price changes if we use another approximation called convexity in addition to duration. The convexity is a proxy for the convexity of the price/yield relationship.[19] The convexity is defined as:

$$\text{Convexity} = C = \frac{1}{2} \cdot \frac{1}{P} \cdot \frac{1}{(1+k)^2} \cdot \sum_{t=1}^{T} \frac{(t) \cdot (t+1) \cdot CF_t}{(1+k)^t}$$

Note that there exist different convexity definitions. Convexity is often defined without the term ½ as:

$$\text{Convexity} = C^* = \frac{1}{P} \cdot \frac{1}{(1+k)^2} \cdot \sum_{t=1}^{T} \frac{(t) \cdot (t+1) \cdot CF_t}{(1+k)^t}$$

19 See for a mathematical derivation of the convexity Appendix A.2 of this module.

Example:
A 10-year bond has a face value of 100 CHF, pays a 6% annual coupon rate. The required market yield is 6.5%. What is its convexity?

The following table represents the cash flows from the bond:

Time	Cash Flow	Present Value (PV)	PV · t · (t + 1)
1	6	5.63	11.27
2	6	5.29	31.74
3	6	4.97	59.61
4	6	4.66	93.28
5	6	4.38	131.38
6	6	4.11	172.70
7	6	3.86	216.22
8	6	3.63	261.03
9	6	3.40	306.37
10	106	56.47	6211.59
Total		**96.41**	**7495.18**

The price P of this bond is 96.41.

The convexity of this bond is:

$$\text{Convexity} = C = \frac{1}{2} \cdot \frac{1}{P} \cdot \frac{1}{(1+k)^2} \cdot \frac{\sum_{t=1}^{10} t \cdot (t+1) \cdot CF_t}{(1+k)^t} = \frac{1}{2} \cdot \frac{1}{96.41} \cdot \frac{1}{1.065^2} \cdot 7'495.18 = 34.27$$

With convexity, we can calculate price changes as follows:

$$\Delta P = -D \cdot P \cdot \frac{\Delta k}{1+k} + C \cdot P \cdot (\Delta k)^2$$

or in relative terms:

$$\frac{\Delta P}{P} = -D \cdot \frac{\Delta k}{1+k} + C \cdot (\Delta k)^2$$

If we use the other definition of convexity, we have to modify the above equations with the term ½.

$$\Delta P = -D \cdot P \cdot \frac{\Delta k}{1+k} + \frac{1}{2} \cdot C^* \cdot P \cdot (\Delta k)^2$$

With the above equations one can show that an option-free bond always has a positive convexity for every kind of yield changes. For positive and negative changes in the market yield k, the effect of the convexity term for the price change ΔP is always positive.

We also see from the above equations that the first term is the approximation based on duration and the second term is a proxy for the convexity of the price/yield relationship.

We can define the **price convexity**[20] as the convexity multiplied by the price of the bond.

$$\text{Price convexity } = C^p = C \cdot P$$

Using the price duration and the price convexity, one can estimate the price change of a bond in value in CHF rather than as a percentage:

$$\Delta P = -D^p \cdot \Delta k + C^p \cdot (\Delta k)^2$$

Using both duration and convexity, we should have a more accurate approximation of the bond's price changes for a small variation in the market yield.

Example:
A 10-year bond has a face value of 1'000 CHF, pays a 6% annual coupon rate and is traded at 102%. The market yield is 5.73%. What are its duration and convexity? What happens if the required yield changes by +200 basis points?

The bond's cash flows are as follows:

t	CF	PV(CF)	PV·t	PV · t · (t + 1)
1	60	56.75	56.75	113.50
2	60	53.67	107.34	322.03
3	60	50.76	152.29	609.17
4	60	48.01	192.05	960.26
5	60	45.41	227.05	1'362.33
6	60	42.95	257.70	1'803.90
7	60	40.62	284.35	2'274.84
8	60	38.42	307.35	2'766.29
9	60	36.34	327.05	3'270.47
10	1'060	607.19	6'071.89	66'790.74
		1'020.13	**7'983.85**	**80'273.52**

The bond price P is 1'020.06 CHF.

The duration is:

$$\text{Duration} = D = \sum_{t=1}^{T} \frac{PV(CF_t)}{P} \cdot t = \frac{7'983.85}{1'020.13} = 7.83 \text{ years}$$

and the convexity is:

$$\text{Convexity} = C = \frac{1}{2} \cdot \frac{1}{P} \cdot \frac{1}{(1+k)^2} \cdot \sum_{t=1}^{T} \frac{(t) \cdot (t+1) \cdot CF_t}{(1+k)^t} = \frac{1}{2} \cdot \frac{1}{(1.0573)^2} \cdot \frac{80'273.52}{1'020.13} = 35.20$$

If there is a yield increase of 200 basis points, the new market yield is 7.73%. The duration predicts a price change of

$$\frac{\Delta P}{P} = -D \cdot \frac{\Delta k}{1+k} = -7.83 \cdot \frac{+0.02}{(1+0.0573)} = -14.81\%$$

and the new bond price should be 1'020.13 · (1 – 0.1481) = 869.05 CHF.

20 In the United States: the **dollar convexity**.

The convexity predicts an additional price change of:

$$\frac{\Delta P}{P} = C \cdot (\Delta k)^2 = 35.20 \cdot (0.02)^2 = +1.41\%$$

Thus, the total price change should be –14.81% + 1.41%= –13.4 %, and the new price should be 1'020.13 · (1 – 0.1340) = 883.43 CHF.

The actual price, using a yield of 7.73%, is 882.43 CHF.

What does convexity exactly measure? The convexity measures the rate of change of the slope of the price-yield curve with respect to yield changes. Just as duration, convexity changes with time.

Graphically, duration and convexity can be shown as follows:

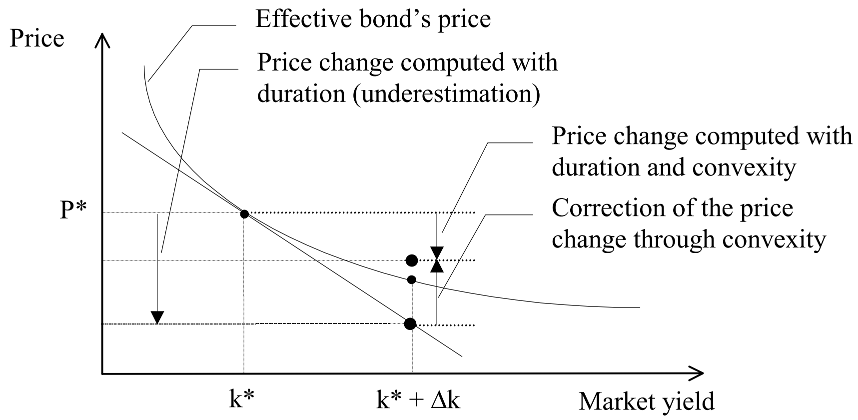


Figure 2-18: Estimating bond price changes using duration and convexity

From the above graph it should be clear that convexity is beneficial to the investor: it has a positive price effect for both increasing and decreasing rates. Thus, all other things being equal, bonds with a larger convexity should be preferred to those with a smaller convexity. Mathematically, we can see this with the already known price change formula:

$$\Delta P = \underbrace{-\text{Duration} \cdot P \cdot \frac{\Delta k}{1+k}}_{\text{negative or positive}} + \underbrace{\text{Convexity} \cdot P \cdot (\Delta k)^2}_{\text{always positive}}$$

In the following figure, we have two bonds A and B with the same duration. But bond B has a smaller convexity than bond A.

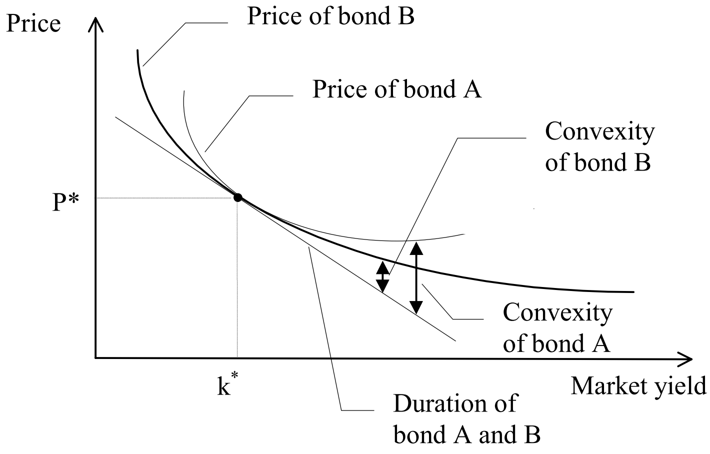
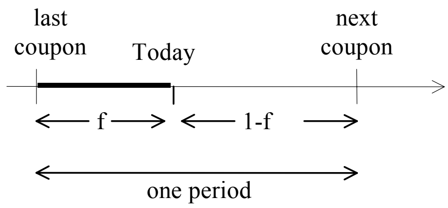


Figure 2-19: Two bonds with different convexities

2.5.4 Duration and convexity between coupon payment dates

Determining the duration and convexity of a bond between two coupon payments is very simple. Consider the following situation



P_{ex} denotes the quoted price of the bond (without accrued interests) and C the coupon payment. Starting with the yield to maturity definition formula from chapter 2:

$$P_{cum} = P_{ex} + f \cdot C = \sum_{t=1}^{T} \frac{CF_t}{(1+k)^{t-f}}$$

If we differentiate the right hand side once with respect to the yield k, we get

$$\frac{dP_{cum}}{dk} = \frac{dP_{ex}}{dk} = -\frac{(1+k)^f}{(1+k)} \cdot \left[\sum_{t=1}^{T} \frac{(t-f) \cdot CF_t}{(1+k)^t} \right]$$

The second derivative with respect to the yield is:

$$\frac{d^2 P_{cum}}{dk^2} = \frac{(1+k)^f}{(1+k)^2} \cdot \left[\sum_{t=1}^{T} \frac{(t-f) \cdot (t-f+1) \cdot CF_t}{(1+k)^t} \right]$$

From the definition of duration, that is

$$D = -\frac{dP_{cum}}{dk} \cdot \frac{1+k}{P_{cum}}$$

we get by replacing $\frac{dP_{cum}}{dk}$ by its value:

$$D = \frac{(1+k)^f}{P_{cum}} \cdot \left[\sum_{t=1}^{T} \frac{(t-f) \cdot CF_t}{(1+k)^t} \right]$$

From the definition of convexity, that is

$$C = -\frac{d^2 P_{cum}}{dk^2} \cdot \frac{1}{P_{cum}}$$

we get by replacing $\frac{d^2 P_{cum}}{dk^2}$ by its value:

$$C = \frac{(1+k)^f}{(1+k)^2} \cdot \frac{1}{P} \cdot \left[\sum_{t=1}^{T} \frac{(t-f) \cdot (t-f+1) \cdot CF_t}{(1+k)^t} \right]$$

Thus, we have derived the formulas for the duration and the convexity of a bond between two coupon payments.

2.5.5 Impact of coupon payments and time lapse on duration

It can be proved that a coupon payment has a positive effect on duration, that is, **duration will suddenly increase at the coupon payment**.

Just before the coupon payment, the duration of the bond is

$$D_{cum} = -\frac{dP_{cum}}{dk} \cdot \frac{1+k}{P_{cum}} \quad \Leftrightarrow \quad D_{cum} \cdot P_{cum} = -\frac{dP_{cum}}{dk} \cdot (1+k)$$

and just after the coupon payment

$$D_{ex} = -\frac{dP_{ex}}{dk} \cdot \frac{1+k}{P_{ex}} \quad \Leftrightarrow \quad D_{ex} \cdot P_{ex} = -\frac{dP_{ex}}{dk} \cdot (1+k)$$

But as

$$\frac{dP_{cum}}{dk} = \frac{dP_{ex}}{dk}$$

one may write

$$D_{cum} \cdot P_{cum} = D_{ex} \cdot P_{ex}$$

that is

$$D_{cum} = D_{ex} \cdot \frac{P_{ex}}{P_{cum}}$$

As $P_{cum} = P_{ex}$ + Coupon, we have $P_{cum} > P_{ex}$. Thus, $D_{cum} < D_{ex}$, which implies that duration will increase just after a coupon payment by an amount of

$$\Delta D = D_{ex} - D_{cum} = D_{ex} - D_{ex} \cdot \frac{P_{ex}}{P_{cum}}$$
$$= \frac{D_{ex} \cdot (P_{cum} - P_{ex})}{P_{cum}} = \frac{D_{ex} \cdot \text{Coupon}}{P_{cum}} > 0$$

One can also prove that **duration will decrease linearly with time** between two coupon payments. Starting from the following equation

$$P_{cum} = P_{ex} + f \cdot I = (1+k)^f \cdot \sum_{t=1}^{T} \frac{CF_t}{(1+k)^t}$$

which gives the yield k of a bond between two coupon payment dates (when a fraction f of a year is elapsed since the last coupon payment date), and replacing P_{cum} in the following formula:

$$D = \frac{(1+k)^f}{P_{cum}} \cdot \sum_{t=1}^{T} \frac{(t-f) \cdot CF_t}{(1+k)^t}$$

which gives the duration of a bond between two coupon payment dates, we get:

$$D = \frac{\sum_{t=1}^{T} \frac{(t-f) \cdot CF_t}{(1+k)^t}}{\sum_{t=1}^{T} \frac{CF_t}{(1+k)^t}}$$

If we derive D with respect to f, we get:

$$\frac{dD}{df} = \frac{\frac{d}{df} \sum_{t=1}^{T} \frac{t \cdot CF_t}{(1+k)^t} - \frac{d}{df} \sum_{t=1}^{T} \frac{f \cdot CF_t}{(1+k)^t}}{\sum_{t=1}^{T} \frac{CF_t}{(1+k)^t}} = \frac{0 - \sum_{t=1}^{T} \frac{CF_t}{(1+k)^t}}{\sum_{t=1}^{T} \frac{CF_t}{(1+k)^t}} = -1$$

that is, duration decreases linearly with f between two coupon payment dates, while f itself increases linearly with time.

In conclusion, one should remember that all other things being equal, the duration of a portfolio will vary linearly over time, except when there is a coupon payment.

2.5.6 Restrictions on using the duration and convexity

The usefulness of the (modified) duration as a bond risk proxy is predicated on three assumptions:

- **a small change in the yield**
- **a parallel shift in the yield, whatever the maturity**
- **and instantaneous change in yield**

Furthermore, as we have already seen, it assumes a **flat yield curve**.

Several methods have been suggested to measure the exposure of a bond to a particular rate change, with all other rates being held constant. A new direction is **functional duration**, which is defined as the price sensitivity of a bond to a particular rate change, all other rates being held constant. Using functional duration, we have the ability to deal with any type of term structure shift[21].

2.5.7 Portfolio duration and convexity

The **duration of a bond portfolio** is simply the weighted average of the durations of the individual bonds.

$$\text{Portfolio duration} = \sum_{i=1}^{n} w_i \cdot D_i$$

where:

w_i	weight (in market value terms) of security i in the portfolio
D_i	duration of security i
n	number of securities in the portfolio

Final Level

The **convexity of a bond portfolio** is simply the weighted average of the convexities of the individual bonds.

$$\text{Portfolio convexity} = \sum_{i=1}^{n} w_i \cdot C_i$$

where:

wi	weight (in market value terms) of security i in the portfolio
Ci	convexity of security i
n	number of securities in the portfolio

21 See for example REITANO R., 1992, “Non-Parallel Yield Curve Shifts and Immunization”, Journal of Portfolio Management, pp. 36-43.

2.6 Credit risk

Credit risk (or **default risk**) reflects the likelihood that the security's issuer will default on payments of interest and/or principal. If the issuing firm is experiencing financial difficulties, it may not be able to honor the bond indenture and may default on a coupon (or principal) payment. To protect himself against such an occurrence, it is essential for the bond investor to know the issuer's future financial and business prospects.

While government bonds of OECD countries may be deemed as default risk free, the same is not true for corporate bonds. Since the actual payment on due time of corporate bond is more or less uncertain, it is important to know the **factors affecting credit risk**, in order to measure it and to try to manage it. The ability to repay debt is ultimately linked to the borrower's ability to generate **adequate cash flows**, the economic and financial current and prospective conditions of the firm are thus of primary interest and concern for bond's investors.

Usually they shape their judgment about the firm's ability to repay its debt both on economic and industry wide considerations and on firm specific considerations.

2.6.1 *Industry considerations*

As far as economic and industry wide considerations are concerned, the main factors which have to be taken into account are

- Economic cyclicality: the more an industry is cyclical the higher is the credit risk of firms operating in that industry, because cyclicality means variability of returns and this increases the likelihood of default;
- Growth prospects: the better the growth prospects of an industry, the lower the credit risk of companies operating within it;
- Research and development expenses: are in large part sunk costs that can be recouped only if the firm is viable. Different industries have different incidence of R&D expenses;
- Competition: the higher the competition in an industry the lower and more volatile the profit margins of the firms in it, and thus the cash flows that can be employed to service the debt
- Sources of supply: shortages in supply can be very harmful to the smooth conduct of operations by the issuer, to its profitability and thus to its ability to repay debt;
- Degree of regulation: the regulation of a particular industry tends to limit competition and thus to level the profitability of the subjects in that industry;
- Labor: the degree of unionization of an industry affects the profitability of firms operating in that industry and thus their ability to repay debt.

2.6.2 Ratio analysis

The financial resources needed to service the debt issued can come from three sources: cash flow from operations; liquidation of some asset; another source of financing. On the long run, however, the ability to repay debt comes essentially from **cash flows from operations**. Consequently, bondholders need to know something about the cash flows that the issuer is likely to generate. The main source of information relies on the analysis of balance sheet and income statement accounts and is synthesized in the analysis of **financial ratios** that relate different items of income statement and balance sheet.

Financial ratios are usually organized into categories: **common size ratios; profitability ratios; liquidity ratios; solvency (financial leverage) ratios; turnover ratios**.

a) Common size ratios express the relevant items of the balance sheet account as a percentage of total asset and relevant income statement account as a percentage of total revenue.

b) Profitability ratios express the firm's profitability compared to some investment base or to net sales and so measure the overall efficiency of the firm. The most common profitability ratios are:

ROE (Retun on Equity) $= \dfrac{\text{Net income}}{\text{Net worth}}$;
it is a summary measure of profitability;

ROA (Retun on Assets) $= \dfrac{\text{EBIT}}{\text{Total assets}}$;
it expresses the efficiency with which the management employs the total capital;

Profit margin $= \dfrac{\text{Net income}}{\text{Net sales}}$;
it measures the profit per dollar of net sales. Its complement to 1 (1 – profit margin) gives the expenses incurred in order to generate 1 dollar of revenues.

c) Liquidity ratios underline the firm's ability to meet its short term liabilities. Two main ratios are used:

Current ratio $= \dfrac{\text{Current asset}}{\text{Current liability}}$;
it shows the extent to which the claims of short term creditors are covered by assets readily convertible into cash;

Quick ratio $= \dfrac{\text{Current assets - inventory}}{\text{Current liability}}$,
in this case only already liquid assets are compared with current liabilities so that the effect of inventory valuation is purged.

d) Solvency (financial leverage) ratios underline the degree with which the creditors are financing the firm. From the bondholder point of view, the amount of equity invested in the firm represents a buffer against the decline in value of total assets. There are three main indices: debt ratio; interest coverage ratio; fixed-charge coverage ratio.

Debt ratio (Leverage) $= \dfrac{\text{Total debt}}{\text{Total assets}}$;
it represents the portion of assets financed by creditors. The higher the leverage the higher the risk of the firm since its net earnings are more volatile.

Interest Coverage ratio = $\frac{\text{Pre - tax income plus interest (EBIT)}}{\text{Interest expenses}}$;

it measures the degree by which the EBIT is absorbed by interest expenses; the lower the ratio, the higher the credit risk.

Fixed-charge Coverage ratio = $\frac{\text{Pre - tax income plus interest (EBIT)} + \text{lease payments}}{\text{Interest} + \text{lease expenses}}$;

it measures the degree by which the EBIT is absorbed by interest expenses and contractual commitments under leasing agreements.

e) **Activity ratios** (also called turnover ratios) measure the intensity with which the main assets are used to reach a given production. Three widely used activity ratios are: the average collection period; the fixed-asset turnover and the inventory turnover ratio.

Average collection period = $\frac{\text{Receivable}}{\text{Sales per day}}$;

it indicates the average time the firm has to wait to collect after making a sale; it measures the quality of the commercial credit extended by the firm and the effectiveness of its collections

Fixed asset turnover = $\frac{\text{Sales}}{\text{Fixed assets}}$;

it measures the sales per dollar of fixed assets. A low value indicates a below-capacity operations (slack of productive capacity) a high value may mean underinvestment in plant and equipment.

Inventory turnover ratio = $\frac{\text{Cost of sales}}{\text{Average inventory}}$;

it shows the effectiveness of inventory management and gives the number of times in a year that the firms rolls over the whole inventory. A too high level means a less than optimal inventory level (and risk of lost sales because of insufficient stock); a too low level may mean poor inventory and production management or even obsolete products.

A useful technique to summarize the main factors affecting the ROE (the main index, showing the ultimate source of income for the firm and so its capacity to refund debt) is given by the so called **Du Pont System** as follows

$$\textbf{ROE} = \frac{\text{Net income}}{\text{Pre - tax profits}} \cdot \frac{\text{Pre - tax profits}}{\text{EBIT}} \cdot \frac{\text{EBIT}}{\text{Sales}} \cdot \frac{\text{Sales}}{\text{Assets}} \cdot \frac{\text{Assets}}{\text{Equity}}$$

where the first term gives the so called tax-burden ratio (the average tax rate paid by the firm); the second gives the incidence of interest payments of the firm (the lower the interest paid, the higher the ratio); the third is the profit margin (also called ROS – Return on Sales); the fourth is the asset turnover ratio and the last is a measure of leverage.

2.6.3 Credit rating and rating agencies

The relative credit-risks of long-term bonds are assessed by various independent financial services firms which are known as the rating agencies. The two major rating agencies in the United states are Moody's Investors Service (Moody's), and Standard & Poor's Corporation (S&P). Their analysts analyse the various financial data such as fundamentals of the company, industry data and the macro-economic data to determine the possibility of the default in interest and/or principal payments. Finally, based on these analyses, rating agencies assign a **rating** to the issuer which is published in various publications available to investors.

The following table provides the brief definitions of Standard & Poor's and Moody's ratings.

S & P	Definition	Moody's corresponding rating
Investment-grade bonds		
AAA	Bonds rated AAA have the highest rating assigned to a debt obligation. Capacity to pay interest and repay principal is extremely strong.	**Aaa**
AA+ **AA** **AA-**	Bonds rated AA have a very strong capacity to pay interest and repay principal and differ from the highest rated issues only in a small degree.	**Aa1** **Aa2** **Aa3**
A+ **A** **A-**	Bonds rated A have a very strong capacity to pay interest and repay principal although they are somewhat more susceptible to the adverse effects of changes in the circumstances and economic conditions than bonds in higher rated categories.	**A1** **A2** **A3**
BBB+ **BBB** **BBB-**	Bonds rated BBB are regarded as having sufficient capacity to pay interest and repay principal. Whereas they normally exhibit adequate protection parameters, adverse economic conditions or changing circumstances are more likely to lead to a weakened capacity to pay interest and to repay principal for bonds in this category than for bonds in higher rated categories.	**Baa1** **Baa2** **Baa3**
Speculative / Low Creditworthiness		
BB+ **BB** **BB-** **B+** **B** **B-** **CCC+** **CCC** **CCC-** **CC+** **CC** **CC-**	Bonds rated BB, B, CCC and CC are regarded, on average, as predominantly speculative with respect to capacity to pay interest and repay principal in accordance with the terms of the obligation. BB indicates the lowest degree of speculation, and CC the highest degree of speculation. While such bonds will likely have some quality and protective characteristics, these are outweighed by large uncertainties or major risk exposures to adverse conditions	**Ba1** **Ba2** **Ba3** **B1** **B2** **B3** **Caa** **Ca**
Predominantly speculative / Substantial risk or in default		
C	The rating C is reserved for income bonds on which no interest is being paid.	**C**
D	Bonds rated D are in default and payment of interest and/or repayment of principal is in arrears.	

Figure 2-20: Standard & Poor's and Moody's ratings[22]

These ratings are used by market participants as a factor in the valuation of securities because of their independent and unbiased nature and because they directly reflect the probability of

22 Source: Standard & Poor's Corporation.

default. The following table shows the cumulative mortality losses (in %) by original Standard & Poor's rating, covering default and issues from 1981 to 2001.

Original rating	Years after issuance 1	2	3	4	5	6	7	8	9	10
AAA	0.00	0.00	0.03	0.07	0.10	0.18	0.27	0.41	0.46	0.52
AA	0.01	0.03	0.08	0.16	0.26	0.37	0.51	0.63	0.71	0.83
A	0.05	0.14	0.24	0.40	0.57	0.74	0.93	1.13	1.36	1.58
BBB	0.26	0.62	0.99	1.57	2.16	2.78	3.30	3.79	4.17	4.66
BB	1.22	3.49	6.14	8.50	10.59	12.65	14.10	15.30	16.49	17.40
B	5.96	12.68	18.25	22.28	25.06	27.18	29.09	30.56	31.63	32.61
CCC	24.72	33.06	38.40	42.60	46.87	48.48	49.62	50.02	51.28	52.22

Figure 2-21: Percentage of cumulative mortality losses[23]

Most investors view bonds rated Aaa-Baa as "investment" quality, with rating Ba-B designating "speculative", "high yield", or "junk" bonds. A rating of Caa-C is assigned to an extremely risky bond that may have already defaulted and be moving towards bankruptcy. Until recently, most junk bonds were "fallen angels", i.e. were issued by companies that had fallen on hard times. But in the 1980s a lot of new junk bonds have been issued with a low rating at issue (to finance or fight against a take-over).

Many organisations are not allowed to invest in bonds with ratings lower than a certain category. For example, commercial banks in the United States, many pension funds and other financial institutions are not allowed to invest in bonds which are rated lower than investment-grade.

Bond ratings have a direct influence on the borrowing costs of the issuer, as shown in the following table:

Date	AAA	AA	A	BBB	BB	B
Q1 99	5.62	6.01	6.12	6.44	7.72	10.87
Q2 99	6.18	6.50	6.63	7.00	8.24	10.74
Q3 99	6.63	6.99	7.15	7.50	8.97	11.31
Q4 99	6.93	7.13	7.29	7.71	9.26	11.61
Q1 00	7.42	7.60	7.73	8.06	9.68	11.85
Q2 00	7.46	7.68	7.91	8.42	10.26	12.78
Q3 00	7.08	7.37	7.61	8.03	9.85	12.88
Q4 00	6.67	7.05	7.46	7.90	10.20	14.86
Q1 01	5.81	6.22	6.81	7.37	9.12	13.47
Q2 01	5.79	6.25	6.68	7.43	8.98	14.36
Q3 01	5.49	5.82	6.31	7.08	9.28	14.45
Q4 01	5.13	5.45	6.06	6.97	9.63	13.73
Q1 02	5.37	5.65	6.13	7.20	9.02	11.98
Q2 02	5.28	5.55	6.04	7.30	9.12	11.90
Q3 02	4.68	4.90	5.45	7.34	9.97	13.36
Q4 02	4.31	4.47	5.15	7.08	9.80	13.66

Figure 2-22: Cost of borrowing and issuer quality[24]

Even in high quality bonds, there is a difference of 16 to 46 basis points (depending on the quarter) between AAA and AA bonds, thus ratings are very important for bond issuers.

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23 Source: www.standardandpoors.com

24 Source: Bloomberg. The table gives the average yields in percentages for bonds with different ratings denominated in USD.