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Mortar spectral element discretization of the Stokes problem in domain with corners

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Abstract

The solution of the Stokes problem in a polygonal domain of $\mathbb{R}^2$ is in general not regular. But it can be written as the sum of a regular part and a linear combination of singular functions. We propose a numerical analysis of the Strang and Fix algorithm by mortar spectral element methods which leads to an Inf-Sup condition on the pressure in a non-conforming decomposition. We prove optimal error estimates for the velocity and the pressure.

Keywords: Stokes problem; singularity; mortar spectral element method

1 Introduction

In a polygonal domain of $\mathbb{R}^2$, the solution of the Stokes problem is decomposed into a regular part and a singular one. Since the singular functions of elliptic problem of fourth order (bilaplacian with homogeneous boundary conditions) are explicitly known (see [1, 2]), we deduce the singularities of the velocity and the pressure [3]. For the approximation of these singular functions we mention the work of Babuška and Suri [4] for the p-version of the finite elements method. In their work [5], Bernardi and Maday extended the approximation to the spectral method. They proved that the polynomial approximation order is double contrary to what is expected by the general approximation theory.

This work is an extension of our previous one where we dealt with Laplacian operator [6]. We carried out the numerical analysis of the mortar spectral element method for the stokes problem. The domain is decomposed into a union of finite number of disjoint rectangles. The discrete functions are polynomials of high degree on each rectangle and are enforced to satisfy a matching condition on the interfaces. This technique is non-conforming because the discrete functions are not continuous. We refer to Bernardi *et al.* [7] for the introduction of the mortar spectral element method.

We define the discrete spaces of the velocity and the pressure. The latter is free of spurious modes since we adopt the $\mathbb{P}_N \times \mathbb{P}_{N-2}$ method. We prove the Inf-Sup condition in a non-conforming geometry [8]. We also prove that the order of the error is doubled if we consider separately the regular part and the two first singular functions of the velocity and the pressure. We handle the Strang and Fix algorithm [9] that permits one to add the first singular function to the discrete space of the velocity. This algorithm was widely used for the finite element method (see [4, 10, 11]).

We define the discrete problem with two bilinear forms. The bilinear form, defined only in the velocity space, contains singular integrals. This integrals are not well estimated by the Gauss-Lobatto quadrature formula on each sub-domain. We prove that the continuity and the coercivity of the bilinear form which are obtained by two different equivalent norms. The equivalence constant tends to 0 when N tends to infinity. When using the continuity norm, we prove two Inf-Sup conditions on the two bilinear forms. One of them is proven on the kernel, the other gives the compatibility between the new space of the velocity and the pressure space. We obtain an optimal estimation of the error with the norm H^1 which is the approximation order of the second singular function.

An outline of this paper is as follows. In Section 2, we present the geometry of the domain and the continuous problem. In Section 3, we define the discrete problem. Section 4 is devoted to the numerical analysis and the error estimation of the mortar spectral element method of the Strang and Fix algorithm. Finally, in Section 5, we conclude our paper.

2 Geometry of the domain and continuous problem

We denote by Ω a polygonal domain in $\mathbb{R}^2$ such that there exist a finite number of open rectangles Ω_k, $1 \le k \le K$, satisfying

$$\bar{\Omega} = \bigcup_{k=1}^{K} \bar{\Omega}_k \quad \text{and} \quad \Omega_k \cap \Omega_l = \emptyset \quad \text{for } k \neq l, \tag{2.1}$$

and such that the intersection of each $\bar{\Omega}_k$, $1 \le k \le K$, with the boundary $\partial\Omega$ is either empty or a corner or one of several entire edges of Ω_k. We choose the coordinate axes parallel to the edge of the Ω_k. We are interested in non-convex domains; we assume that there exists an angle equal either to $\frac{3\pi}{2}$ or to 2π (case of the crack). Handling the singular function is a local process, so that there is no restriction on the supposition that the non-convex corner is unique.

Assumption 1 Let ω be the value of the non-convex angle equal either to $\frac{3\pi}{2}$ or to 2π, $\mathbf{a}$ be the corresponding corner of Ω and Δ be the open domain in Ω such that $\bar{\Delta}$ is the union of the $\bar{\Omega}_k$ which contains $\mathbf{a}$. We choose the origin of the coordinate axes at the point $\mathbf{a}$, we introduce a system of polar coordinates (r, θ) where r stands for the distance starting from $\mathbf{a}$ and θ is such that the line $\theta = 0$ contains an edge of $\partial\Omega$. For reasons which will appear later, we are led to assume the following. If the intersection of $\bar{\Omega}_k$ and $\bar{\Omega}_l$, $k \neq l$, contains $\mathbf{a}$, it contains either or both an edge of Ω_k and Ω_l.

We consider the velocity-pressure formulation of the Stokes problem on the domain Ω. Find the velocity $\mathbf{u}$ and the pressure p such that

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \Gamma, \end{cases} \tag{2.2}$$

ν is the viscosity of the fluid that we suppose to be a positive constant, $\mathbf{f}$ is the data which represents a density of body forces. Then for $\mathbf{f}$ in $[H^{-1}(\Omega)]^2$, the functional spaces are $[H_0^1(\Omega)]^2$ for the velocity and $L_0^2(\Omega)$ for the pressure where

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega), \int_\Omega q(x)\,dx\,dy = 0 \right\}.$$

The problem (2.2) is equivalent to the following variational formulation.

For $\mathbf{f}$ in $[H^{-1}(\Omega)]^2$, find $\mathbf{u}$ in $[H_0^1(\Omega)]^2$ and p in $L_0^2(\Omega)$ such that for all $\mathbf{v}$ in $[H_0^1(\Omega)]^2$ and for all q in $L^2(\Omega)$:

$$\begin{cases} \mathbf{a}(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = \langle f,\mathbf{v}\rangle, \\ b(\mathbf{u},q) = 0, \end{cases} \tag{2.3}$$

where

$$\mathbf{a}(\mathbf{u},\mathbf{v}) = \nu \int_\Omega \nabla\mathbf{u}\nabla\mathbf{v}\,dx\,dy$$

and

$$b(\mathbf{u},q) = -\int_\Omega (\operatorname{div}\mathbf{u})q\,dx\,dy,$$

where the $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. The bilinear form $\mathbf{a}(\cdot,\cdot)$ is continuous on the space $[H_0^1(\Omega)]^2 \times [H_0^1(\Omega)]^2$ and elliptic on $[H_0^1(\Omega)]^2$. The bilinear form $b(\cdot,\cdot)$ is continuous and verifies the following Inf-Sup condition [12, 13]: there exists a positive constant β such that

$$\forall q \in L_0^2(\Omega), \qquad \sup_{\mathbf{v}\in[H_0^1(\Omega)]^2} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_{[H^1(\Omega)]^2}} \geq \beta\|q\|_{L^2(\Omega)}.$$

Then we conclude that for all $\mathbf{f}$ in the space $[H^{-1}(\Omega)]^2$, the problem (2.3) has a unique solution $(\mathbf{u},p)$ in $[H_0^1(\Omega)]^2 \times L_0^2(\Omega)$. This solution verifies the following stability condition ([14], Chapter 1):

$$\|\mathbf{u}\|_{[H^1(\Omega)]^2} + \beta\|p\|_{L^2(\Omega)} \leq C\|\mathbf{f}\|_{[H^{-1}(\Omega)]^2}, \tag{2.4}$$

where C is a positive constant.

3 The discrete problem

In this section, we recall some basic notions related to the spectral element method and the mortar matching condition. Since the discretization is essentially a Galerkin method, we have to define the discrete space and give the quadrature formula which is used to compute the integrals of polynomials.

The discretization parameter is a K-tuple of integers $N_1,\ldots,$ and N_K larger than or equal to 2, denoted by δ. For any nonnegative integer n and for $1 \leq k \leq K$, we denote by $\mathbb{P}_n(\Omega_k)$ the space of polynomials on Ω_k such that their degree with respect to each variable x and y is less than or equal to n. The restriction of discrete functions to Ω_k will belong to $\mathbb{P}_{N_k}(\Omega_k)$. Let us recall the Gauss-Lobatto quadrature formula, for any positive integer n: there exists a unique set of $(n+1)$ nodes $\xi_0 = -1$, $\xi_n = 1$, $\xi_j^n \in]-1,1[$, $1 \leq j \leq (n-1)$, and of $(n+1)$ positive weights ρ_j^n, $0 \leq j \leq n$, such that the following equality holds for any polynomial ϕ with degree less than or equal to $2n-1$,

$$\int_{-1}^{1} \phi(z)\,dz = \sum_{j=0}^{n} \phi(\xi_j^n)\rho_j^n. \tag{3.1}$$

If T^k denotes an affine mapping from $]-1,1[^2$ onto Ω_k, we define a bilinear form on continuous functions on $\bar{\Omega}_k$ by

$$(u,v)_{N_k} = \frac{|\Omega_k|}{4} \sum_{i=0}^{N_k} \sum_{j=0}^{N_k} \left(uoT^k\right)\left(\xi_i^{N_k}, \xi_j^{N_k}\right)\left(voT^k\right)\left(\xi_i^{N_k}, \xi_j^{N_k}\right)\rho_i^{N_k}\rho_j^{N_k}, \tag{3.2}$$

where $|\Omega_k|$ stands for the area of Ω_k.

We need some more notations to enforce the matching condition. Let $\surd$ be the set of all the corners of the Ω_k, $1 \le k \le K$. We denote by $\Gamma^{k,j}$, $1 \le j \le 4$ the edges of Ω_k and $\gamma^{kl} = \bar{\Omega}_k \cap \bar{\Omega}_l$, $k \ne l$. We make the further assumption that the boundary $\partial\Omega$ consists of entire edges of the Ω_k. We introduce the skeleton of the decomposition:

$$S = \left(\bigcup_{k=1}^{K} \partial\Omega_k\right) \Big\backslash \partial\Omega, \tag{3.3}$$

and we assume that it is a disjoint union of mortars (γ_m), $1 \le m \le M$ (M is a positive integer),

$$S = \bigcup_{m=1}^{M} \gamma_m \quad \text{and} \quad \gamma_m \cap \gamma_{m'} = \emptyset \quad \text{for } m \ne m', \tag{3.4}$$

where each mortar γ_m is an entire edge of one rectangle Ω_k, denoted by $\Omega_{k(m)}$. For any nonnegative integer n and for any segment γ, we denote by $\mathbb{P}_n(\gamma)$ the space of polynomials with degree less than or equal to n on γ. The mortar space W_δ is then defined by

$$W_\delta = \left\{\phi \in L^2(S)/\forall m, 1 \le m \le M, \phi/_{\gamma_m} \in \mathbb{P}_{N_{k(m)}}(\gamma_m)\right\}. \tag{3.5}$$

The space X_δ is then defined as in the standard mortar method ([7], Chapter 3, Section 1). It represents the space of function v_δ in $L^2(\Omega)$ such that

- the restriction of v_δ to Ω_k, $1 \le k \le K$, belongs to $\mathbb{P}_{N_k}(\Omega_k)$,
- v_δ vanishes on $\partial\Omega$,
- the mortar function φ defined on S by

$$\varphi/_{\gamma_m} = v_\delta/\Omega_{k(m)}/\gamma_m, \quad 1 \le m \le M,$$

verifies, for $1 \le k \le K$ and for any edge Γ of Ω_k contained in S,

$$\forall\psi \in \mathbb{P}_{N_k-2}(\Gamma), \quad \int_\Gamma (v_\delta/_{\Omega_k} - \varphi)(\tau)\psi(\tau)\,d\tau = 0. \tag{3.6}$$

Let the space

$$X_\delta^- = \left\{v_\delta \in X_\delta/v_{\delta/\Omega_k} \in \mathbb{P}_{N_k-1}(\Omega_k), 1 \le k \le K\right\}.$$

Then we define $Y_\delta = X_\delta \times X_\delta$ as the discrete space of the velocity. Later we will need to define the space $Y_\delta^- = X_\delta^- \times X_\delta^-$. For the discrete pressure we consider the space

$$M_\delta = \left\{p_\delta \in L^2(\Omega)/p_{\delta/\Omega_k} \in \mathbb{P}_{N_k-2}(\Omega_k) \text{ and } \int_\Omega p_\delta(x,y)\,dx\,dy = 0\right\}.$$

This corresponds to the case where the pressure has no spurious modes [15]. The space Y_δ is provided by the norm $\|\cdot\|$ defined by

$$\|\mathbf{v}_\delta\| = \left(\sum_{k=1}^{K} \|\mathbf{v}_{\delta/\Omega_k}\|_{[H^1(\Omega_k)]^2}\right)^{1/2}.$$

We define the following scalar product on Ω, for all u and v continuous on each $\bar{\Omega}_k$:

$$(u,v)_\delta = \sum_{k=1}^{K} (u,v)_{N_k}.$$

The discrete bilinear forms are defined as follows: for all u and v, in X_δ

$$a_\delta(u,v) = (\nabla u, \nabla v)_\delta$$

and for w, in X_δ and q, in M_δ,

$$b_\delta(w,q) = -(\operatorname{div} w, q)_\delta.$$

Then we define the discrete problem. For the data $\mathbf{f} = (f_1, f_2)$ continuous on $\bar{\Omega}$, find $\mathbf{u}_\delta = (u_{\delta 1}, u_{\delta 2})$ in the space Y_δ and p_δ in the space M_δ such that, for all $\mathbf{w}_\delta = (w_{\delta 1}, w_{\delta 2})$ in Y_δ and for all q_δ in M_δ

$$\begin{cases} \mathbf{a}_\delta(\mathbf{u}_\delta, \mathbf{w}_\delta) + b_\delta(\mathbf{w}_\delta, p_\delta) = (\mathbf{f}, \mathbf{w}_\delta)_\delta, \\ b_\delta(\mathbf{u}_\delta, q_\delta) = 0, \end{cases} \tag{3.7}$$

where $\mathbf{a}_\delta(\mathbf{u}_\delta, \mathbf{w}_\delta) = a_\delta(u_{\delta 1}, w_{\delta 1}) + a_\delta(u_{\delta 2}, w_{\delta 2})$ and $(\mathbf{f}, \mathbf{w}_\delta)_\delta = (f_1, w_{\delta 1})_\delta + (f_2, w_{\delta 2})_\delta$. The bilinear form $\mathbf{a}_\delta(\cdot,\cdot)$ is continuous and elliptic on the space Y_δ with a norm and a constant of ellipticity independent of δ. The bilinear form $b_\delta(\cdot,\cdot)$ is continuous on $Y_\delta \times M_\delta$, and its norm is independent of δ. Indeed, by the exactness of the quadrature formula on each sub-domain Ω_k, we conclude that for all $\mathbf{w}_\delta$ in Y_δ and for all q_δ in M_δ

$$b_\delta(\mathbf{w}_\delta, q_\delta) = -\sum_{k=1}^{K} \int_{\Omega_k} \operatorname{div} \mathbf{w}_{\delta/\Omega_k} q_{\delta/\Omega_k} \, dx\, dy = b(\mathbf{w}_\delta, q_\delta).$$

To show that the problem (3.7) is well posed, we need to verify the existence of a global Inf-Sup condition of the form $b_\delta(\cdot,\cdot)$.

For $q_\delta \in M_\delta$, let $q_k = q_{\delta/\Omega_k}$ for all k, $1 \le k \le K$, q_k is decomposed as

$$q_k = \tilde{q}_k + \bar{q}_k, \tag{3.8}$$

where $\tilde{q}_k \in M_k(\Omega_k) = \mathbb{P}_{N_k-2}(\Omega_k) \cap L_0^2(\Omega_k)$ and $\bar{q}_k = \frac{1}{|\Omega_k|}\int_{\Omega_k} q_k(x,y)\, dx\, dy$. We consider the space

$$Y_k(\Omega_k) = \left\{\mathbf{v}_k \in \left[H^1(\Omega_k)\right]^2, \mathbf{v}_k \in \left[\mathbb{P}_{N_k}(\Omega_k)\right]^2, \ \mathbf{v}_{k/\partial\Omega_k \cap \partial\Omega} = 0\right\}.$$

We know that there is a local discrete Inf-Sup condition verified by the restriction of $b_\delta(\cdot,\cdot)$ on $Y_k(\Omega_k) \times M_k(\Omega_k)$ ([16], Chapter IV, Proposition 7.2).

Then using an argument of Boland and Nicolaides [17], we prove the global Inf-Sup condition. Since $q_\delta \in M_\delta$, thus $\sum_{k=1}^K \bar{q}_k|\Omega_k| = 0$. By the local Inf-Sup condition, we deduce that there exists $\tilde{\mathbf{v}}_k \in Y_k(\Omega_k)$ satisfying

$$\int_{\Omega_k} (\operatorname{div} \tilde{\mathbf{v}}_k)\tilde{q}_k \, dx = -\|\tilde{q}_k\|^2_{L^2(\Omega_k)} \tag{3.9}$$

and

$$\|\tilde{\mathbf{v}}_k\|_{[H^1(\Omega_k)]^2} \leq \frac{1}{\beta_k}\|\tilde{q}_k\|_{L^2(\Omega_k)}. \tag{3.10}$$

This reduces the problem dealing with the constant pressure on sub-domains, where we define the space below:

$$\bar{M}(\Omega) = \left\{ \bar{q} = (\bar{q}_k)_{1\leq k\leq K} \in \mathbb{R}^K, \int_\Omega \bar{q}\, dx = \sum_{k=1}^K \bar{q}_k|\Omega_k| = 0 \right\}. \tag{3.11}$$

For an integer I, let X_I be the reduced space for the velocity. X_I is the set of the functions v such that

- for all k, $1 \leq k \leq K$, the restriction of v on Ω_k belongs to $\mathbb{P}_{2(I-2)}(\Omega_k)$;
- $v_{/\Omega_k}$, $1 \leq k \leq K$, vanishes on $\partial\Omega_k \cap \Gamma$;
- the function v is continuous on $\bar{\Omega}$.

We consider $Y_I = X_I \times X_I$, and we note that the condition of continuity is sufficient to have the space X_I included in the mortar space. The main idea is to prove that the spaces X_I and $\bar{M}$ are compatible under an Inf-Sup condition for an appropriate choice of I. Since these spaces are for finite dimension and do not depend on the discretization, we need to prove that the pressure has no spurious modes. We begin by announcing the following lemma (see [8] for the proof).

Lemma 3.1 *There exists an integer J not depending on the decomposition such that the set of the functions q of $\bar{M}$ for which*

$$\forall \mathbf{v} \in Y_J, \quad b(\mathbf{v}, q) = 0 \tag{3.12}$$

is reduced to $\{0\}$.

The result of the global Inf-Sup condition is given by the following proposition.

Proposition 3.2 *For all q_δ in M_δ,*

$$\sup_{\mathbf{w}_\delta \in Y_\delta} \frac{b_\delta(\mathbf{w}_\delta, q_\delta)}{\|w_\delta\|} \geq \beta_\delta \|q_\delta\|_{L^2(\Omega)}, \tag{3.13}$$

where $\beta_\delta = \inf(\frac{1}{\sqrt{N_k}})$, $1 \leq k \leq K$.

Proof From Lemma 3.1 we conclude that if $\bar{q}$ in $\bar{M}$ there exists $\bar{\mathbf{v}}$ in Y_d such that

$$\int_\Omega (\operatorname{div} \bar{\mathbf{v}})\bar{q}\, dx = -\|\bar{q}\|^2_{L^2(\Omega)} \tag{3.14}$$

and

$$\|\bar{\mathbf{v}}\|_{[H^1(\Omega)]^2} \le C_1\|\bar{q}\|_{L^2(\Omega)}, \tag{3.15}$$

where C_1 is a constant independent of δ.

With $q_\delta \in M_\delta$, we associate the function $\mathbf{v}_\delta \in Y_\delta$ such that

$$\mathbf{v}_{\delta/\Omega_k} = \bar{\mathbf{v}}_{\delta/\Omega_k} + \alpha_k\tilde{\mathbf{v}}_k, \tag{3.16}$$

for a fixed constant α_k.

Using an argument of Boland and Nicolaides, we find the value of α_k such that $(\mathbf{v}_\delta, q_\delta)$ satisfies the Inf-Sup condition,

$$\begin{aligned} b_\delta(\mathbf{v}_\delta, q_\delta) &= -\int_\Omega q_\delta \operatorname{div}\mathbf{v}_\delta\,dx \\ &= -\sum_{k=1}^K \int_{\Omega_k} (\tilde{q}_k + \bar{q}_k)\operatorname{div}(\bar{\mathbf{v}}_{/\Omega_k} + \alpha_k\tilde{\mathbf{v}}_k)\,dx \\ &= -\sum_{k=1}^K \biggl(\alpha_k\biggl(\int_{\Omega_k} \bar{q}_k \operatorname{div}\tilde{\mathbf{v}}_k\,dx + \int_{\Omega_k} \tilde{q}_k \operatorname{div}\tilde{\mathbf{v}}_k\,dx\biggr)\biggr) \\ &\quad + \sum_{k=1}^K \biggl(\int_{\Omega_k} \bar{q}_k \operatorname{div}\bar{\mathbf{v}}\,dx + \int_{\Omega_k} \tilde{q}_k \operatorname{div}\bar{\mathbf{v}}\,dx\biggr). \end{aligned} \tag{3.17}$$

We evaluate each term of the equality (3.17).

(1) $\int_{\Omega_k} \bar{q}_k \operatorname{div}\tilde{\mathbf{v}}_k\,dx$.

Since $\bar{q}_k$ is a constant, this implies that

$$\int_{\Omega_k} \bar{q}_k \operatorname{div}\tilde{\mathbf{v}}_k\,dx = \bar{q}_k \int_{\Omega_k} \operatorname{div}\tilde{\mathbf{v}}_k\,dx = \bar{q}_k \int_{\partial\Omega_k} \tilde{\mathbf{v}}_k \cdot n\,d\tau = 0,$$

since $\tilde{\mathbf{v}}_k \in Y_k(\Omega_k)$.

(2) $\int_{\Omega_k} \tilde{q}_k \operatorname{div}\tilde{\mathbf{v}}_k\,dx$.

According to (3.9)

$$\int_{\Omega_k} \tilde{q}_k \operatorname{div}\tilde{\mathbf{v}}_k\,dx = -\|\tilde{q}_k\|^2_{L^2(\Omega_k)}. \tag{3.18}$$

(3) $\sum_{k=1}^K \int_{\Omega_k} \bar{q}_k \operatorname{div}\bar{\mathbf{v}}_\delta\,dx$.

From (3.14) we have

$$\sum_{k=1}^K \int_{\Omega_k} \bar{q}_k \operatorname{div}\bar{\mathbf{v}}\,dx = -\|\bar{q}_\delta\|^2_{L^2(\Omega)}. \tag{3.19}$$

We insert (3.18) and (3.19) in (3.17) and obtain

$$b_\delta(\mathbf{v}_\delta, q_\delta) = \sum_{k=1}^K \biggl(\alpha_k\|\tilde{q}_k\|^2_{L^2(\Omega_k)} - \int_{\Omega_k} \tilde{q}_k \operatorname{div}\bar{\mathbf{v}}_\delta\,dx\biggr) + \|\bar{q}\|^2_{L^2(\Omega)},$$

and from (3.15) we have

$$b_\delta(\mathbf{v}_\delta, q_\delta) \le \sum_{k=1}^{K}\left(\alpha_k\|\tilde{q}_k\|^2_{L^2(\Omega_k)} - \frac{C_1^2}{2}\|\tilde{q}\|^2_{L^2(\Omega_k)}\right) + \frac{\|\bar{q}\|^2_{L^2(\Omega)}}{2}.$$

Then if $\alpha_k = C_1^2$, considering (3.8), and that $\tilde{q}_k$ and $\bar{q}_k$ are orthogonal in the sense of the scalar product of $L^2(\Omega_k)$, we have

$$\|q_\delta\|^2_{L^2(\Omega)} = \|\bar{q}_\delta\|^2_{L^2(\Omega)} + \|\tilde{q}_\delta\|^2_{L^2(\Omega)},$$

implying that

$$b_\delta(\mathbf{v}_\delta, q_\delta) = \|q_\delta\|^2_{L^2(\Omega)}.$$

From (3.15) we bound $\mathbf{v}_\delta$,

$$\|\mathbf{v}_\delta\| \le \left(\sum_{k=1}^{K}\|\mathbf{v}_{\delta/\Omega_k}\|^2_{[H^1(\Omega_k)]^2}\right)^{\frac{1}{2}} \le C\left(\sum_{k=1}^{K}\left\{\|\bar{q}_\delta\|_{L^2(\Omega_k)} + C_1^2\|\tilde{\mathbf{v}}_{k/\Omega_k}\|_{[H^1(\Omega_k)]^2}\right\}^2\right)^{\frac{1}{2}},$$

and from (3.10) we conclude that

$$\begin{aligned}\|\mathbf{v}_\delta\| &\le C\left(\sum_{k=1}^{K}\left\{\|\bar{q}_{\delta/\Omega_k}\|_{L^2(\Omega_k)} + \frac{C_1^2}{\beta_k}\|\tilde{q}\|_{L^2(\Omega_k)}\right\}^2\right)^{\frac{1}{2}} \\ &\le C\left(\sum_{k=1}^{K}\left\{\sup\left(1, \frac{C_1^2}{\beta_k}\right)\|q_{\delta/\Omega_k}\|_{L^2(\Omega_k)}\right\}^2\right)^{\frac{1}{2}}.\end{aligned}$$

If $\beta_k^* = \sup(1, \frac{C_1^2}{\beta_k})$, then

$$\|\mathbf{v}_\delta\| \le C \sup_{1\le k\le K}\left(\beta_k^*\right)\|q_\delta\|_{L^2(\Omega)},$$

which completes the demonstration. □

Proposition 3.3 *For all* $\mathbf{f}$ *in* $[L^2(\Omega)]^2$, *the problem* (2.2) *has a unique solution* $(\mathbf{u}_\delta, p_\delta)$ *in* $Y_\delta \times M_\delta$ *verifying*

$$\|\mathbf{u}_\delta\| + \beta_\delta\|p_\delta\|_{L^2(\Omega)} \le C\|\mathbf{f}\|_{[L^2(\Omega)]^2}.$$

We consider the discrete kernel of the bilinear form $b_\delta(\cdot,\cdot)$

$$V_\delta = \left\{\mathbf{v}_\delta \in Y_\delta, b_\delta(\mathbf{v}_\delta, q_\delta) = 0, \forall q_\delta \in M_\delta\right\}.$$

We have the case of the approximation of a saddle point problem by non-conforming discretization. Using Strang's lemma we consider the following error estimate ([18], Chapter 16) and ([19], Chapter 1, Section 4).

Proposition 3.4 *Let* $\mathbf{f}$ *such that* $\mathbf{f}_{/\Omega_k}$ *belongs to* $[H^{\rho_k}(\Omega_k)]^2$, $\rho_k > 1$. *The error estimate between the solution* $(\mathbf{u}, p)$ *of the continuous problem* (2.2) *and the solution* $(\mathbf{u}_\delta, p_\delta)$ *of the discrete problem* (3.7) *is as follows*:

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_\delta\| + \beta_\delta \|p - p_\delta\|_{L^2(\Omega)} \leq C \Biggl(& \inf_{\mathbf{v}_\delta \in V_\delta} \|\mathbf{u} - \mathbf{v}_\delta\| + \inf_{q_\delta \in M_\delta} \|p - q_\delta\|_{L^2(\Omega)} \\ & + \sum_{k=1}^{K} N_k^{-\rho_k} \|\mathbf{f}\|_{H^{\rho_k}(\Omega_k)} \\ & + \sup_{\mathbf{w}_\delta \in V_\delta} \frac{\sum_{k=1}^{K} \sum_{k=1}^{K} \int_{\gamma^{kl}} (-\nu \frac{\partial \mathbf{u}}{\partial n} + pn)[\mathbf{w}_\delta]}{\|\mathbf{w}_\delta\|} \Biggr), \end{aligned} \tag{3.20}$$

where C *is a constant independent of* δ, *the* $[\mathbf{w}_\delta]$ *represents the jump of* $\mathbf{w}_\delta$ *through* γ^{kl}.

To estimate each term in the inequality (3.20), we start by estimating the best approximation of the velocity by elements of the kernel V_δ. We introduce the orthogonal projection operator $\tilde{\Pi}_N^2$ from $H_0^2(\Omega)$ to $\mathbb{P}_N^{2,0}(\Omega)$, the space of polynomial functions vanishing with its normal derivative on the boundary of Ω. $\tilde{\Pi}_N^2$ preserves the trace, the trace of the normal derivative on Γ, and the values on the corners; see ([20], Chapter II) for the properties of this operator. Let us consider the following lemma; see ([20], Chapter II, Theorem 6.2) for its proof.

Lemma 3.5 *For any real number* s *such that* $s \geq 4$, *there exists a positive constant* C *such that for any function* v *in the space* $H^s(\Omega) \cap H_0^2(\Omega)$ *we have*

$$\left\| v - \tilde{\Pi}_N^2 v \right\|_{H^2(\Omega)} \leq C N^{2-s} \|v\|_{H^s(\Omega)}. \tag{3.21}$$

We introduce the lifting operator R_N stable for the norm $H^2(\Omega)$. We consider the following lemma; see [21] for its proof.

Lemma 3.6 *For all edge* γ *of* $\bar{\Omega}$ *with a measure different of zero, there exists a lifting operator* R_N *from* $\mathbb{P}_N^0(\gamma) \times \mathbb{P}_N^{2,0}(\gamma)$, *to* $\mathbb{P}_N(\Omega)$ *such that* $\forall \phi = (\varphi_0, \varphi_1) \in \mathbb{P}_N^0(\gamma) \times \mathbb{P}^{2,0}(\gamma)$ *we have*

$$\partial_n^k R_N(\phi) = \begin{cases} \varphi_k & \text{in } \gamma, \\ 0 & \text{on } \Omega \backslash \gamma, \end{cases}$$

where $k \in \{0, 1\}$, *and for all real* s *such that* $\frac{3}{2} < s \leq \frac{5}{2}$,

$$\left\| R_N(\phi) \right\|_{H^s(\Omega)} \leq C \bigl(\|\varphi_0\|_{H^{s-\frac{1}{2}}(\gamma)} + \|\varphi_1\|_{H^{s-\frac{3}{2}}(\gamma)} \bigr).$$

The following proposition gives us the error bound of the best approximation.

Proposition 3.7 *We suppose that the geometry of the decomposition is conforming, then*

$$\inf_{\mathbf{v}_\delta \in V_\delta} \|\mathbf{u} - \mathbf{v}_\delta\| \leq \sum_{k=1}^{K} N_k^{1-s} \|\mathbf{u}\|_{[H^s(\Omega_k)]^2}. \tag{3.22}$$

Proof The demonstration involves to associate the stream function of $\mathbf{u}$. On each subdomain Ω_k, we apply a first approximation of $\mathbf{u}$ and then adjust to satisfy the condition on the interfaces.

Let $\mathbf{u}$ belongs to $[H^s(\Omega_k)]^2$ for $s \geq 3$, $\operatorname{div}(\mathbf{u}) = 0$ then there exists a stream function ψ in $H^{s+1}(\Omega_k) \cap H_0^2(\Omega_k)$ such that [19]

$$\mathbf{u} = \mathbf{curl}(\psi)$$

and

$$\|\psi/_{\Omega_k}\|_{H^{s+1}(\Omega_k)} \leq C\|\mathbf{u}/_{\Omega_k}\|_{[H^s(\Omega_k)]^2}.$$

We define $\psi_\delta = (\psi_\delta^k)$ such that

$$\psi_\delta^k = \tilde{\Pi}_{N_k}^2(\psi/_{\Omega_k}),$$

where $\tilde{\Pi}_{N_k}^2$ is the projection operator defined in Lemma 3.5, then

$$\mathbf{u}_\delta^k = \mathbf{curl}\big(\psi_\delta^k\big)$$

and it satisfies

$$\big\|\mathbf{u}/_{\Omega_k} - \mathbf{u}_\delta^k\big\|_{[H^1(\Omega_k)]^2} \leq \big\|\mathbf{curl}(\psi) - \mathbf{curl}\big(\psi_\delta^k\big)\big\|_{[H^1(\Omega_k)]^2} \leq \big\|\psi - \psi_\delta^k\big\|_{H^2(\Omega_k)}$$

and by Lemma 3.5

$$\big\|\mathbf{u}/_{\Omega_k} - \mathbf{u}_\delta^k\big\|_{[H^1(\Omega_k)]^2} \leq CN_k^{1-s}\big\|\psi_\delta^k\big\|_{H^{s+1}(\Omega_k)},$$

which gives us

$$\big\|\mathbf{u}/_{\Omega_k} - \mathbf{u}_\delta^k\big\|_{[H^1(\Omega_k)]^2} \leq CN_k^{1-s}\|\mathbf{u}/_{\Omega_k}\|_{[H^s(\Omega_k)]^2}. \tag{3.23}$$

$\mathbf{u}_\delta = (\mathbf{u}_\delta^k)$ is a good approximation of $\mathbf{u}$ on each Ω_k, but it does not check the compatibility condition on the interfaces. To find a solution, we define on each mortar γ_m, r_δ^m by

$$r_\delta^m = \begin{cases} R_{N_k}^{k(m)}(\tilde{\Pi}_{N_k}^2(\psi_\delta), \tilde{\Pi}_{N_k}^2(\frac{\partial\psi_\delta}{\partial n})) & \text{in } \Omega_{k(m)}, \\ 0 & \text{on } \Omega\backslash\bar{\Omega}_k, \end{cases}$$

where $R_{N_k}^{k(m)}$ is the traces lifting operator on $\Omega_{k(m)}$ deduced from R_N, by translation and dilation; see Lemma 3.6.

We check that $\mathbf{v}_\delta = \mathbf{u}_\delta + \mathbf{curl}(r_\delta^m)$ is in the space Y_δ. We obtain the next estimation:

$$\begin{aligned}\big\|r_\delta^m\big\|_{H^2(\Omega_k)} \leq C\bigg(&\|\psi_\delta\|_{H_0^{\frac{1}{2}}(\gamma_m)} + \big\|\big(id - \tilde{\Pi}_{N_k}^2\big)\psi_\delta\big\|_{H_0^{\frac{1}{2}}(\gamma_m)} \\ &+ \bigg\|\frac{\partial\psi_\delta}{\partial n}\bigg\|_{H_{00}^{\frac{3}{2}}(\gamma_m)} + \bigg\|\big(id - \tilde{\Pi}_{N_k}^2\big)\frac{\partial\psi_\delta}{\partial n}\bigg\|_{H_{00}^{\frac{3}{2}}(\gamma_m)}\bigg).\end{aligned}$$

According to the definition of the operator $\tilde{\Pi}^2$ we obtain

$$\left\| r_\delta^m \right\|_{H^2(\Omega_k)} \leq C N_k^{1-s} \|\psi\|_{H^{s+1}(\Omega_k)} \leq C N_k^{1-s} \|\mathbf{u}\|_{[H^s(\Omega_k)]^2}, \tag{3.24}$$

and applying the triangle inequality we have

$$\|\mathbf{u} - \mathbf{v}_\delta\| \leq \|\mathbf{u} - \mathbf{u}_\delta\| + \sum_{m \in M} \left\| \mathbf{curl}\left(r_\delta^m\right) \right\|_{[H^1(\Omega_k)]^2} \leq \|\mathbf{u} - \mathbf{u}_\delta\| + \sum_{m \in M} \left\| r_\delta^m \right\|_{H^2(\Omega_k)},$$

and we conclude by (3.23) and (3.24). □

In the case of the non-conforming decomposition of Ω, we know (see [19] Chapter III, Section 1) that

$$\inf_{\mathbf{v}_\delta \in V_\delta} \|\mathbf{u} - \mathbf{v}_\delta\| \leq C \bar{N}^{\frac{1}{2}} \inf_{\mathbf{v}_\delta \in Y_\delta} \|\mathbf{u} - \mathbf{v}_\delta\|,$$

where $\bar{N} = \sup(N_k)$, $1 \leq k \leq K$, and since we have an optimal estimation of $\inf_{v_\delta \in X_\delta} \|u - v_\delta\|$ (see [7]),

$$\inf_{\mathbf{v}_\delta \in Y_\delta} \|\mathbf{u} - \mathbf{v}_\delta\| \leq C \sum_{k=1}^{K} N_k^{1-s} \|\mathbf{u}\|_{[H^s(\Omega_k)]^2}.$$

The positive number s is the global regularity of $\mathbf{u}$ in the neighborhood of $\mathbf{a}$, $\mathbf{u}$ is in the space $[H^s(\Omega)]^2$ for $s < 1 + \eta(\omega)$ [3], then

$$\inf_{\mathbf{v}_\delta \in V_\delta} \|\mathbf{u} - \mathbf{v}_\delta\| \leq C \bar{N}^{\frac{1}{2}} \sum_{k=1}^{K} N_k^{1-s} \|\mathbf{u}\|_{[H^s(\Omega_k)]^2}. \tag{3.25}$$

In this case we lose $\bar{N}^{\frac{1}{2}}$ on the error of the best approximation of $\mathbf{u}$ by elements of the space V_δ. However, decomposing $\mathbf{u}$ into a regular part plus a singular part, we find an optimal error estimate.

For the error of the best approximation of p by elements of the space M_δ, for any positive real s, such that $s < 1 + \eta(\omega)$:

$$\inf_{q_\delta \in M_\delta} \|p - q_\delta\|_{L^2(\Omega)} \leq \sum_{k=1}^{K} N_k^{1-s} \|p\|_{H^{s-1}(\Omega_k)}. \tag{3.26}$$

The last term of (3.20) is the error due to the nonconformity of the method. This error has the same order as the errors of the best approximation of $\mathbf{u}$, by elements of the space Y_δ, and of p, by the elements of the space M_δ [7]. Thus by combining (3.25) and (3.26), we may conclude by the following theorem.

Theorem 3.8 *Let* $\mathbf{f}$ *in* $[H^{s-2}(\Omega)]^2$, $s > 0$, *such that* $\mathbf{f}/_{\Omega_k}$ *belongs to* $[H^{\rho_k}(\Omega_k)]^2$ *where* $\rho_k > 1$, *then for all* ϵ *positive, the error between the solution* $(\mathbf{u}, p)$ *of the continuous problem* (2.2) *and the solution* $(\mathbf{u}_\delta, p_\delta)$ *of the discrete problem* (3.7) *is*

$$\|\mathbf{u} - \mathbf{u}_\delta\| + \beta_\delta \|p - p_\delta\|_{L^2(\Omega)} \leq C \bar{N}^{\frac{1}{2}} \sup\left\{ \sum_{k=1}^{K} N_k^{-\sigma_k}, \sum_{k=1}^{K} N_k^{-\rho_k} \right\} \|\mathbf{f}\|_{[H^{s-2}(\Omega)]^2}, \tag{3.27}$$

where σ_k is given by

$$\sigma_k = \begin{cases} s-1 & \textit{if } \bar{\Omega}_k \textit{ contains no corner of } \bar{\Omega}_k, \\ \inf\{s-1, 2.739-\epsilon\} & \textit{if } \bar{\Omega}_k \textit{ contains no corner different of } \mathbf{a}, \\ \inf\{s-1, \eta(\omega)-\epsilon\} & \textit{if } \bar{\Omega}_k \textit{ contains } \mathbf{a}. \end{cases}$$

Remark 3.9 For a regular data function $\mathbf{f}$, $N = \inf_{1\leq k\leq K} N_k$ and $\omega = \frac{3\pi}{2}$ then for all ϵ positive we obtain an order of convergence $N^{\epsilon-0,044484}$ for non-conforming decomposition. However, in the conforming case (3.27) will not contain the term $\bar{N}^{\frac{1}{2}}$; the order of convergence is $N^{\epsilon-0,544484}$.

We decompose $(\mathbf{u}, p)$ as a regular part plus a linear combination of its k first singular functions $(\mathbf{S}_i, S_{pi})$, $i \in \{1,\dots,k\}$, where k is an integer [3]

$$\mathbf{u} = \mathbf{u}_r + \lambda_1 \mathbf{S}_1 + \lambda_2 \mathbf{S}_2 + \cdots + \lambda_k \mathbf{S}_k$$

and

$$p = p_r + \beta_1 S_{p1} + \beta_2 S_{p2} + \cdots + \beta_k S_{pk}.$$

The (λ_i, β_i), $i \in \{1,\dots,k\}$ are the singular coefficients associated to $(\mathbf{S}_i, S_{pi})$. To improve the order of the error, we need to estimate separately each of the following terms:

- $\inf_{\mathbf{v}_\delta \in V_\delta} \|\mathbf{u}_r - \mathbf{v}_\delta\|$;
- $\inf_{p_\delta \in M_\delta} \|p_r - p_\delta\|_{L^2(\Omega)}$;
- $\inf_{\mathbf{v}_\delta \in V_\delta} \|\mathbf{S}_i - \mathbf{v}_\delta\|$;
- $\inf_{p_\delta \in M_\delta} \|S_{pi} - p_\delta\|_{L^2(\Omega)}$;
- $\sup_{\mathbf{w}_\delta \in V_\delta} \frac{\sum_{k=1}^{K}\sum_{k=1}^{K}\int_{\bar{\Omega}_k\cap\bar{\Omega}_l}(-\nu\frac{\partial \mathbf{u}}{\partial n}+pn)(\mathbf{w}_{\delta/\Omega_k}-\mathbf{w}_{\delta/\Omega_l})}{\|\mathbf{w}_\delta\|}$.

The estimation of the two first terms is immediate from (3.25) and (3.26), for a positive real s such that $s < 1 + \eta_k(\omega)$,

$$\inf_{\mathbf{v}_\delta \in V_\delta} \|\mathbf{u}_r - \mathbf{v}_\delta\| \leq C\bar{N}^{\frac{1}{2}} \sum_{k=1}^{K} N_k^{1-s} \|\mathbf{u}_r\|_{[H^s(\Omega_k)]^2}, \tag{3.28}$$

$$\inf_{p_\delta \in M_\delta} \|p_r - p_\delta\|_{L^2(\Omega)} \leq C \sum_{k=1}^{K} N_k^{1-s} \|p_r\|_{H^{s-1}(\Omega_k)}. \tag{3.29}$$

The estimation of the third and fourth terms is given by the following lemma.

Lemma 3.10 *For all ϵ positive, there exist a function $\mathbf{v}_\delta$ in Y_δ and p_δ in M_δ such that*

$$\|\mathbf{S}_i - \mathbf{v}_\delta\| \leq CN^{\epsilon-2\eta_i(\omega)}, \tag{3.30}$$

$$\|\mathbf{S}_{pi} - p_\delta\|_{L^2(\Omega)} \leq CN^{\epsilon-2\eta_i(\omega)}, \tag{3.31}$$

for $i \in \{1,\dots,k\}$

Proof Since the support of the cut-off function is included in $\bar{\Delta}$, we will restrict the number to rectangles contained in $\bar{\Delta}$, three for $\omega = \frac{3\pi}{2}$ and four for the crack. We suppose that the degree of the polynomials on the rectangles included in $\bar{\Delta}$ is the same. Since $\mathrm{div}(\mathbf{S}_i) = 0$, there exists φ_{si} such that $\mathbf{S}_i = \mathbf{curl}(\varphi_{si})$. We know that $\varphi_{si}(r,\theta) = r^{1+\eta_i(\omega)}\psi(\theta)$ in the neighborhood of $\mathbf{a}$ where ψ is in $C^\infty(]0,\omega[)$ ([1], Chapter V). Using the results of the approximation of singular functions there exists φ_δ in Y_δ, which gives us a double order on the convergence for the approximation of φ_{si} (see [5], Remark 18),

$$\mathbf{v}_\delta = \mathbf{curl}(\varphi_\delta),$$

thus

$$\|\mathbf{S}_i - \mathbf{v}_\delta\| \le CN^{\epsilon - 2\eta_i(\omega)}. \qquad \square$$

To estimate the last term, we know that the jump $(\mathbf{w}_{\delta/\Omega_k} - \mathbf{w}_{\delta/\Omega_l})$ vanishes on the edges contained in $\bar{\Delta}$ by the conformity assumption $(\mathbf{u},p)$ is equal to $(\mathbf{u}_r,p_r)$ on $\Omega \setminus \bar{\Delta}$. Then, for $s < 1 + \eta_k(\omega)$,

$$\begin{aligned} &\sup_{\mathbf{w}_\delta \in V_\delta} \frac{\sum_{k=1}^K \sum_{k=1}^K \int_{\bar{\Omega}_k \cap \bar{\Omega}_l} (-\nu \frac{\partial \mathbf{u}_r}{\partial n} + p_r n)[\mathbf{w}_\delta]\, d\tau}{\|\mathbf{w}_\delta\|} \\ &\quad \le C\left(\sum_{k=1}^K N_k^{1-s} \|\mathbf{u}_r\|_{H^s(\Omega_k)} + \sum_{k=1}^{K}{}^{K} N_k^{1-s} \|p_r\|_{H^{s-1}(\Omega_k)}\right). \end{aligned} \tag{3.32}$$

We remark that the order of convergence is determined by the order of convergence of the first singular function. To improve the estimation (3.32) we need to push the decomposition of $\mathbf{u}$ to find an integer k such that $(\eta_k(\omega) - \frac{1}{2}) \ge 2\eta_1(\omega)$.

By collecting (3.28), (3.29), (3.30), (3.31), and (3.32) in (3.20), we announce the following theorem.

Theorem 3.11 *Let* $\mathbf{f}$ *in* $[H^{s-2}(\Omega)]^2$, $s > 0$, *such that* $\mathbf{f}/_{\Omega_k}$ *belongs to* $[H^{\rho_k}(\Omega_k)]^2$ *where* $\rho_k > 1$, *then, for all* ϵ *positive, the error between the solution* $(\mathbf{u},p)$ *of the continuous problem* (2.2) *and the solution* $(\mathbf{u}_\delta,p_\delta)$ *of the discrete problem* (3.7) *is*

$$\|\mathbf{u} - \mathbf{u}_\delta\| + \beta_\delta \|p - p_\delta\|_{L^2(\Omega)} \le C \sup\left\{\sum_{k=1}^K N_k^{-\sigma_k}, \sum_{k=1}^K N_k^{-\rho_k}\right\} \|\mathbf{f}\|_{[H^{s-2}(\Omega)]^2}, \tag{3.33}$$

where σ_k *is given by*

$$\sigma_k = \begin{cases} s-1 & \text{if } \bar{\Omega}_k \text{ contains no corner of } \bar{\Omega}_k; \\ \inf\{s-1, 2.739 - \epsilon\} & \text{if } \bar{\Omega}_k \text{ contains no corner different of } \mathbf{a}; \\ \inf\{s-1, \eta(\omega) - \epsilon\} & \text{if } \bar{\Omega}_k \text{ contains } \mathbf{a}. \end{cases}$$

Remark 3.12 For a regular data function $\mathbf{f}$, let $N = \inf_{1 \le k \le K} N_k$, then for all ϵ positive

- the convergence in the case of the velocity is $N^{\epsilon - 1}$ for $\omega = 2\pi$;
- the convergence in the case of the velocity is $N^{\epsilon - 1{,}0888}$ for $\omega = \frac{3\pi}{2}$.

Compared with the results of Remark 3.9 it is clear that the order of convergence is improved.

4 Algorithm of Strang and Fix

4.1 The discrete problem

In this section, we will enlarge the discrete space of the velocity Y_δ. We keep the space M_δ for the pressure. We will only handle the case where $\omega = \frac{3\pi}{2}$.

Let $\mathbf{S}_1 = (S_1^1, S_1^2)$ the first singular function of the velocity [3], we define the space Y_δ^*,

$$Y_\delta^* = Y_\delta + \mathbb{R}\mathbf{S}_1 \tag{4.1}$$

if $\mathbf{u}_\delta^*$ is in Y_δ^* there exist $\mathbf{u}_\delta$ in Y_δ and λ_1 in $\mathbb{R}$ such that

$$\mathbf{u}_\delta^* = \mathbf{u}_\delta + \lambda_1 \mathbf{S}_1.$$

Since $\mathbf{S}_1$ belongs to $[H^1(\Omega)]^2$, we define the following norm on the space Y_δ^* for all $\mathbf{u}_\delta^* = \mathbf{u}_\delta + \lambda_1 \mathbf{S}_1$ in Y_δ^*:

$$\|\mathbf{u}_\delta\|_* = \left(\|\mathbf{u}_\delta\|^2 + |\lambda_1|^2 \|\mathbf{S}_1\|^2\right)^{1/2},$$

where $\|\cdot\|$ is the norm defined on Y_δ.

Thus, we define the discrete problem as follows.

For a continuous data function $\mathbf{f} = (f_1, f_2)$ on $\bar{\Omega}$, find $\mathbf{u}_\delta^* = (u_{\delta 1}^*, u_{\delta 2}^*)$ in Y_δ^* and p_δ in M_δ such that for all $\mathbf{v}_\delta^* = (v_{\delta 1}^*, v_{\delta 2}^*)$ in Y_δ^* and for all q_δ in M_δ,

$$\begin{cases} \mathbf{a}_\delta^*(\mathbf{u}_\delta^*, \mathbf{v}_\delta^*) + b_\delta^*(\mathbf{v}_\delta^*, p_\delta) = (\mathbf{f}, \mathbf{v}_\delta^*)_\delta, \\ b_\delta^*(\mathbf{u}_\delta^*, q_\delta) = 0, \end{cases} \tag{4.2}$$

where

$$\mathbf{a}_\delta^*\left(\mathbf{u}_\delta^*, \mathbf{v}_\delta^*\right) = a_{1\delta}^*\left(u_{\delta 1}^*, v_{\delta 1}^*\right) + a_{2\delta}^*\left(u_{\delta 2}^*, v_{\delta 2}^*\right)$$

such that $a_{i\delta}^*(\cdot,\cdot)$, $i \in \{1,2\}$ is the bilinear form defined by [6],

$$\begin{aligned} a_{i\delta}^*\left(u_\delta^*, v_\delta^*\right) = \sum_{k=1}^{K} \biggl(&\left(\nabla u_\delta^k, \nabla v_\delta^k\right)_{N_k} + \lambda \int_{\Omega_k} \nabla S_1^i \nabla v_\delta^k \, dx + \mu \int_{\Omega_k} \nabla u_\delta^k \nabla S_1^i \, dx \\ &+ \lambda\mu \int_{\Omega_k} \left(\nabla S_1^i\right)^2 dx \biggr). \end{aligned} \tag{4.3}$$

Since S_1^i are explicitly known, the integral $\int_{\Omega_k} (\nabla S_1^i)^2 \, dx$ is exactly computed (see [22]):

$$\left(\mathbf{f}, \mathbf{v}_\delta^*\right)_\delta = \left(f_1, v_{\delta 1}^*\right)_\delta + \left(f_2, v_{\delta 2}^*\right)_\delta$$

and $b_\delta^*(\cdot,\cdot)$ is defined as follows. For $\mathbf{u}_\delta^* = \mathbf{u}_\delta + \lambda_1 \mathbf{S}_1$ in Y_δ^* and q_δ in M_δ:

$$b_\delta^*\left(\mathbf{u}_\delta^*, q_\delta\right) = -\left(\sum_{k=1}^{K} (\operatorname{div} \mathbf{u}_\delta, q_\delta)_{N_k} + \lambda_1 \int_{\Omega_k} \operatorname{div} \mathbf{S}_1 q_\delta \, dx\right) = b_\delta(\mathbf{u}_\delta, q_\delta), \tag{4.4}$$

since $\operatorname{div} \mathbf{S}_1 = 0$. Let

$$V_\delta^* = \{\mathbf{v}_\delta^* \in Y_\delta^*, b_\delta^*(\mathbf{v}_\delta^*, q_\delta) = 0, \forall q_\delta \in M_\delta\},$$

the kernel of the bilinear form $b_\delta^*(\cdot,\cdot)$. To prove that the problem (4.2) is well posed, we need to show the following properties:

(1) $\mathbf{a}_\delta^*(\cdot,\cdot)$ is continuous on Y_δ^* with a norm independent of δ [6].
(2) $b_\delta^*(\cdot,\cdot)$ is continuous on $Y_\delta^* \times M_\delta$ and the constant of continuity is independent of δ. This is due to the exactness of the quadrature formula on each sub-domain Ω_k, $1 \leq k \leq K$, since p_{δ/Ω_k} is in the space $\mathbb{P}_{N_k-2}(\Omega_k)$.
(3) The property of global compatibility is checked between the spaces Y_δ^* and M_δ. This results in the existence of an Inf-Sup condition on the form $b_\delta^*(\cdot,\cdot)$.

Proposition 4.1 *There exists a constant* $\beta_\delta = \inf(N_k^{-\frac{1}{2}})$, $1 \leq k \leq K$ *such that the following Inf-Sup condition holds*:

$$\forall q_\delta \in M_\delta, \quad \sup_{\mathbf{u}_\delta^* \in Y_\delta^*} \frac{b_\delta^*(\mathbf{u}_\delta^*, q_\delta)}{\|\mathbf{u}_\delta^*\|_*} \geq \beta_\delta \|q_\delta\|_{L^2(\Omega)}. \tag{4.5}$$

Proof Since the space Y_δ is included in Y_δ^*,

$$\sup_{\mathbf{u}_\delta^* \in Y_\delta^*} \frac{b_\delta^*(\mathbf{u}_\delta^*, q_\delta)}{\|\mathbf{u}_\delta^*\|_*} \geq \sup_{\mathbf{u}_\delta \in Y_\delta} \frac{b_\delta(\mathbf{u}_\delta, q_\delta)}{\|\mathbf{u}_\delta\|},$$

and we conclude using Proposition 3.2. □

(4) $\mathbf{a}_\delta^*(\cdot,\cdot)$ verifies an Inf-Sup condition on V_δ^* which is independent of δ.

Proposition 4.2 *We have the following Inf-Sup condition on the form* $\mathbf{a}_\delta^*(\cdot,\cdot)$.
There exists a constant $\beta > 0$ *such that*

$$\forall \mathbf{u}_\delta^* \in V_\delta^*, \quad \sup_{\mathbf{v}_\delta^* \in V_\delta^*} \frac{\mathbf{a}_\delta^*(\mathbf{u}_\delta^*, \mathbf{v}_\delta^*)}{\|\mathbf{v}_\delta^*\|_*} \geq \beta \|\mathbf{u}_\delta^*\|_{L^2(\Omega)}. \tag{4.6}$$

Proof Let $\mathbf{u}_\delta^* = \mathbf{u}_\delta + \lambda_1 \mathbf{S}_1$ in V_δ^*; there exists $\mathbf{v}_\delta^* = \mathbf{u}_\delta + 3\lambda_1 \mathbf{S}_1$ in Y_δ^* for which the inequality (4.6) is checked [6], and since $\operatorname{div}(\mathbf{S}_1) = 0$ we see that $\mathbf{v}_\delta^*$ is in V_δ^*. □

After having proved the last four properties we conclude by the following proposition.

Proposition 4.3 *For all* $\mathbf{f}$ *belonging to* $[L^2(\Omega)]^2$, *the problem* (4.2) *has a unique solution* $(\mathbf{u}_\delta^*, p_\delta)$ *in* $Y_\delta^* \times M_\delta$ *satisfying*

$$\|\mathbf{u}_\delta^*\|_* + \beta_\delta \|p_\delta\|_{L^2(\Omega)} \leq C\|\mathbf{f}\|_{L^2(\Omega)}.$$

4.2 The error estimate

In this section, we are interested in improving the error on the velocity. From the Inf-Sup condition on the bilinear form $\mathbf{a}_\delta^*(\cdot,\cdot)$, we state the following proposition (see [6] for its proof).

Proposition 4.4 *If* $\mathbf{u}$ *is the solution of the problem* (2.2) *and* $\mathbf{u}_\delta^*$ *is the solution of the problem* (4.2) *then*

$$\|\mathbf{u}-\mathbf{u}_\delta^*\|_* \le C\Bigg(\inf_{v_\delta^*\in V_\delta^*}\left\{\|\mathbf{u}-\mathbf{v}_\delta^*\|_* + \sup_{\mathbf{w}_\delta^*\in Y_\delta^*}\frac{(\mathbf{a}-\mathbf{a}_\delta^*)(\mathbf{v}_\delta^*,\mathbf{w}_\delta^*)}{\|\mathbf{w}_\delta^*\|_*}\right\} + \sup_{\mathbf{w}_\delta^*\in Y_\delta^*}\frac{\int_{\Omega_k}\mathbf{f}\mathbf{w}_\delta^*\,dx\,dy-(f,\mathbf{w}_\delta^*)_\delta}{\|\mathbf{w}_\delta^*\|_*} + \sup_{\mathbf{w}_\delta\in V_\delta^*}\frac{\sum_{k=1}^K\sum_{k=1}^K\int_{\gamma^{kl}}(-\nu\frac{\partial\mathbf{u}}{\partial n}+pn)[\mathbf{w}_\delta^*]}{\|\mathbf{w}_\delta^*\|_*}\Bigg); \tag{4.7}$$

$[\mathbf{w}_\delta^*]$ *is the jump of* $\mathbf{w}_\delta^*$ *through* γ^{kl} *and C is a positive constant independent of* δ.

To obtain an estimation of the error, we have to estimate each term of the inequality (4.7). We start by the error of the best approximation on the velocity, if $\mathbf{u}=\mathbf{u}_r+\lambda_1\mathbf{S}_1$, we choose $\mathbf{v}_\delta^*$ in V_δ^* such that $\mathbf{v}_\delta^*=\mathbf{v}_\delta+\lambda_1\mathbf{S}_1$. Since $\mathbf{S}_1$ is in the kernel of $b_\delta^*(\cdot,\cdot)$, we conclude that

$$\inf_{\mathbf{v}_\delta^*\in V_\delta^*}\|\mathbf{u}-\mathbf{v}_\delta^*\|_* \le \inf_{v_\delta\in V_\delta}\|\mathbf{u}_r-\mathbf{v}_\delta\|.$$

To get a better order of convergence, we decompose the regular part of the solution, thus:

$$\mathbf{u}_r=\tilde{\mathbf{u}}_r+\lambda_2\mathbf{S}_2+\lambda_3\mathbf{S}_3+\cdots+\lambda_k\mathbf{S}_k.$$

This gives us

$$\inf_{\mathbf{v}_\delta^*\in V_\delta^*}\|\mathbf{u}_r-\mathbf{v}_\delta\|_* \le \left\{\inf_{\mathbf{v}_\delta\in V_\delta}\|\tilde{\mathbf{u}}_r-\mathbf{v}_\delta\|+\sum_{i=2}^k\inf_{\mathbf{v}_\delta\in V_\delta}\|\mathbf{S}_i-\mathbf{v}_\delta\|\right\}, \tag{4.8}$$

we will bound each term of the inequality (4.8) separately. If $\mathbf{f}$ belongs to $[H^{s-2}(\Omega)]^2$, $\tilde{\mathbf{u}}_r$ belongs to $[H^s(\Omega)]^2$ such that $s<1+\eta_k(\omega)$ [3], then we have from (3.25)

$$\inf_{\mathbf{v}_\delta\in V_\delta}\|\tilde{\mathbf{u}}_r-\mathbf{v}_\delta\| \le \bar{N}^{\frac{1}{2}}\sum_{k=1}^K N_k^{1-s}\|\tilde{\mathbf{u}}_r\|_{[H^s(\Omega_k)]^2}. \tag{4.9}$$

We estimate the second term of the inequality (4.8) by (3.12). The second term of the inequality (4.7) vanishes if we choose $\mathbf{v}_\delta^*=\mathbf{v}_\delta+\lambda_1\mathbf{S}_1$ in Y_δ^* such that $\mathbf{v}_\delta$ belongs to Y_δ^-. For the estimation of the third term see [6]. The last term is the consistency error related to the term $(\nu\frac{\partial\mathbf{u}}{\partial n}+pn)$; by the conformity assumption of $\bar{\Delta}$, we conclude that

$$E(\mathbf{w}_\delta)=\sup_{\mathbf{w}_\delta\in V_\delta^*}\sum_{k=1}^K\sum_{l>k}^K\int_{\gamma^{kl}}\left(-\nu\frac{\partial\tilde{\mathbf{u}}_r}{\partial n}+p_r n\right)[\mathbf{w}_\delta],$$

thus if $s<1+\eta_k(\omega)$

$$\sup_{\mathbf{w}_\delta^*\in Y_\delta^*}\frac{E(\mathbf{w}_\delta)}{\|\mathbf{w}_\delta^*\|_*} \le C\bar{N}^{\frac{1}{2}}\sum_{k=1}^K\left(N_k^{1-s}\|\tilde{\mathbf{u}}_r\|_{H^s(\Omega_k)}+N_k^{1-s}\|\tilde{p}_r\|_{H^{s-1}(\Omega_k)}\right). \tag{4.10}$$

Remark 4.5 We remark that since the first singular function is in the space Y_δ^*, the second singular function imposes its order of convergence because it is the worst. Then we decompose $\mathbf{u}_r$ until we find an integer k such that $(\eta_k(\omega) - \frac{1}{2}) \geq 2\eta_2(\omega)$.

If we combine (4.8), (4.9), and (4.10) in (4.7) we consider the following theorem.

Theorem 4.6 *Let* $\mathbf{f}$ *in* $[H^{s-2}(\Omega)]^2$, $s > 0$, *such that* $\mathbf{f}/_{\Omega_k}$ *belongs to* $[H^{\rho_k}(\Omega_k)]^2$ *where* $\rho_k > 1$, *then, for all* $\epsilon > 0$, *the error between the velocity* $\mathbf{u}$ *of the continuous problem* (2.2) *and the discrete velocity* $\mathbf{u}_\delta^*$, *of the problem* (4.2) *is*

$$\|\mathbf{u} - \mathbf{u}_\delta^*\|_* \leq C \sup\left\{\sum_{k=1}^{K} N_k^{-\sigma_k}, \sum_{k=1}^{K} N_k^{-\rho_k}\right\} \|\mathbf{f}\|_{[H^{s-2}(\Omega)]^2},$$

where σ_k *is given by*

$$\sigma_k = \begin{cases} s-1 & \text{if } \bar{\Omega}_k \text{ contains no corner } \bar{\Omega}_k; \\ \inf\{s-1, 2\eta_2(\frac{\pi}{2}) - \epsilon\} & \text{if } \bar{\Omega}_k \text{ contains different corners of } \mathbf{a}; \\ \inf\{s-1, 2\eta_2(\omega) - \epsilon\} & \text{if } \bar{\Omega}_k \text{ contains } \mathbf{a}. \end{cases} \tag{4.11}$$

Remark 4.7 For a regular data $\mathbf{f}$ and $N = \inf_{1\leq k\leq K} N_k$, then for all ϵ positive, the order of convergence is $N^{\epsilon-1,816}$ for $\omega = \frac{3\pi}{2}$. We remark that we double the order of the convergence when compared with the results of Remark 3.9 for $\omega = \frac{3\pi}{2}$. This shows the importance of the algorithm of Strang and Fix.

Corollary 4.8 *Let* $\mathbf{f}$ *in* $[H^{s-2}(\Omega)]^2$, $s > 0$, *such that* $\mathbf{f}/_{\Omega_k}$ *belongs to* $[H^{\rho_k}(\Omega_k)]^2$ *where* $\rho_k > 1$, *then, for all* $\epsilon > 0$, *the error between the velocity* $\mathbf{u}$ *of the continuous problem* (2.2) *and the discrete velocity* $\mathbf{u}_\delta^*$, *of the problem* (4.2) *is*

$$\|\mathbf{u} - \mathbf{u}_\delta^*\|_{[L^2(\Omega)]^2} \leq C \sup\left\{N^{-1}\left(\sum_{k=1}^{K} N_k^{-\sigma_k}\right), \sum_{k=1}^{K} N_k^{-\rho_k}\right\} \|\mathbf{f}\|_{[H^{s-2}(\Omega)]^2},$$

where σ_k *verifies* (4.11), $N = \inf_{1\leq k\leq K} N_k$.

Proof For the proof, we use the Aubin-Nische duality, which is not standard in this case because the domain is not convex,

$$\|\mathbf{u} - \mathbf{u}_\delta^*\|_{[L^2(\Omega)]^2} = \sup_{\mathbf{g}\in[L^2(\Omega)]^2} \frac{\int_\Omega (\mathbf{u} - \mathbf{u}_\delta^*)\mathbf{g}\,dx}{\|\mathbf{g}\|_{L^2(\Omega)}};$$

for all $\mathbf{g}$ belonging to $[L^2(\Omega)]^2$, we consider the solution $(\mathbf{w}, t)$ in $[H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ of the problem

$$\begin{cases} \mathbf{a}(\mathbf{w}, \mathbf{v}) + b(\mathbf{v}, t) = \int_\Omega \mathbf{g}(x)\mathbf{v}(x)\,dx, & \forall \mathbf{v} \in [H_0^1(\Omega)]^2, \\ b(\mathbf{w}, q) = 0, \quad \forall q \in L^2(\Omega). \end{cases}$$

From the result in [3] we have

$$\mathbf{w} = \tilde{\mathbf{w}}_r + \lambda_1 \mathbf{S}_1 + \lambda_2 \mathbf{S}_2, \qquad t = \tilde{t}_r + \beta_1 S_{p1} + \beta_2 S_{p2}, \tag{4.12}$$

with

$$\|\tilde{\mathbf{w}}_r\|_{[H^2(\Omega)]^2} + \|\tilde{t}_r\|_{H^1(\Omega)} + |\lambda_1| + |\lambda_2| + |\beta_1| + |\beta_2| \le C\|\mathbf{g}\|_{[L^2(\Omega)]^2},$$

so

$$\int_\Omega (\mathbf{u} - \mathbf{u}_\delta^*)\mathbf{g}\,dx\,dy = \mathbf{a}(\mathbf{u} - \mathbf{u}_\delta^*, \mathbf{w}) + b(\mathbf{u} - \mathbf{u}_\delta^*, t).$$

We notice that the last term is not equal to 0 because $\operatorname{div}(\mathbf{u}_\delta^*) \neq 0$. Let $\mathbf{w}_\delta^* = \mathbf{w}_\delta + \mu\mathbf{S}_1$ in Y_δ^* such that $\mathbf{w}_\delta$ belongs to Y_δ^-, and $\operatorname{div}(\mathbf{w}_\delta) = 0$, then by the exactness of the quadrature formula:

$$\begin{aligned}\int_\Omega (\mathbf{u} - \mathbf{u}_\delta^*)g\,dx\,dy = \sum_{k=1}^{K}\biggl(&\int_{\Omega_k} \nabla(\mathbf{u} - \mathbf{u}_\delta^*)\nabla(\mathbf{w} - \mathbf{w}_\delta^*)\,dx \\ &+ \int_{\Omega_k} \operatorname{div}(\mathbf{u} - \mathbf{u}_\delta^*)(t - \Pi_{N_k-2}t)\,dx + \int_{\Omega_k} \mathbf{f}\mathbf{w}_\delta^*\,dx - (\mathbf{f}, \mathbf{w}_\delta^*)_{N_k}\biggr).\end{aligned}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\int_\Omega (\mathbf{u} - \mathbf{u}_\delta^*)\mathbf{g}\,dx &\le C\sum_{k=1}^{K}\|\mathbf{u} - \mathbf{u}_\delta^*\|_{[H^1(\Omega_k)]^2}\Bigl(\inf_{\mathbf{w}_\delta^* \in Y_\delta^*}\|\mathbf{w} - \mathbf{w}_\delta^*\|_{[H^1(\Omega_k)]^2} \\ &\quad + \|t - \Pi_{N_k-2}t\|_{L^2(\Omega_k)}\Bigr) + C\sum_{k=1}^{K} N_k^{-\rho_k}\|\mathbf{f}\|_{H^{\rho_k}(\Omega_k)} \\ &\le C\|\mathbf{u} - \mathbf{u}_\delta^*\|_*\sum_{k=1}^{K}\Bigl(\inf_{\mathbf{w}_\delta^* \in X_\delta^*}\|\mathbf{w} - \mathbf{w}_\delta^*\|_{[H^1(\Omega_k)]^2} \\ &\quad + \|t - \Pi_{N_k-2}t\|_{L^2(\Omega_k)}\Bigr) + C\sum_{k=1}^{K} N_k^{-\rho_k}\|\mathbf{f}\|_{[H^{\rho_k}(\Omega_k)]^2},\end{aligned}$$

where C is positive constant. It remains now to find an estimation of the term

$$\inf_{\mathbf{w}_\delta^* \in X_\delta^*}\|\mathbf{w} - \mathbf{w}_\delta^*\|_{[H^1(\Omega_k)]^2} + \|t - \Pi_{N_k-2}t\|_{L^2(\Omega_k)}.$$

If we choose $\mathbf{w}_\delta^* = \mathbf{w}_\delta + \lambda_1\mathbf{S}_1$, then

$$\inf_{\mathbf{w}_\delta^* \in Y_\delta^*}\|\mathbf{w} - \mathbf{w}_\delta^*\|_{[H^1(\Omega_k)]^2} \le \inf_{\mathbf{w}_\delta \in Y_\delta^-}\|\mathbf{w}_r - \mathbf{w}_\delta\|_{[H^1(\Omega_k)]^2}.$$

Since $\mathbf{w}_r = \tilde{\mathbf{w}}_r + \lambda_2\mathbf{S}_2$, and considering the decomposition (4.12),

$$\begin{aligned}&\inf_{\mathbf{w}_\delta^* \in X_\delta^*}\|\mathbf{w} - \mathbf{w}_\delta^*\|_{[H^1(\Omega_k)]^2} + \|t - \Pi_{N_k-2}t\|_{L^2(\Omega_k)} \\ &\quad\le \inf_{\mathbf{w}_\delta \in X_\delta^-}\|\tilde{\mathbf{w}}_r - \mathbf{w}_\delta\|_{[H^1(\Omega_k)]^2} + |\lambda_2|\inf_{\mathbf{w}_\delta \in X_\delta^-}\|\mathbf{S}_2 - \mathbf{w}_\delta\|_{[H^1(\Omega_k)]^2} \\ &\qquad + \|\tilde{t}_r - \Pi_{N_k-2}\tilde{t}_r\|_{L^2(\Omega_k)} + |\beta_1|\|S_{p1} - \Pi_{N_k-2}S_{p1}\|_{L^2(\Omega_k)} + |\beta_2|\|S_{p2} - \Pi_{N_k-2}S_{p2}\|_{L^2(\Omega_k)}.\end{aligned}$$

If $N = \inf_{1 \leq k \leq K} N_k$:

$$\inf_{\mathbf{w}_\delta^* \in X_\delta^*} \|\mathbf{w} - \mathbf{w}_\delta^*\|_{[H^1(\Omega_k)]^2} \leq C N_k^{-1} \big(\|\tilde{\mathbf{w}}_r\|_{[H^2(\Omega_k)]^2} + \|\tilde{t}_r\|_{H^1(\Omega_k)} + |\lambda_2| + |\beta_1| + |\beta_2| \big)$$
$$\leq C N^{-1} \|\mathbf{g}\|_{[L^2(\Omega)]^2},$$

which completes the proof. □

Remark 4.9 To improve the order of the convergence in the case of the crack, we need to add the two first singular functions to the discrete space of the velocity. However, the implementation will be complicated.

5 Conclusion

We considered, in this paper, the system of Stokes equation in velocity and pressure formulation in a non-regular domain of $\mathbb{R}^2$. We presented the discrete problem using the mortar spectral element method. We proved the Inf-Sup condition in a non-conforming domain. We showed that if we consider the decomposition of the solution into a regular part and a singular one, we improve the order of the error. Using the Strang and Fix algorithm, which consists of adding the singular function in the discrete space, we prove an optimal order of the error on the velocity.

We intend to do the implementation of those results in our next work. The extension of this discretization to the three dimensional axisymmetric domain is presently under consideration.

Competing interests
The author declares that they have no competing interests.

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References

1. Grisvard, P: Elliptic Problems in Nonsmooth Domains. Pitman, New York (1985)
2. Kondratiev, VA: Boundary value problems for elliptic equations in domain with conical or angular points. Trans. Mosc. Math. Soc. **16**, 227-313 (1967)
3. Chorfi, N: Geometric singularities of the Stokes problem. Abstr. Appl. Anal. **2014**, Article ID 491326 (2014)
4. Babuška, I, Suri, M: The optimal convergence rate of the *P*-version of the finite element method. SIAM J. Numer. Anal. **24**, 750-776 (1987)
5. Bernardi, C, Maday, Y: Polynomial approximation of some singular functions. Appl. Anal. **42**, 1-32 (1991)
6. Chorfi, N: Handling geometric singularities by mortar spectral element method case of the Laplace equation. J. Sci. Comput. **18**, 25-48 (2003)
7. Bernardi, C, Maday, Y, Patera, AT: A new nonconforming approach to domain decomposition: the mortar element method. In: Brézis, H, Lions, J-L (eds.) Collège de France Seminar XI, pp. 13-51 Pitman, New York (1994)
8. Ben Belgacem, F, Bernardi, C, Chorfi, N, Maday, Y: Inf-Sup condition for the mortar spectral element discretization of the Stokes problem. Numer. Math. **85**, 257-281 (2000)
9. Strang, G, Fix, GJ: An Analysis of the Finite Element Methods. Prentice-Hall, Englewood Cliffs (1973)
10. Babuška, I, Rosenzweig, MB: A finite element scheme for domains with corners. Numer. Math. **20**, 1-21 (1972)
11. Lelièvre, J: Sur les éléments finis singuliers. C. R. Math. Acad. Sci. Paris, Sér. I **283**, 1029-1032 (1967)
12. Babuška, I: The finite element method for Lagrangian multilpiers. Numer. Math. **29**, 179-192 (1973)
13. Brezzi, F, Rappaz, J, Raviard, PA: Finite dimensional approximation of nonlinear problems. Part I. Branches of nonsingular solution. Numer. Math. **36**, 1-25 (1980)
14. Temam, R: Theory and Numerical Analysis of the Navier-Stokes Equations. North-Holland, Amsterdam (1977)
15. Maday, Y, Meiron, D, Patera, AT, Rønquist, EM: Analysis of iterative methods for the steady and unsteady Stokes problem: application to spectral element discretizations. SIAM J. Sci. Comput. **14**(2), 310-337 (1993)
16. Bernardi, C, Maday, Y: Approximations spectrales de problèmes aux limites elliptiques. Collection 'Mathématiques et Applications', vol. 10. Springer, Paris (1992)

17. Boland, J, Nicolaides, R: Stability of finite element under divergence constraints. SIAM J. Numer. Anal. **20**, 722-731 (1983)
18. Ciarlet, PG: Basic error estimates for elliptic problems. In: Handbook of Numerical Analysis. Vol. II, pp. 17-351 (1991)
19. Girault, V, Raviard, PA: Finite Element Methods for the Navier-Stokes Equations, Theory and Algorithms. Springer, New York (1986)
20. Bernardi, C, Maday, Y: Spectral method. In: Ciarlet, PG, Lions, J-L (eds.) Handbook of Numerical Analysis. Vol. V, pp. 209-485. North-Holland, Amsterdam (1996)
21. Bernardi, C, Dauge, M, Maday, Y: Polynomials in weighted sobolev spaces: basics and trace liftings. Publications de Laboratoire d'Analyse Numérique R92039 (1992)
22. Amara, M, Bernardi, C, Moussaoui, M: Handling corner singularities by mortar elements method. Appl. Anal. **46**, 25-44 (1992)