which, by defining $c_j = \Sigma_{h=0}^{j} d_h$ for $j = 0, 1, \ldots, m$, is in the form of (18.2.3). For these c's, $c_0 = d_0$ and $\Delta c_j = d_{j+1}$ for $j = 0, 1, \ldots, m-1$, thus $\Delta^j c_0 = \Delta^{j-1}(\Delta c_0) = \Delta^{j-1} d_1$ for $j = 1, 2, \ldots, m$. Then we have, from the right-hand side of (18.2.3),

$$\sum_{j=0}^{m} d_j \; {}_tp_{\overline{x_1 x_2 \cdots x_m}}^{\;\;j} = d_0 + \sum_{j=1}^{m} \Delta^{j-1} d_1 \; {}_tD_j. \qquad \blacksquare$$

Corollary 18.2.1 can be used to express the survival function of the k-survivor status in terms of joint- and single-life survival functions.

Corollary 18.2.3

$$ {}_tp_{\overline{x_1 x_2 \cdots x_m}}^{\;\;k} = \sum_{j=k}^{m} \left[(-1)^{j-k} \tbinom{j-1}{k-1}\right] {}_tD_j. \tag{18.2.7}$$

Proof:

In Corollary 18.2.1, set $d_k = 1$ and $d_j = 0$, for $j \neq k$. For these d's, $\Delta^{j-1} d_1 = (\mathrm{E} - 1)^{j-1} d_1 = (-1)^{j-k} \binom{j-1}{k-1}$, $j = k, k+1, \ldots, m$. $\blacksquare$

From the expression for its survival function in (18.2.7) we can obtain, by differentiation, a parallel expression for the p.d.f. of the future lifetime of the k-survivor status, T, as

$$f_T(t) = \frac{d}{dt}\left(1 - {}_tp_{\overline{x_1 x_2 \cdots x_m}}^{\;\;k}\right) = \sum_{j=k}^{m} (-1)^{j-k} \tbinom{j-1}{k-1}(-{}_tD_j'). \tag{18.2.8}$$

The actuarial present value and other characteristics of the probability distribution of the present value of a set of payments that depend on T can be determined using (18.2.7) or (18.2.8). In such determinations we use the fact that $-{}_tD_j'$ is the sum of the p.d.f.'s of the future lifetimes of the $\binom{m}{j}$ joint j-life statuses of the m lives.

Example 18.2.4

Let T denote the future lifetime of the last-survivor status of three lives. Exhibit in terms of joint- and single-life functions

a. The survival function
b. $\mathrm{E}[v^T]$
c. $\mathrm{E}[\bar{a}_{\overline{T}|}]$.

Solution:

a. By (18.2.7),

$$\begin{aligned} {}_tp_{\overline{x_1 x_2 x_3}} &= \sum_{j=1}^{3} (-1)^{j-1} \tbinom{j-1}{0} \, {}_tD_j \\ &= {}_tD_1 + (-1) \, {}_tD_2 + {}_tD_3 \end{aligned}$$

where

$$ {}_tD_1 = {}_tp_{x_1} + {}_tp_{x_2} + {}_tp_{x_3}, $$

$$ {}_tD_2 = {}_tp_{x_1x_2} + {}_tp_{x_1x_3} + {}_tp_{x_2x_3}, $$

$$ {}_tD_3 = {}_tp_{x_1x_2x_3}. $$

b. We denote $\mathrm{E}[v^T]$ by $\bar{A}_{\overline{x_1x_2x_3}}$ and use (18.2.8) to obtain

$$ \bar{A}_{\overline{x_1x_2x_3}} = \int_0^\infty v^t(-1)({}_tD_1' - {}_tD_2' + {}_tD_3')\,dt $$

$$ = \bar{A}_{x_1} + \bar{A}_{x_2} + \bar{A}_{x_3} - (\bar{A}_{x_1x_2} + \bar{A}_{x_1x_3} + \bar{A}_{x_2x_3}) + \bar{A}_{x_1x_2x_3}. $$

c. Replacing v^T by $\bar{a}_{\overline{T}|}$ in part (b) and denoting $\mathrm{E}[\bar{a}_{\overline{T}|}]$ by $\bar{a}_{\overline{x_1x_2x_3}}$, we have

$$ \bar{a}_{\overline{x_1x_2x_3}} = \bar{a}_{x_1} + \bar{a}_{x_2} + \bar{a}_{x_3} - (\bar{a}_{x_1x_2} + \bar{a}_{x_1x_3} + \bar{a}_{x_2x_3}) + \bar{a}_{x_1x_2x_3}. $$

For any survival status, $v^T + \delta\bar{a}_{\overline{T}|} = 1$, so we can calculate either of the expected values from the other using $\bar{A}_{\overline{x_1x_2x_3}} + \delta\bar{a}_{\overline{x_1x_2x_3}} = 1$. ▼

By differentiating both sides of (18.2.6), we can extend the relationship to the corresponding p.d.f.'s. This can be used for insurances paying an amount upon each death among the m lives.

Example 18.2.5

Consider an insurance on (x), (y), and (z) paying 1 on the first death, 2 on the second death, and 3 on the third death. Express the actuarial present value for the insurance in terms of actuarial present values for unit amount insurances on single- and joint-life statuses.

Solution:

Let $f_j(t)$ be the p.d.f. for the future lifetime of the j-survivor status. The actuarial present value is

$$ \int_0^\infty v^t[1f_3(t) + 2f_2(t) + 3f_1(t)]\,dt. $$

In the notation of (18.2.6) we have the following:

j	d_j	Δd_j	$\Delta^2 d_j$	$\Delta^3 d_j$
0	0	3	−4	4
1	3	−1	0	—
2	2	−1	—	—
3	1	—	—	—

Hence the actuarial present value is

$$ \int_0^\infty v^t(-1)(3\,{}_tD_1' - {}_tD_2')\,dt = 3\,(\bar{A}_x + \bar{A}_y + \bar{A}_z) - (\bar{A}_{xy} + \bar{A}_{xz} + \bar{A}_{yz}). $$ ▼

18.3 Compound Statuses

In the previous section we defined statuses for several lives by means of the k-survivor status. Others statuses can be defined by compounding. A ***compound status*** is said to exist if the status is based on a combination of statuses, at least one of which is itself a status involving more than one life. We examine some possibilities in Example 18.3.1.

Example 18.3.1

Describe the conditions of payment for the annuities and insurances corresponding to the following actuarial present-value symbols:

a. $\bar{a}_{\overline{wx}:\overline{yz}}$ b. $\bar{a}_{\overline{\overline{wx}:(yz)}}$ c. $\bar{a}_{\overline{(x:\overline{n|}):(yz:\overline{m|})}}$

d. $\bar{A}_{\overline{wx}:yz}$ e. $\bar{A}_{\overline{(wx):(yz)}}$ f. $\bar{A}_{\overline{(wx):y:z}}$.

Solution:

a. The annuity is payable continuously at the rate of 1 per year while at least one of (w) and (x) and at least one of (y) and (z) survive. Thus the annuity is payable while three or four of the lives survive and while two survive if one is from the pair (w), (x) and the other from the pair (y), (z).
b. The annuity is payable continuously at the rate of 1 per year while at least two of the four lives survive and also while only one survives if that survivor is either (w) or (x).
c. The annuity is payable continuously at the rate of 1 per year while either (x) is alive and an n-year period has not elapsed, or while both (y) and (z) are alive and an m-year period has not elapsed.
d. A unit amount is payable at the moment of the first death if (y) or (z) dies first, and otherwise on the second death.
e. A unit amount is payable at the moment of the second death if two deaths consist of one from the (w), (x) group and the other from the (y), (z) group. If not, the payment is made at the time of the third death.
f. A unit amount is payable only after (y), (z), and one of (w) and (x) have died. In other words, it is payable at the moment of the third death if (w) or (x) remains alive, but otherwise at the moment of the fourth death. ▼

In applications, a numerical value for any one of these actuarial present values would most likely be obtained by first expressing it in terms of those for single- and joint-life statuses. The relationships in Section 9.4 among $T(xy)$, $T(\overline{xy})$, $T(x)$, and $T(y)$, and among $K(xy)$, $K(\overline{xy})$, $K(x)$, and $K(y)$, hold for survival statuses (u), (v). For example,

$$v^{T(uv)} + v^{T(\overline{uv})} = v^{T(u)} + v^{T(v)}. \tag{18.3.1}$$

Employing parts of Example 18.3.1, we will illustrate the process of using (18.3.1) and similar identities. First, we consider part (e):

$$\bar{A}_{\overline{(wx):(yz)}} = \bar{A}_{wx} + \bar{A}_{yz} - \bar{A}_{wxyz}.$$

Here, $(u) = (wx)$ and $(v) = (yz)$. To write $\bar{A}_{(wx):(yz)}$ as $\bar{A}_{wxyz}$ we have used

$$\min\{\min[T(w), T(x)], \min[T(y), T(z)]\} = \min[T(w), T(x), T(y), T(z)]. \quad (18.3.2)$$

For part (c) of Example 18.3.1, we have

$$\bar{a}_{\overline{(x:\overline{n|})(yz:\overline{m|})}} = \bar{a}_{x:\overline{n|}} + \bar{a}_{yz:\overline{m|}} - \bar{a}_{xyz:\overline{n|}}$$

where the last term is obtained from

$$\min[T(x), T(y), T(z), T(\overline{n|}), T(\overline{m|})] = \min[T(x), T(y), T(z), T(\overline{n|})]$$

for the case $n \le m$.

Other arrangements, as in parts (a), (b), (d), and (f) of Example 18.3.1, require the use of other relationships. For part (a), we want

$$\bar{a}_{\overline{wx}:\overline{yz}} = \mathrm{E}[\bar{a}_{\overline{T|}}] \quad (18.3.3)$$

where

$$\begin{aligned} T &= \min[T(\overline{wx}), T(\overline{yz})] \\ &= \min\{\max[T(w), T(x)], \max[T(y), T(z)]\}. \end{aligned}$$

A simple answer, like (18.3.2), is not available for this random variable. To proceed, let us first assume that $T(\overline{wx})$ and $T(\overline{yz})$ are independent and look at $s(t)$, the survival function of T. Thus

$$\begin{aligned} s(t) &= \Pr(T > t) = \Pr(\min[T(\overline{wx}), T(\overline{yz})] > t) \\ &= \Pr[T(\overline{wx}) > t, T(\overline{yz}) > t] \\ &= \Pr[T(\overline{wx}) > t] \Pr[T(\overline{yz}) > t] \\ &= {}_tp_{\overline{wx}}\ {}_tp_{\overline{yz}} \\ &= ({}_tp_w + {}_tp_x - {}_tp_{wx})({}_tp_y + {}_tp_z - {}_tp_{yz}) \\ &= {}_tp_{wy} + {}_tp_{wz} + {}_tp_{xy} + {}_tp_{xz} - {}_tp_{wyz} - {}_tp_{xyx} \\ &\quad - {}_tp_{wxy} - {}_tp_{wxz} + {}_tp_{wxyz} \end{aligned} \quad (18.3.4)$$

for the independent case. Now, using (18.3.4), we obtain

$$\begin{aligned} \bar{a}_{\overline{wx}:\overline{yz}} &= \int_0^\infty v^t\, s(t)\, dt \\ &= \bar{a}_{wy} + \bar{a}_{wz} + \bar{a}_{xy} + \bar{a}_{xz} - \bar{a}_{wyz} - \bar{a}_{xyz} - \bar{a}_{wxy} - \bar{a}_{wxz} + \bar{a}_{wxyz}. \end{aligned} \quad (18.3.5)$$

We return to (18.3.4) and show that a parallel relationship for the random variables holds, and then that (18.3.4) is true without the independence assumption. We start with the assertion that for all possible outcomes,

$$\begin{aligned} T(\overline{wx}:\overline{yz}) &= T(wy) + T(wz) + T(xy) + T(xz) - T(wyz) \\ &\quad - T(xyz) - T(wxy) - T(wxz) + T(wxyz). \end{aligned} \quad (18.3.6)$$

The outcomes can be collected into 24 mutually exclusive events according to the order of $T(w)$, $T(x)$, $T(y)$, and $T(z)$. Since the given assertion is symmetric in w and

x and symmetric in y and z, only six different outcomes require verification. As an example, consider $T(w) < T(x) < T(y) < T(z)$ for which the left-hand side of (18.3.6) is $T(\overline{wx}:\overline{yz}) = T(x)$ and the right-hand side is, on a term-by-term basis,

$$T(w) + T(w) + T(x) + T(x) - T(w) - T(x) - T(w) - T(w) + T(w) = T(x)$$

as required. The other cases can be verified in the same way.

An expression in annuities that is parallel to (18.3.6) can be established by similar reasoning. Thus,

$$\begin{aligned} \bar{a}_{\overline{T(\overline{wx}:\overline{yz})}|} &= \bar{a}_{\overline{T(wy)}|} + \bar{a}_{\overline{T(wz)}|} + \bar{a}_{\overline{T(xy)}|} + \bar{a}_{\overline{T(xz)}|} \\ &\quad - \bar{a}_{\overline{T(wyz)}|} - \bar{a}_{\overline{T(xyz)}|} - \bar{a}_{\overline{T(wxy)}|} - \bar{a}_{\overline{T(wxz)}|} + \bar{a}_{\overline{T(wxyz)}|} . \end{aligned} \tag{18.3.7}$$

Taking expectations of both sides of this expression we have (18.3.5).

We emphasize two aspects of the independence assumption for this case. It would not be used to establish (18.3.7), nor is it required in the expectation calculation used to obtain (18.3.5) from (18.3.7). Again, however, to obtain joint-life status functions from single-life life tables, for convenience, we do assume that individual future lifetimes are independent.

18.4 Contingent Probabilities and Insurances

In this section we extend the notion of contingent functions (Section 9.9) to more than two lives. We start with an integral expression for the required probability, or actuarial present value, which can then be rewritten in terms of probabilities or actuarial present value defined on the first death. It is then possible to use some of the techniques of Section 9.10 to complete the evaluation. In any case, numerical integration methods can be used.

To obtain an integral expression for a probability, we use

$$\Pr(A) = \int_{-\infty}^{\infty} \Pr(A \mid T = t)\, f_T(t)\, dt \tag{18.4.1}$$

where T will usually mean the time of death of an individual life.

Example 18.4.1

Express ${}_nq_{wx\overset{2}{y}z}$ in terms of functions contingent on the first death.

Solution:

Here A is the event that (y) is the second life among (w), (x), (y), and (z) to die and does so within n years. Since A is defined by $T(y)$, we use $T(y)$ as T in (18.4.1) to obtain

$${}_nq_{wx\overset{2}{y}z} = \int_0^n \Pr(A \mid T(y) = t)\; {}_tp_y\; \mu_y(t)\, dt.$$

The integral's limits follow from

$$f_{T(y)}(t) = 0 \qquad t < 0$$

and

$$\Pr[A|T(y) = t] = 0 \qquad t > n.$$

Now, (y) will be the second to die if and only if there are exactly two of (w), (x), and (z) surviving at that time. If we assume that $T(y)$ is independent of $T(w)$, $T(x)$, and $T(z)$, then

$$\Pr[A|T(y) = t] = {}_tp_{\overline{wxz}}^{[2]} \qquad t < n$$

and

$$\begin{aligned} {}_nq_{wx\overset{2}{y}z} &= \int_0^n {}_tp_{\overline{wxz}}^{[2]}\ {}_tp_y\ \mu_y(t)\ dt \\ &= \int_0^n ({}_tD_2 - 3\ {}_tD_3)\ {}_tp_y\ \mu_y(t)\ dt \\ &= {}_nq_{wx\overset{1}{y}} + {}_nq_{w\overset{1}{y}z} + {}_nq_{x\overset{1}{y}z} - 3\ {}_nq_{wx\overset{1}{y}z}. \end{aligned}$$

(The second integral comes from applying Theorem 18.2.1.) ▼

The similarity of the final expression of Example 18.4.1 to previous results that did not require independence suggests the assumed independence was not necessary. Alternative derivations in Exercises 18.18 and 18.38 will verify that this is the case.

A contingent insurance can be analyzed by a similar procedure based on

$$\mathrm{E}[Z] = \int_{-\infty}^{\infty} \mathrm{E}[Z|T = t]\, f_T(t)\ dt. \tag{18.4.2}$$

Example 18.4.2

Express $\bar{A}_{wx\overset{2}{y}}$ in terms of actuarial present values for insurances contingent on the first death only.

Solution:

Let Z be the random variable representing the present value at issue of the insurance benefit. Since the insurance is payable on the death of (y), we choose $T(y)$ to play the role of T in the conditional expectation of (18.4.2):

$$\bar{A}_{wx\overset{2}{y}} = \mathrm{E}[Z] = \int_0^{\infty} \mathrm{E}[Z|T(y) = t]\ {}_tp_y\ \mu_y(t)\ dt.$$

If, at the death of (y) at duration t, there is exactly one of (w) and (x) surviving, the unit benefit will be paid; otherwise no benefit will be paid. Thus, we have

$$\mathrm{E}[Z|T(y) = t] = v^t\ {}_tp_{\overline{wx}}^{[1]}$$

and

$$\bar{A}_{wx\overset{2}{y}} = \int_0^\infty v^t \, {}_tp_{\overline{wx}}^{[1]} \, {}_tp_y \, \mu_y(t) \, dt$$

$$= \int_0^\infty v^t({}_tD_1 - 2 \, {}_tD_2) \, {}_tp_y \, \mu_y(t) \, dt$$

$$= \bar{A}_{x\overset{1}{y}} + \bar{A}_{w\overset{1}{y}} - 2 \, \bar{A}_{wx\overset{1}{y}} \, .$$

Because the benefit is 1, Var(Z) can be obtained by the rule of moments. ▼

18.5 Compound Contingent Functions

The functions in this section are distinguished from those in the previous section by specifications on the order of deaths prior to the death when the benefits are paid, or the event is defined. These specifications on the prior deaths are indicated by numbers placed below the symbols for the lives involved. We examine two such symbols and note the distinctions possible in the notation.

The symbols ${}_nq_{\underset{1}{x}\overset{2}{y}z}$ and ${}_nq_{\underset{1}{x}y\overset{3}{z}}$ both refer to events in which $T(x) < T(y) < T(z)$. They differ, though, in that the second death must occur before time n in the first event, while the third death must precede time n in the second event.

It is not always possible to express compound contingent functions completely in terms of functions depending only on the first death. On the other hand, the function can always be expressed as one or more multiple integrals of the joint p.d.f. of the future-lifetime random variables of the lives involved. The example of this general procedure (Example 18.5.1) is more complex than others in this section.

Example 18.5.1

Derive an expression for the probability that (w), (x), (y), and (z) die in that order with less than 10 years between the deaths of (w) and (z) and less than 5 years between the deaths of (x) and (y).

Solution:

We first define the event A for use in a multivariate version of (18.4.1):

$$A = \left\{ \begin{array}{c} T(w) < T(x) < T(y) < T(z) \\ T(z) - T(w) < 10 \\ T(y) - T(x) < 5 \end{array} \right\}. \tag{18.5.1}$$

We choose to condition on $T(w)$ and $T(x)$ because these are involved in both the upper and lower bounds for $T(y)$ and $T(z)$. That is,

$$\Pr(A) = \int_0^\infty \int_0^\infty \Pr(A \,|\, [T(w) = r] \cap [T(x) = s]) \; g_{T(w),T(x)}(r, s) \, ds \, dr \tag{18.5.2}$$

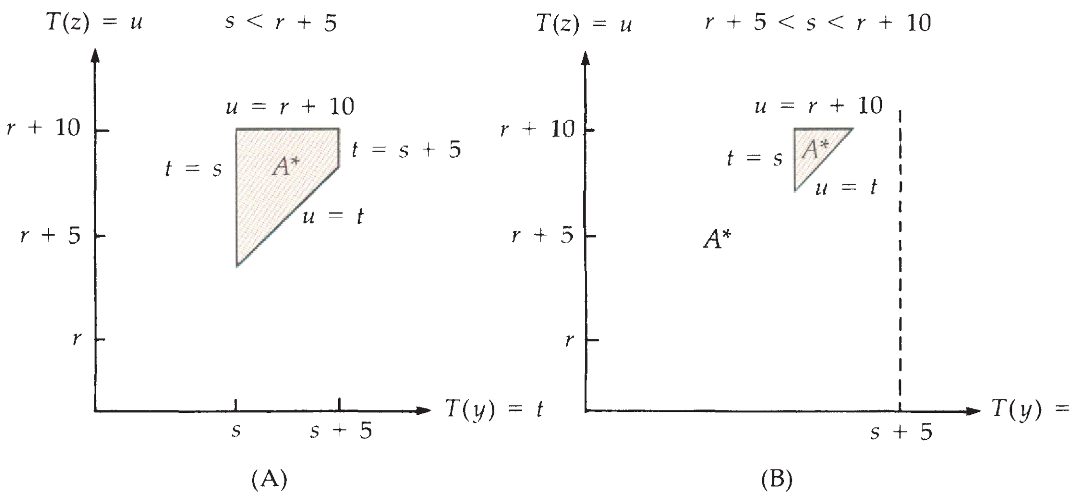
where $g_{T(w),T(x)}(r, s)$ is the joint p.d.f. of $T(w)$ and $T(x)$. Now, $\Pr[A|(T(w) = r) \cap (T(x) = s)]$ is equal to $\Pr(A^*)$ where

$$A^* = \begin{Bmatrix} r < s < T(y) < T(z) < r + 10 \\ T(y) < s + 5 \end{Bmatrix},$$

and the probability is calculated by the conditional distribution of $T(y)$ and $T(z)$, given $T(x) = s$ and $T(w) = r$. Thus, $\Pr(A^*)$ can be set up in the sample space of the random variables $T(y)$ and $T(z)$. Two cases are displayed in Figure 18.5.1.

FIGURE 18.5.1

Cases (A), $s < r + 5$, and (B), $r + 5 < s < r + 10$



Using the abbreviated notation $h(t, u)$ for the conditional p.d.f. of $T(y)$ and $T(z)$ given $T(w) = r$ and $T(x) = s$, we have

$$\Pr(A^*) = \begin{cases} \displaystyle\int_s^{s+5} \int_t^{r+10} h(t, u)\, du\, dt & r < s < r + 5 \\ \displaystyle\int_s^{r+10} \int_t^{r+10} h(t, u)\, du\, dt & r + 5 < s < r + 10 \\ 0 & s > r + 10 \text{ or } s < r. \end{cases}$$

Substituting into (18.5.2), we have

$$\Pr(A) = \int_0^{\infty} \int_r^{r+5} \left[\int_s^{s+5} \int_t^{r+10} h(t, u) g_{T(w),T(x)}(r, s)\, du\, dt \right] ds\, dr$$
$$+ \int_0^{\infty} \int_{r+5}^{r+10} \left[\int_s^{r+10} \int_t^{r+10} h(t, u) g_{T(w),T(x)}(r, s)\, du\, dt \right] ds\, dr.$$

Under the assumption of mutually independent future lifetimes, the integrand can be replaced by

$${}_rp_w\ \mu_w(r)\ {}_sp_x\ \mu_x(s)\ {}_tp_y\ \mu_y(t)\ {}_up_z\ \mu_z(u).$$ ▼

We now examine some compound contingent probabilities that can be written in terms of single integrals. We will first obtain equivalent forms for a probability by applying (18.4.1).

Example 18.5.2

Write three different integrals for ${}_nq_{\underset{1}{x}y\overset{3}{z}}$ and reduce one of them to probability functions dependent on only the first death. Assume mutually independent future lifetimes.

Solution:

Here $A = \{T(x) < T(y) < T(z) < n\}$. We set three integrals by conditioning on each of the future lifetimes:

$$ {}_nq_{\underset{1}{x}y\overset{3}{z}} = \int_0^\infty \Pr[A \,|\, T(x) = t]\ {}_tp_x\ \mu_x(t)\ dt $$

and

$$ \Pr[A \,|\, T(x) = t] = \begin{cases} 0 & t > n \\ {}_tp_{yz}\ {}_{n-t}q_{y+t:\overset{2}{z+t}} & t \le n; \end{cases} $$

thus

$$ {}_nq_{\underset{1}{x}y\overset{3}{z}} = \int_0^n {}_tp_{yz}\ {}_{n-t}q_{y+t:\overset{2}{z+t}}\ {}_tp_x\ \mu_x(t)\ dt. $$

Similarly,

$$ \begin{aligned} {}_nq_{\underset{1}{x}y\overset{3}{z}} &= \int_0^\infty \Pr[A \,|\, T(y) = t]\ {}_tp_y\ \mu_y(t)\ dt \\ &= \int_0^n {}_tq_x\ {}_tp_z\ {}_{n-t}q_{z+t}\ {}_tp_y\ \mu_y(t)\ dt \end{aligned} $$

and

$$ \begin{aligned} {}_nq_{\underset{1}{x}y\overset{3}{z}} &= \int_0^\infty \Pr[A \,|\, T(z) = t]\ {}_tp_z\ \mu_z(t)\ dt \\ &= \int_0^n {}_tq_{x\overset{2}{y}}\ {}_tp_z\ \mu_z(t)\ dt. \end{aligned} $$

The second of these integrals can be expressed in terms of first-death probabilities as follows:

$$ \begin{aligned} {}_nq_{\underset{1}{x}y\overset{3}{z}} &= \int_0^n (1 - {}_tp_x)({}_tp_z - {}_np_z)\ {}_tp_y\ \mu_y(t)\ dt \\ &= {}_nq_{\overset{1}{y}z} - {}_nq_{x\overset{1}{y}z} - {}_np_z\,({}_nq_y - {}_nq_{x\overset{1}{y}}). \end{aligned} $$

▼

Example 18.5.3

Use (18.4.1) to write four different integral expressions for ${}_nq_{w\underset{1}{x}\underset{2}{y}\overset{3}{z}}$. Assume mutually independent future lifetimes.

Solution:

Here $A = \{T(w) < T(x) < T(y) < T(z) \text{ and } T(y) < n\}$. Then

$$
\begin{aligned}
{}_nq_{w\underset{1}{x}\underset{2}{y}\overset{3}{z}} &= \int_0^n {}_tp_{xyz}\ {}_{n-t}q_{\underset{1}{x+t}:\overset{2}{y+t}:z+t}\ {}_tp_w\ \mu_w(t)\ dt \\
&= \int_0^n {}_tq_w\ {}_tp_{yz}\ {}_{n-t}q_{\overset{1}{y+t}:z+t}\ {}_tp_x\ \mu_x(t)\ dt \\
&= \int_0^n {}_tq_{w\overset{2}{x}}\ {}_tp_z\ {}_tp_y\ \mu_y(t)\ dt \\
&= \int_0^n {}_tq_{w\underset{1}{x}\overset{3}{y}}\ {}_tp_z\ \mu_z(t)\ dt + {}_nq_{w\underset{1}{x}\overset{3}{y}}\ {}_np_z.
\end{aligned}
\tag{18.5.3}
$$

The last line, obtained by conditioning on $T(z)$, requires one expression for $T(z) < n$ and another for $T(z) > n$. ▼

In the application of (18.4.1) to the examples of this section we have used the assumption of independent future lifetimes in writing the $\Pr[A|T = t]$ factors of the integrands. We now consider the numerical evaluation of these probabilities when a single Gompertz mortality law is used for each life involved.

Example 18.5.4

Under a Gompertz law, show that

$$
{}_\infty q_{w\underset{1}{x}\underset{2}{y}\overset{3}{z}} = {}_\infty q^{1}_{wxyz}\ {}_\infty q^{1}_{xyz}\ {}_\infty q^{1}_{yz}.
$$

Solution:

Letting $n \to \infty$ in (18.5.3), we have

$$
{}_\infty q_{w\underset{1}{x}\underset{2}{y}\overset{3}{z}} = \int_0^\infty {}_tq_w\ {}_tp_{yz}\ {}_\infty q_{\overset{1}{y+t}:z+t}\ {}_tp_x\ \mu_x(t)\ dt. \tag{18.5.4}
$$

In Example 9.10.1(b) it was shown that, under the Gompertz mortality law,

$$
{}_nq^{1}_{xy} = \frac{c^x}{c^w}\ {}_nq_w \tag{18.5.5}
$$

where $c^w = c^x + c^y$. Adapting this and substituting it for ${}_\infty q_{\overset{1}{\overline{y+t}}:z+t}$ in the integrand of (18.5.4), we obtain

$$
\begin{aligned}
{}_{\infty}q_{wx\overset{3}{y}z \atop 12} &= \int_0^{\infty} \frac{c^{y+t}}{c^{y+t} + c^{z+t}}\ {}_tq_w\ {}_tp_{yz}\ {}_tp_x\ \mu_x(t)\ dt \\
&= \frac{c^y}{c^y + c^z} \left({}_{\infty}q^{\ 1}_{xyz} - {}_{\infty}q^{\ \ 1}_{wxyz}\right).
\end{aligned}
$$

Formula (18.5.5) can be extended to more than two lives and then used in this expression; therefore,

$$
\begin{aligned}
{}_{\infty}q_{wx\overset{3}{y}z \atop 12} &= \frac{c^y}{c^y + c^z} \left(\frac{c^x}{c^x + c^y + c^z} - \frac{c^x}{c^w + c^x + c^y + c^z} \right) \\
&= \left(\frac{c^w}{c^w + c^x + c^y + c^z} \right) \left(\frac{c^x}{c^x + c^y + c^z} \right) \left(\frac{c^y}{c^y + c^z} \right) \\
&= {}_{\infty}q^{1}_{wxyz}\ {}_{\infty}q^{1}_{xyz}\ {}_{\infty}q^{1}_{yz}.
\end{aligned}
$$

▼

18.6 More Reversionary Annuities

In Section 9.7 we examined a number of insurance and annuity contracts involving more than one life. Included in that discussion were the more common types of reversionary annuities, those involving only two lives and some examples with terms certain. We consider examples with terms certain measured from a date of death and examples with contingent events defining the start of the annuity payments. We also restrict our discussion to continuous annuities.

Let us examine two reversionary annuities with a term certain measured from the date of death. For a reversionary annuity paying an n-year temporary annuity to (y) after the death of (x), the term certain is a deferred status, so we go back to first principles. The present value at policy issue, Z, is

$$
Z = \begin{cases} 0 & T(y) \le T(x) \\ v^{T(x)}\ \bar{a}_{\overline{T(y)-T(x)}|} & T(x) < T(y) \le T(x) + n \\ v^{T(x)}\ \bar{a}_{\overline{n}|} & T(x) + n \le T(y). \end{cases}
$$

Using (18.4.2) with conditioning on $T(x) = t$, we can write the actuarial present value as

$$
\begin{aligned}
\mathrm{E}[Z] &= \int_0^{\infty} \mathrm{E}[Z|T(x) = t]\ {}_tp_x\ \mu_x(t)\ dt \\
&= \int_0^{\infty} {}_tp_y\ v^t\ \bar{a}_{y+t:\overline{n}|}\ {}_tp_x\ \mu_x(t)\ dt. \qquad (18.6.1)
\end{aligned}
$$

By substituting

$$
\bar{a}_{y+t:\overline{n}|} = \int_t^{t+n} v^{s-t}\ {}_{s-t}p_{y+t}\ ds
$$

into (18.6.1) we obtain

$$E[Z] = \int_0^\infty \int_t^{t+n} v^s \, {}_sp_y \, {}_tp_x \, \mu_x(t) \, ds \, dt.$$

Next we interchange the order of integration so that

$$\begin{aligned} E[Z] &= \int_0^n \int_0^s v^s \, {}_sp_y \, {}_tp_x \, \mu_x(t) \, dt \, ds + \int_n^\infty \int_{s-n}^s v^s \, {}_sp_y \, {}_tp_x \, \mu_x(t) \, dt \, ds \\ &= \int_0^n v^s \, {}_sp_y \, (1 - {}_sp_x) \, ds + \int_n^\infty v^s \, {}_sp_y \, ({}_{s-n}p_x - {}_sp_x) \, ds \\ &= \bar{a}_{y:\overline{n}|} - \bar{a}_{xy} + v^n \, {}_np_y \, \bar{a}_{x:y+n}. \end{aligned} \tag{18.6.2}$$

The second display in (18.6.2) is the current payment form for this actuarial present value.

Another reversionary annuity of this type would be one where the annuity starts n years after the death of (x) and pays only as long as (y) remains alive. The present value at policy issue, Z, is

$$Z = \begin{cases} 0 & T(y) \le T(x) + n \\ (v^{T(x)+n})\bar{a}_{\overline{T(y)-T(x)-n}|} & T(x) + n < T(y). \end{cases}$$

Using (18.4.2) with conditioning on $T(x) = t$, we can write the actuarial present value as

$$\begin{aligned} E[Z] &= \int_0^\infty E[Z|T(x) = t] \, {}_tp_x \, \mu_x(t) \, dt \\ &= \int_0^\infty {}_{t+n}p_y \, v^{t+n} \, \bar{a}_{y+n+t} \, {}_tp_x \, \mu_x(t) \, dt. \end{aligned} \tag{18.6.3}$$

By substituting

$${}_{t+n}p_y \, v^{t+n} \, \bar{a}_{y+n+t} = \int_{t+n}^\infty v^s \, {}_sp_y \, ds$$

into (18.6.3), we obtain

$$E[Z] = \int_0^\infty \int_{t+n}^\infty v^s \, {}_sp_y \, {}_tp_x \, \mu_x(t) \, ds \, dt.$$

Next we interchange the order of integration, so that

$$\begin{aligned} E[Z] &= \int_n^\infty \int_0^{s-n} v^s \, {}_sp_y \, {}_tp_x \, \mu_x(t) \, dt \, ds \\ &= \int_n^\infty v^s \, {}_sp_y \, (1 - {}_{s-n}p_x) \, ds \\ &= v^n \, {}_np_y \, (\bar{a}_{y+n} - \bar{a}_{x:y+n}) = v^n \, {}_np_y \, \bar{a}_{x|y+n}. \end{aligned} \tag{18.6.4}$$

Another class of reversionary annuities that we consider is of, perhaps, limited commercial interest: those where some contingent event must occur before payments start. We consider two such examples and proceed from first principles.

Example 18.6.1

Express the reversionary annuity's actuarial present value, which has symbol $\bar{a}^{1}_{xy|z}$, (a) by definition and (b) in the current payment form by interchanging the order of integration in your answer to (a).

Solution:

Using (18.4.2) and conditioning on $T(x) = t$,

$$\begin{aligned}
\bar{a}^{1}_{xy|z} &= \int_0^\infty v^t \, {}_tp_x \, \mu_x(t) \, {}_tp_y \, {}_tp_z \, \bar{a}_{z+t} \, dt \\
&= \int_0^\infty {}_tp_x \, \mu_x(t) \, {}_tp_y \left(\int_t^\infty v^s \, {}_sp_z \, ds \right) dt \\
&= \int_0^\infty v^s \, {}_sp_z \left[\int_0^s {}_tp_x \, \mu_x(t) \, {}_tp_y \, dt \right] ds \\
&= \int_0^\infty v^s \, {}_sp_z \, {}_sq^{1}_{xy} \, ds.
\end{aligned}$$

▼

This result can be considered as the current payment form of the actuarial present value. It shows that the general form for reversionary annuities can be interpreted quite broadly with the possibility that the failure of status (u) can involve a contingent probability. In general, we have

$$\bar{a}_{u|v} = \int_0^\infty v^t \, {}_tp_v \, {}_tq_u \, dt. \tag{18.6.5}$$

Our next example shows a particularly simple case involving two lives where the actuarial present value can be reduced to a form not involving integrals.

Example 18.6.2

Express the actuarial present-value symbol $\bar{a}^{1}_{x:\overline{n}|\,|y}$ in a form free of integrals.

Solution:

By (18.6.5),

$$\begin{aligned}
\bar{a}^{1}_{x:\overline{n}|\,|y} &= \int_0^\infty v^t \, {}_tp_y \, {}_tq^{1}_{x:\overline{n}|} \, dt \\
&= \int_0^n v^t \, {}_tp_y \left[\int_0^t {}_sp_x \, \mu_x(s) \, ds \right] dt + \int_n^\infty v^t \, {}_tp_y \left[\int_0^n {}_sp_x \, \mu_x(s) \, ds \right] dt \\
&= \int_0^n v^t \, {}_tp_y \, (1 - {}_tp_x) \, dt + (1 - {}_np_x) \int_n^\infty v^t \, {}_tp_y \, dt \\
&= \bar{a}_y - \bar{a}_{xy:\overline{n}|} - v^n \, {}_np_{xy} \, \bar{a}_{y+n}.
\end{aligned}$$

▼

18.7 Benefit Premiums and Reserves

Here we examine benefit premiums and benefit reserves for the insurances of this chapter. As in Chapter 6, the benefit premium is defined by the equivalence principle. Following the development in Chapter 7, benefit reserves are defined prospectively as the conditional expectation of the future loss, given survival to the duration of the reserve.

The premium payment period must end no later than the time of claim payment and, in the case of contingent insurances, when it is clear that no claim payment can be made. The period may be shorter.

In the case of insurances payable on the first death, the premiums are payable only while all lives survive. Using the equivalence principle we have, for example, the following:

$$P_{xy}\, \ddot{a}_{xy} = A_{xy},$$

$$_{10}P^{\{4\}}(\bar{A}_{\overset{1}{xy}:\overline{20}|})\; \ddot{a}^{\{4\}}_{xy:\overline{10}|} = \bar{A}_{\overset{1}{xy}:\overline{20}|},$$

and

$$P(\bar{A}^{1}_{xyz})\ddot{a}_{xyz} = \bar{A}^{1}_{xyz}.$$

Insurances payable on the second or a later death give rise to more than one possible premium payment period. To minimize the benefit premium that can be charged for a particular insurance benefit, we use the longest period. The following example illustrates the process for a number of cases.

Example 18.7.1

Using the equivalence principle, write the equation for the following benefit premiums:

a. $P_{\overline{xy}}$ b. $P(\bar{A}_{\overset{2}{\overline{xyz}}})$ c. $P(\bar{A}_{\overline{wx}:yz})$

d. $P(\bar{A}_{x\overset{2}{y}z})$ e. $P(\bar{A}_{\underset{1}{x}\overset{2}{y}z})$.

Solution:

a. $P_{\overline{xy}}\, \ddot{a}_{\overline{xy}} = A_{\overline{xy}}$

b. $P(\bar{A}_{\overset{2}{\overline{xyz}}})\ddot{a}_{\overset{2}{\overline{xyz}}} = \bar{A}_{\overset{2}{\overline{xyz}}}$

c. $P(\bar{A}_{\overline{wx}:yz})\ddot{a}_{\overline{wx}:yz} = \bar{A}_{\overline{wx}:yz}$

d. As long as (y) and at least one of (x) and (z) are alive, payment of the benefit is still possible. Therefore,

$$P(\bar{A}_{x\overset{2}{y}z})\; \ddot{a}_{y:\overline{xz}} = \bar{A}_{x\overset{2}{y}z}.$$

e. In this case payment of the benefit is still possible if all are alive or if only (y) and (z) are alive. Thus the appropriate premium payment period is the lifetime

of (yz), and

$$P(\bar{A}_{\substack{xyz \\ 1}}^{\;2})\, \ddot{a}_{yz} = \bar{A}_{\substack{xyz \\ 1}}^{\;2}. \qquad \blacktriangledown$$

As the conditional expectation of the future loss, the benefit reserve will depend on the condition of the status used in the calculation. The reserve is unique for an insurance payable on the first death because all lives must survive until termination of the insurance. We illustrate reserve formulas for two of these insurances:

$${}_5V_{\overline{xy}:\overline{10|}}^{\;1} = A_{\overline{x+5:y+5}:\overline{5|}}^{\;1} - P_{\overline{xy}:\overline{10|}}^{\;1}\, \ddot{a}_{x+5:y+5:\overline{5|}}$$

where

$$P_{\overline{xy}:\overline{10|}}^{\;1}\, \ddot{a}_{xy:\overline{10|}} = A_{\overline{xy}:\overline{10|}}^{\;1}$$

and

$${}_5V_{xyz}^{1} = A_{x+5:y+5:z+5}^{1} - P_{xyz}^{1}\, \ddot{a}_{x+5:y+5:z+5}\,.$$

For an insurance payable on the second or later death, the benefit reserve can be calculated with the given condition of the expectation being either (a) which lives are surviving or (b) only that the insurance has not terminated through the last death.

Consider the simple case of a fully continuous unit insurance payable upon the failure of $(\overline{xy})$ with premiums payable until the second death. Let ${}_tL$ be the future loss at t. Given the information about which of x and y (or both) are surviving at t, we would have

$$\mathrm{E}[{}_tL \mid T(x) > t \cap T(y) > t] = \bar{A}_{\overline{x+t:y+t}} - \bar{P}(\bar{A}_{\overline{xy}})\, \bar{a}_{\overline{x+t:y+t}}, \tag{18.7.1}$$

$$\mathrm{E}[{}_tL \mid T(x) > t \cap T(y) \le t] = \bar{A}_{x+t} - \bar{P}(\bar{A}_{\overline{xy}})\, \bar{a}_{x+t}, \tag{18.7.2}$$

or

$$\mathrm{E}[{}_tL \mid T(x) \le t \cap T(y) > t] = \bar{A}_{y+t} - \bar{P}(\bar{A}_{\overline{xy}})\, \bar{a}_{y+t}. \tag{18.7.3}$$

On the other hand if the given information is only that the survival status $(\overline{xy})$ has not failed, the benefit reserve is

$${}_t\bar{V}(\bar{A}_{\overline{xy}}) = \mathrm{E}[{}_tL \mid T(\overline{xy}) > t]$$

which we can calculate by the law of total probability as the sum

$$\begin{aligned} &\mathrm{E}[{}_tL \mid T(x) > t \cap T(y) \le t]\ \Pr[T(x) > t \cap T(y) \le t] \\ &+ \mathrm{E}[{}_tL \mid T(x) \le t \cap T(y) > t]\ \Pr[T(x) \le t \cap T(y) > t] \\ &+ \mathrm{E}[{}_tL \mid T(x) > t \cap T(y) > t]\ \Pr[T(x) > t \cap T(y) > t]. \end{aligned} \tag{18.7.4}$$

In this expression the conditional expectations are given by (18.7.1)–(18.7.3). On the assumption of independent $T(x)$ and $T(y)$, the probabilities are of the form

$$\Pr[T(x) > t \cap T(y) \le t \mid T(\overline{xy}) > t] = \frac{{}_tp_x\,(1 - {}_tp_y)}{{}_tp_x\,(1 - {}_tp_y) + {}_tp_y\,(1 - {}_tp_x) + {}_tp_x\,{}_tp_y}.$$

Combining these we have

$$ {}_t\bar{V}(\bar{A}_{\overline{xy}}) = \left[\frac{1}{{}_tp_x\,(1 - {}_tp_y) + {}_tp_y\,(1 - {}_tp_x) + {}_tp_x\,{}_tp_y}\right] \{{}_tp_x(1 - {}_tp_y)[\bar{A}_{x+t} - \bar{P}(\bar{A}_{\overline{xy}})\bar{a}_{x+t}] $$

$$ + {}_tp_y(1 - {}_tp_x)[\bar{A}_{y+t} - \bar{P}(\bar{A}_{\overline{xy}})\bar{a}_{y+t}] + {}_tp_x\,{}_tp_y\,[\bar{A}_{\overline{x+t:y+t}} - \bar{P}(\bar{A}_{\overline{xy}})\bar{a}_{\overline{x+t:y+t}}]\}. \tag{18.7.5} $$

We now use the results from Section 9.7,

$$ \bar{a}_{\overline{xy}} = \bar{a}_x + \bar{a}_y - \bar{a}_{xy} $$

and

$$ \bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}\,, $$

in the final bracketed term of (18.7.5) to establish the equality

$$ {}_tp_x\,{}_tp_y[\bar{A}_{\overline{x+t:y+t}} - \bar{P}(\bar{A}_{\overline{xy}})\bar{a}_{\overline{x+t:y+t}}] $$

$$ = {}_tp_x\,{}_tp_y[\bar{A}_{x+t} + \bar{A}_{y+t} - \bar{A}_{x+t:y+t} - \bar{P}(\bar{A}_{\overline{xy}})(\bar{a}_{x+t} + \bar{a}_{y+t} - \bar{a}_{x+t:y+t})]. $$

Substituting this into (18.7.5) we have

$$ {}_t\bar{V}(\bar{A}_{\overline{xy}}) = [({}_tp_x\,\bar{A}_{x+t} + {}_tp_y\,\bar{A}_{y+t} - {}_tp_x\,{}_tp_y\,\bar{A}_{x+t:y+t}) $$

$$ - \bar{P}(\bar{A}_{\overline{xy}})({}_tp_x\,\bar{a}_{x+t} + {}_tp_y\,\bar{a}_{y+t} - {}_tp_x\,{}_tp_y\,\bar{a}_{x+t:y+t})]\,/\,({}_tp_x + {}_tp_y - {}_tp_{xy}) \tag{18.7.6} $$

Because $(\overline{xy})$ is a survival status it has a proper conditional survival function, given that it has survived to t, which we will denote by ${}_up_{\overline{xy}+t}$. The benefit reserve for the insurance that was just discussed above can be calculated directly from the conditional survival function if it is first calculated. More precisely,

$$ {}_up_{\overline{xy}+t} = \Pr[T(xy) > u + t \,|\, T(xy) > t] $$

$$ = \frac{{}_{t+u}p_x + {}_{t+u}p_y - {}_{t+u}p_{xy}}{{}_tp_x + {}_tp_y - {}_tp_{xy}}. \tag{18.7.7} $$

We emphasize that only when $t = 0$ is it known that both (x) and (y) are alive. If we assume independence between the future lifetimes of (x) and (y), we have as the corresponding conditional p.d.f. for the last survivor status, given that it has survived to t,

$$ \frac{{}_{u+t}p_x\,\mu_x(t + u) + {}_{u+t}p_y\,\mu_y(t + u) - {}_{u+t}p_{xy}[\mu_x(t + u) + \mu_y(t + u)]}{{}_tp_x + {}_tp_y - {}_tp_{xy}}. \tag{18.7.8} $$

If each of the ${}_{u+t}p$ factors in the numerators of (18.7.7) and (18.7.8) is factored as ${}_{u+t}p_x = {}_up_{x+t}\,{}_tp_x$, for example, then expressions in those equations will appear as weighted averages with the weights being the probabilities of survival to t.

When (18.7.8) is used to calculate $E[v^{T(xy)-t} - \bar{P}(\bar{A}_{\overline{xy}})\,\bar{a}_{\overline{T(xy)-t}|}\,|\,T(xy) > t]$, then (18.7.6) is obtained again.

18.8 Notes and References

The practical applications of the ideas of this chapter have not been as numerous as those in some of the others. Nevertheless, extensive actuarial literature exists on

various topics in multiple-life theory. Parts of Chapters 10, 11, 12, and 13 in Jordan (1967), and parts of Chapters 7 and 8 in Neill (1977) contain material on these topics.

Theorem 18.2.2 is a basic theorem of probability. It combines many of the ideas in Chapter 4 of Feller (1968). The technique used in proving results of this type is often called the ***method of inclusion and exclusion.*** The main results in the field are summarized and an extensive reference list provided by Takács (1967). Credit for the application of these algebraic methods in calculating life annuity values has been given to Waring. Earlier actuarial textbooks gave the results of Corollaries 18.2.1 and 18.2.3 by the so-called ***Z method.*** This was an algebraic mnemonic based on the observation that the coefficients of ${}_tD_j$ in ${}_tp_{\overline{x_1 \cdots x_m}}^{[k]}$ and in ${}_tp_{\overline{x_1 \cdots x_m}}^{k}$ are those in the expansions of $Z^k/(1 + Z)^{k+1}$ and $Z^k/(1 + Z)^k$, respectively.

An earlier version of Theorem 18.2.2 is contained in a discussion by Schuette and Nesbitt of a paper by White and Greville (1959). The use of these methods to determine the actuarial present value of a share in a share-and-share-alike last-survivor annuity is the subject of Exercise 18.36 and a paper by Rasor and Myers (1952). Another proof of Theorem 18.2.2 that avoids the use of ideas from probability is given by Buchta (1994).

Some of the issues regarding premiums and reserves on last-survivor insurances were discussed by Frasier in *The Actuary* (1978).

Life insurance policies with nonforfeiture values contain an embedded option. At each policy anniversary the insured has the option to take the nonforfeiture value and negotiate a new insurance contract, using health and market information available at that time, in an attempt to increase the actuarial present value of life insurance wealth. Reynolds (1994) discusses the cost implications of this option with respect to last survivor policies for which premiums, reserves, and nonforfeiture values are determined, using a conditional survival function as in (18.7.7). Reynolds develops the proposition that mortality antiselection, in the sense that those statuses exercising the withdrawal option will be in "better health," that is, the statuses will have a higher probability of long survival than those continuing, will be significant. Provision for the expected cost of this option should be built into the design of the policy. The argument depends on the observation that nonforfeiture values, like reserves, that are derived from conditional survival functions that assume only the survival of the status will tend to be larger than those that incorporate additional information about the survival of (x) and (y) as in (18.7.1), (18.7.2), and (18.7.3).

Appendix

Theorem 18.2.2

Let $A_1, A_2, \ldots, A_n$ represent the events of interest, and let $P_{[j]}$ denote the probability that exactly j of the n events take place. Further, let D_j be the sum, for all combinations of j events out of the n, of the probabilities that j specified events will occur, irrespective of the occurrence of the other $n - j$ events. Then, for any choice of numbers $c_0, c_1, \ldots, c_n$,

$$c_0P_{[0]} + c_1P_{[1]} + c_2P_{[2]} + \cdots + c_nP_{[n]} = c_0 + D_1\Delta c_0 + D_2\Delta^2 c_0 + \cdots + D_n\Delta^n c_0.$$

Proof:

Let X_i denote the indicator for the event A_i, that is, $X_i = 1$ for sample points in A_i and $X_i = 0$ for sample points not in A_i. Let Y_j be the indicator such that $Y_j = 1$ for sample points in exactly j of the n events $A_1, A_2, \ldots, A_n$ and $Y_j = 0$ for the other sample points. We note that the expectation of Y_j is $P_{[j]}$. Finally, we define an operator, $\phi(\mathrm{E})$, a function of the shift operator, $\mathrm{E} = 1 + \Delta$, by

$$\phi(\mathrm{E}) = (\mathrm{X}_1\mathrm{E} + 1 - \mathrm{X}_1)(\mathrm{X}_2\mathrm{E} + 1 - \mathrm{X}_2) \cdots (\mathrm{X}_n\mathrm{E} + 1 - \mathrm{X}_n).$$

We note that any factor equals E if the corresponding $X_t = 1$ and equals 1 if $X_i = 0$. After multiplying, we have for any single point

$$\phi(\mathrm{E}) = Y_0 + Y_1\mathrm{E} + Y_2\mathrm{E}^2 + \cdots + Y_n\mathrm{E}^n$$

since, in the expansion of the product, the power of E is equal to the number of the X_i equaling 1. Thus the exponent of E is equal to the number of the events $A_1, A_2, A_3, \ldots, A_n$ containing the sample point.

Since a power of the shift operator, E^j, applied to c_0 yields c_j, we obtain

$$\phi(\mathrm{E})c_0 = c_0Y_0 + c_1Y_1 + \cdots + c_nY_n,$$

and then the expectation of $\phi(\mathrm{E})c_0$ is

$$c_0P_{[0]} + c_1P_{[1]} + \cdots + c_nP_{[n]}.$$

Since $\mathrm{E} = 1 + \Delta$, we can also write $\phi(\mathrm{E})$ as

$$\begin{aligned}\phi(\mathrm{E}) &= (1 + \mathrm{X}_1\Delta)(1 + \mathrm{X}_2\Delta) \cdots (1 + \mathrm{X}_n\Delta) \\ &= 1 + \sum_{j=1}^{n} \left(\sum_{i_1, i_2, \ldots, i_j} \mathrm{X}_{i_1} \mathrm{X}_{i_2} \cdots \mathrm{X}_{i_j} \right) \Delta^j,\end{aligned}$$

which displays the coefficient of Δ^j as the sum of all possible products, $\binom{n}{j}$ in number, of the X_i taken j at a time. Since $\mathrm{X}_{i_1} \mathrm{X}_{i_2} \cdots \mathrm{X}_{i_j} = 1$ only if the sample point is in $A_{i_1} A_{i_2} \cdots A_{i_j}$, the expectation of $\mathrm{X}_{i_1} \mathrm{X}_{i_2} \cdots \mathrm{X}_{i_j}$ is $\Pr(A_{i_1} A_{i_2} \cdots A_{i_j})$ and the expectation of

$$\sum_{i_1, i_2, \ldots, i_j} \mathrm{X}_{i_1} \mathrm{X}_{i_2} \cdots \mathrm{X}_{i_j}$$

is D_j. Hence the expectation of $\phi(\mathrm{E})c_0$ can also be written as

$$c_0 + D_1\Delta c_0 + D_2\Delta^2 c_0 + \cdots + D_n\Delta^n c_0.$$

Equating the two forms for the expectation of $\phi(\mathrm{E})c_0$ completes the proof of the theorem. ■

The familiar inclusion-exclusion theorem of probability provides an example of applying Theorem 18.2.2. For $n = 4$,

$$\Pr(A_1 \cup A_2 \cup A_3 \cup A_4) = P_{[1]} + P_{[2]} + P_{[3]} + P_{[4]}.$$

Here $c_0 = 0$, and $c_1 = c_2 = c_3 = c_4 = 1$ in the first form of the expectation of $\phi(\mathrm{E})c_0$. From the table

i	c_i	Δc_i	$\Delta^2 c_i$	$\Delta^3 c_i$	$\Delta^4 c_i$
0	0	1	−1	1	−1
1	1	0	0	0	—
2	1	0	0	—	—
3	1	0	—	—	—
4	1	—	—	—	—

we see that the second form of the expectation is

$$\begin{aligned}\Pr(A_1 \cup A_2 \cup A_3 \cup A_4) &= D_1 - D_2 + D_3 - D_4 \\ &= \sum_{i=1}^{4} \Pr(A_i) - \sum_{\substack{\text{all combinations}\\ \text{of two of}\\ 1,2,3,4}} \Pr(A_iA_j) \\ &\quad + \sum_{\substack{\text{all combinations}\\ \text{of three of}\\ 1,2,3,4}} \Pr(A_iA_jA_k) - \Pr(A_1A_2A_3A_4).\end{aligned}$$

Exercises

Unless otherwise indicated, all lives are subject to the same table of mortality rates, and their time-until-death random variables are independent.

Section 18.2

18.1. Describe the events having probabilities given by the following expressions:

a. ${}_tp_{wx} + {}_tp_{wy} + {}_tp_{wz} + {}_tp_{xy} + {}_tp_{xz} + {}_tp_{yz} - 3({}_tp_{wxy} + {}_tp_{wxz} + {}_tp_{wyz} + {}_tp_{xyz}) + 7\,{}_tp_{wxyz}$

b. ${}_tp_w + {}_tp_x + {}_tp_y + {}_tp_z - 2({}_tp_{wx} + {}_tp_{wy} + {}_tp_{wz} + {}_tp_{xy} + {}_tp_{xz} + {}_tp_{yz}) + 4({}_tp_{wxy} + {}_tp_{wxz} + {}_tp_{wyz} + {}_tp_{xyz}) - 8\,{}_tp_{wxyz}$.

18.2. Use the corollaries of Section 18.2 to verify that ${}_tp_{\overline{x_1x_2\cdots x_m}}^{[0]} = 1 - {}_tp_{\overline{x_1x_2\cdots x_m}}^{1}$.

18.3. An extract from a table of joint-life annuities valued at 3-1/2% interest reads as follows.

Joint-Life Status	Actuarial Present Value of Joint-Life Annuity-Immediate
20:26:28	14.4
20:26:29	14.3
20:28:29	14.0
26:28:29	13.8
20:26:28:29	12.5

a. Calculate the actuarial present value of an annuity payable at the end of each year while exactly three of (20), (26), (28), and (29) are alive.
b. Calculate the actuarial present value for an insurance of 10,000 payable at the end of the year of death of the second life to fail out of (20), (26), (28), and (29).

18.4. Express ${}_tp_{\overline{wxyz}}^{\;2} - {}_tp_{\overline{wxyz}}^{\;[2]}$ in terms of ${}_tD_j$, $j = 1, 2, 3, 4$.

18.5. Express, in terms of annuity symbols, the actuarial present value of an annuity of 1 per year payable at the end of each year while (w) and at most one of (x), (y), and (z) are alive.

18.6. If $\mu_{40}(t) = 0.002$, $0 \le t \le 10$, and $\delta = 0.05$, calculate the value of $\bar{A}_{40:40:40:40:40:\overline{10|}}$.

18.7. A trust is set up to provide income to (x), (y), and (z). The fund is to provide a continuous income at the rate of 8 per year to each while all three are alive, at a rate of 10 per year to each while two are alive, and at a rate of 15 per year to a sole survivor. Calculate the actuarial present values of
a. All the payments to be made
b. All the payments to be made to (x).

18.8. An insurance provides a death benefit of 4 payable immediately upon the first death among four lives age x, a benefit of 3 payable upon the second death, a benefit of 2 payable upon the third death, and a benefit of 1 payable upon the last death. If $\bar{A}_x = 0.4$ and $\bar{A}_{xx} = 0.5$, evaluate the actuarial present value of this insurance.

Section 18.3

18.9. Develop an expression in terms of single- and joint-life annuity symbols for the actuarial present value of an annuity-immediate of 1,000 per month payable
a. While exactly one of (40) and (35) is surviving during the next 25 years
b. While at least one of (40) and (35) survives at an age less than age 65.

18.10. Express the following in terms of symbols of annuities certain and single- and joint-life annuities:

a. $\bar{a}_{\overline{x:y:\overline{n}|}}$

b. $\bar{a}_{\overline{(25:\overline{40}|):\overline{30}|}}$.

Section 18.4

18.11. If at each duration the force of mortality for (x) is $1/2$ that for (y) while the force of mortality for (z) is twice that for (y), what is the probability that of the three lives (x) will die

a. First

b. Second

c. Third.

18.12. Which of the following statements are true? Correct the others as necessary.

a. $\bar{A}_{\overline{wxy\overset{1}{z}}} = \bar{A}_{\overset{1}{w}xyz} + \bar{A}_{w\overset{1}{x}yz} + \bar{A}_{wx\overset{1}{y}z} + \bar{A}_{wxy\overset{1}{z}}$

b. $\bar{A}_{\overline{wxy\overset{3}{z}}} = \bar{A}_{\overset{2}{w}xyz} + \bar{A}_{w\overset{2}{x}yz} + \bar{A}_{wx\overset{2}{y}z} + \bar{A}_{wxy\overset{2}{z}}$

c. $\bar{A}_{wxy\overset{3}{z}} = \bar{A}_{w\overset{1}{z}} + \bar{A}_{x\overset{1}{z}} + \bar{A}_{y\overset{1}{z}} - (\bar{A}_{wx\overset{1}{z}} + \bar{A}_{wy\overset{1}{z}} + \bar{A}_{xy\overset{1}{z}}) + \bar{A}_{wxy\overset{1}{z}}$.

18.13. Write, as a definite integral, the actuarial present value for an insurance to be paid at the moment of death of (x) if (x) survives (y). The benefit amount is equal to the time elapsed between the issue of the policy and the date of death of (y).

18.14. If Gompertz's law applies with $\mu(40) = 0.003$ and $\mu(56) = 0.012$, calculate

a. ${}_{\infty}q_{40:\overset{2:3}{48}:56}$ b. ${}_{\infty}q_{40:\overset{2}{48}:56}$.

[Note: In part (a) the notation 2:3 indicates the event that (48) dies second or third among the lives involved.]

18.15. An insurance of 1 issued on the lives (x), (y), and (z) is payable at the moment of death of (z) only if (x) has been dead at least 10 years and (y) has been dead less than 10 years. Express the actuarial present value of this insurance in terms of actuarial present values for insurances and pure endowments.

18.16. Develop an expression that does not involve integrals for the actuarial present value of an insurance of 1 payable 10 years after the death of (x), provided that either or both of (y) and (z) survive (x) and both are dead before the end of the 10-year period.

18.17. Obtain a formula for the single contract premium for a special contingent, unit insurance payable if (30) dies before (60), or within 5 years after the death of the latter, with return of the contract premium, without interest, 5 years after the death of (60) if no claim under the insurance arises by the death of (30). Assume the loading is 7-1/2% of the benefit premium.

Section 18.5

18.18. Without using the independence assumption, establish relations such as

$$ {}_nq_{\overset{1}{w}xy} = {}_nq_{\overset{1}{w}xyz} + {}_nq_{xwx\overset{2}{y}\underset{1}{z}}, $$

and use them to obtain the result of Example 18.4.1.

18.19. Without using the independence assumption, establish the relations

$$ \bar{A}_{\overset{1}{x}y} = \bar{A}_{\overset{1}{x}yz} + \bar{A}_{x\overset{2}{y}\underset{1}{z}} $$

$$ \bar{A}_{\overset{1}{y}z} = \bar{A}_{x\overset{1}{y}z} + \bar{A}_{\underset{1}{x}\overset{2}{y}z}, $$

and use them to obtain the result of Example 18.4.2.

18.20. Express ${}_\infty q_{\underset{1}{w}\overset{2}{x}yz}$

a. As a definite integral
b. In terms of simple contingent probabilities.

18.21. Assuming that Gompertz's law applies, show that

a. ${}_tq_{x\overset{2}{y}} = {}_tq_y - \dfrac{c^y}{c^x + c^y}\, {}_tq_{xy}$

b. $\bar{A}_{\underset{1}{x}y\overset{3}{z}} = \dfrac{c^x}{c^x + c^y} \bar{A}_z - \dfrac{c^z}{c^y + c^z} \bar{A}_{yz} + \dfrac{c^y}{c^x + c^y} \dfrac{c^z}{c^x + c^y + c^z} \bar{A}_{xyz}.$

18.22. If $\mu(x) = 1/(100 - x)$ for $0 < x < 100$ applies for (20), (40), and (60), evaluate
a. ${}_\infty q_{20:\overset{2}{40}:\underset{1}{60}}$ b. ${}_\infty q_{20:40:\overset{1}{60}}$ c. ${}_\infty q_{20:\overset{1}{40}}$.

This illustrates that ${}_\infty q_{\underset{1}{x}\overset{2}{y}z} = {}_\infty q_{\overset{1}{x}yz}\, {}_\infty q_{\overset{1}{y}z}$, which holds on the basis of Gompertz's law, does not hold in general.

18.23. On the basis of a mortality table following Gompertz's law (with $c^8 = 2$), $\bar{A}_{54} = 0.3$, $\bar{A}_{62} = 0.4$, and $\bar{A}_{70} = 0.52$. Determine $\bar{A}_{\overset{2}{54}:54:\underset{1}{62}}$.

18.24. Given $\bar{A}_w = 0.6$, $\bar{A}_{\overset{1}{w}x} = 0.3$, $\bar{A}_{\overset{1}{w}xx} = 0.2$, and $\bar{A}_{\overset{1}{w}xxx} = 0.1$, evaluate
a. $\bar{A}_{\overset{2}{w}xxx}$ b. $\bar{A}_{\overset{4}{w}xxx}$ c. $\bar{A}_{\overset{4}{w}\underset{1}{x}\underset{2}{x}\underset{2}{x}}$.

18.25. Express in integral form the probability that (x), (y), and (z) will die in that order within the next 25 years with at least 10 years separating the times of any pair of deaths.

18.26. Express in integral form the probability that (10), (20), and (30) will all die before attaining age 60 with (20) being the second to die.

18.27. Given ${}_{\infty}q^{\ 1}_{xy} = 0.5537$, ${}_{\infty}q^{\ 1}_{xz} = 0.6484$, ${}_{\infty}q^{\ 1}_{xyz} = 0.5325$, and ${}_{\infty}q^{\ \ 2}_{xyz} = {}_{\infty}q^{\ \ \ 3}_{xyz}$, calculate ${}_{\infty}q^{\ \ 2}_{x\underset{1}{y}z}$.

18.28. According to a certain mortality table, the probability that three lives age 70, 55, and 40 will die in that order at intervals of not less than 15 years is 0.048, and the probability that at least one of two lives now age 70 will be alive 15 years before the death of a life now age 55 is 0.8. Calculate the probability that neither of two lives now age 40 will survive to age 70.

18.29. Which of the following statements are true? Correct the others as necessary.

a. $\bar{A}_{\underset{1\ 2}{wxyz}^{\ \ 3}} = \int_0^\infty v^t\ {}_tq_w\ {}_tp_{xyz}\ \mu_x(t)\ \bar{A}_{y+t}\ dt$

b. $\displaystyle\int_0^{10} (1 - {}_{t+10}p_{50})\ {}_tp_{60}\ \mu_{60}(t)\ dt + \int_{10}^\infty ({}_{t-10}p_{50} - {}_{t+10}p_{50})\ {}_tp_{60}\ \mu_{60}(t)\ dt$

$$= \int_0^{10} (1 - {}_{t+10}p_{60})\ {}_tp_{50}\ \mu_{50}(t)\ dt + \int_{10}^\infty ({}_{t-10}p_{60} - {}_{t+10}p_{60})\ {}_tp_{50}\ \mu_{50}(t)\ dt$$

c. ${}_{30}q^{\ 1}_{40:50:60} + {}_{30}q^{\ 2}_{40:50:60} = {}_{30}q^{\ 1}_{40:\overline{50:60}}$.

Section 18.6

18.30. Write an expression, free of integrals, for the actuarial present value of a continuous annuity payable at a rate of 1 per year

a. During the lifetime of (y) and for 10 years following the death of (y) with no payments to be made while (x) is alive

b. During the lifetime of (y) and for 10 years following the death of (y) with no payments to be made while (x) is alive or if (y) dies before (x).

18.31. Express in terms of annuity and insurance symbols the single contract premium to provide the following benefits with a loading of 8% of the contract premium:

A last-survivor annuity of 1 per annum on (x) and (y) deferred n years and reducing by 1/3 on the first death: If the death of (x) occurs before the death of (y) and during the deferred period, the annuity on the reduced basis is commenced on the next anniversary. If the death of (x) occurs after the death of (y) and during the deferred period, the contract premium is to be refunded at the end of the year of death.

18.32. In Section 18.6, the reversionary annuities commenced payment if a status (u) was surviving upon the failure of a status (v). This idea can be extended to annuities with payment commencing upon the occurrence of two or more deaths in a prescribed order.

a. Show that

$$a_{\overset{\ 2}{xy}|z} = a_{y|z} - a_{\overset{1}{xy}|z}.$$

b. On the basis of a Gompertz mortality table, prove that

$$a_{\overset{2}{xy}|z} = \frac{c^x}{c^x + c^y} a_z - a_{yz} + \frac{c^y}{c^x + c^y} a_{xyz}.$$

Section 18.7

18.33. Develop an expression for the annual benefit premium for a contingent pure endowment of 1 payable if (x) is alive n years after the death of (y).

18.34. What annuity actuarial present value should be used to obtain the annual benefit premium corresponding to the actuarial present value $A_{\underset{1}{w}\overset{2}{x}yz}$?

Miscellaneous

18.35. An insurance on the event that (x) dies before age $x + n$ and (y) dies before age $y + m$, with $m < n$, pays 1 at the end of the year in which the second death occurs.

a. Show that the actuarial present value can be expressed as

$$A_{\overset{\ 1}{xy}:\overline{m|}} + v^m \, {}_mp_x \, (1 - {}_mp_y) \, A^{\ \ 1}_{x+m:\overline{n-m|}}.$$

b. What is the appropriate annuity actuarial present value to be used to obtain the annual benefit premium?

18.36. A collection of m lives are to share-and-share-alike in the income from a last-survivor annuity of 1 per annum payable continuously. The actuarial present value of (x_1)'s share is

$$\sum_{j=0}^{m-1} \frac{1}{j+1} \bar{a}_{x_1:\overline{x_2 x_3 \cdots x_m}^{[j]}}.$$

Show that this actuarial present value can be expressed as

$$\bar{a}_{x_1} - \frac{1}{2}(\bar{a}_{x_1 x_2} + \cdots + \bar{a}_{x_1 x_m})$$

$$+ \frac{1}{3}(\bar{a}_{x_1 x_2 x_3} + \cdots + \bar{a}_{x_1 x_{m-1} x_m}) - \cdots (-1)^{m-1} \frac{1}{m} \bar{a}_{x_1 x_2 \cdots x_m}.$$

$$\left[\text{Hint: Use Theorem 18.2.1 on } \sum_{j=0}^{m-1} \frac{1}{j+1} \, {}_tp_{\overline{x_2 \cdots x_m}^{[j]}}.\right]$$

18.37. State in words what is represented by

$$\int_0^\infty v^t \, {}_tq_x \, {}_tp_{yz} \, \mu_y(t) \, \bar{A}^{\ \ 1}_{z+t:\overline{10|}} \, dt.$$

18.38. In the notation of Theorem 18.2.2 let

$$A_1 = \{T(y) < \min[n, T(x), T(z)]\},$$

$$A_2 = \{T(y) < \min[n, T(x), T(w)]\},$$

$$A_3 = \{T(y) < \min[n, T(w), T(z)]\}.$$

Show that the event A of Example 18.4.1 is the same as the event that occurs when exactly one of A_1, A_2, and A_3 occurs. Hence use Theorem 18.2.2 to establish the result of Example 18.4.1 without use of the independence assumption. [Hint: Argue that $\Pr(A_1) = {}_{n}q_{wx\overset{1}{y}z}$, $\Pr(A_1A_2) = \Pr(A_1A_3) = \Pr(A_2A_3) = \Pr(A_1A_2A_3) = {}_{n}q_{wx\overset{1}{y}z}$, and $\Pr(A_1\,[\text{not } A_2][\text{not } A_3]) = {}_{n}q_{w\underset{1}{x}\overset{2}{y}z}$.]

18.39. Consider a second-to-die whole life insurance with premiums payable for as long as at least one of (x) and (y) survive. Assume a fully discrete basis.

a. Show that ${}_kV_{\overline{xy}} = 1 - \dfrac{\ddot{a}_{\overline{xy}+k}}{\ddot{a}_{\overline{xy}}}$

where $\ddot{a}_{\overline{xy}+k} = \dfrac{{}_kp_x\,\ddot{a}_{x+k} + {}_kp_y\,\ddot{a}_{y+k} - {}_kp_x\,{}_kp_y\,\ddot{a}_{x+k:y+k}}{{}_kp_x + {}_kp_y - {}_kp_x\,{}_kp_y}$.

b. Show that $\ddot{a}_{\overline{xy}+k}$ can be evaluated by

$$\ddot{a}_{\overline{xy}+k} = \frac{\sum_{j=k}^{\infty} v^j\,{}_jp_{\overline{xy}}}{v^k\,{}_kp_{\overline{xy}}}.$$

19

POPULATION THEORY

19.1 Introduction

Many of the ideas in Chapter 3 are building blocks in the construction of a mathematical theory of populations. For example, the survival function used to define the distribution of the random variable time-until-death, and to trace the progress of a survivorship group, plays a role in constructing models for populations.

The models developed in this chapter are general. They can be applied, with appropriate modifications, to the population of a political unit, a population of workers, or a wildlife population.

We are particularly interested in certain actuarial applications of population theory. In Section 19.5 a population model is used to study the progress of a system that provides life insurance benefits to a population. In Chapter 20 a population model is used as a component of a model for studying the progress of a system that provides retirement income benefits to a population.

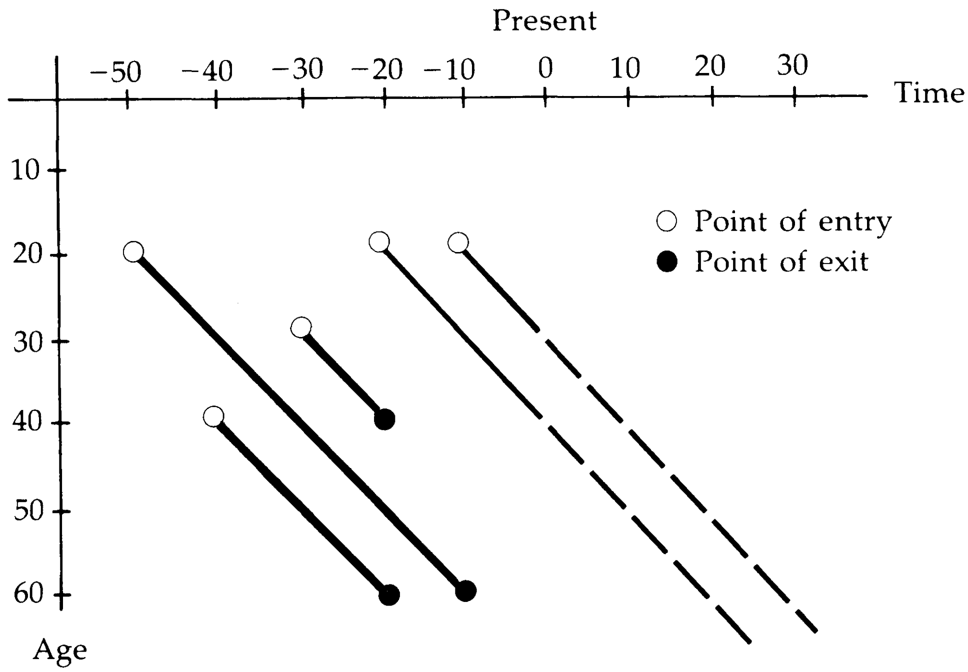
19.2 The Lexis Diagram

In this section we introduce a convenient method of picturing the progress of a population. For example, the history of a workforce's participation can be represented by parallel line segments in a two-dimensional diagram called a ***Lexis diagram.*** (See Figure 19.2.1.) The pictured point of entry of an individual into the workforce population (with coordinates time of entry and age at entry) represents one end point of the line segment associated with that individual. The line segment then follows a diagonal path to the terminal end point representing exit from the workforce population (with coordinates time of exit and age at exit).

Figure 19.2.1 illustrates, for the population of workers depicted, that at time -25 measured from the present ($t = 0$) there were three active workers. At the present time there are two active workers. One might be interested in making statements

FIGURE 19.2.1

A Lexis Diagram



about their future working lifetimes. The dashed line segments in Figure 19.2.1 denote the prospective working lifetimes of the two currently active workers.

The following observations summarize features of a Lexis diagram.

Observations:

1. A fixed point in time is represented by a vertical line. The number of members of a population at that time is given by the number of parallel line segments (each representing an individual) that intersect the vertical line.
2. A fixed age is represented by a horizontal line. If a line segment associated with an individual intersects a horizontal line at age x_0, then that individual attained age x_0 while a member of the population.
3. If a member attains age x at time t, the member's time of birth is $u = t - x$. While x and t are used as coordinates in a Lexis diagram, we frequently use the variables x and u in our developments. One of the reasons for this is that while u is constant for each member of a population, it is not constant over the population.

There are many extensions of these ideas. For example, Lexis diagrams are used to picture the progress of cohorts of lives rather than of individuals. A cohort is a collection of individuals with a common birth period. In a model for a population of workers, several modes of exit may be recognized and entries can occur at different ages. These possibilities were discussed in Chapters 10 and 11.

The demographic models developed in the next two sections utilize only one mode of exit, interpreted as death. Likewise, only birth is considered as a mode of entry. A deterministic approach is taken.

19.3 A Continuous Model

For the remainder of this chapter we use a continuous model for populations rather than a discrete set of parallel line segments (each corresponding to a member) as was used for the illustration in Figure 19.2.1. This shift permits us to use calculus as well as many of the tools developed in earlier chapters. A parallel development based on a discrete model, and using tools from linear algebra, could have been used.

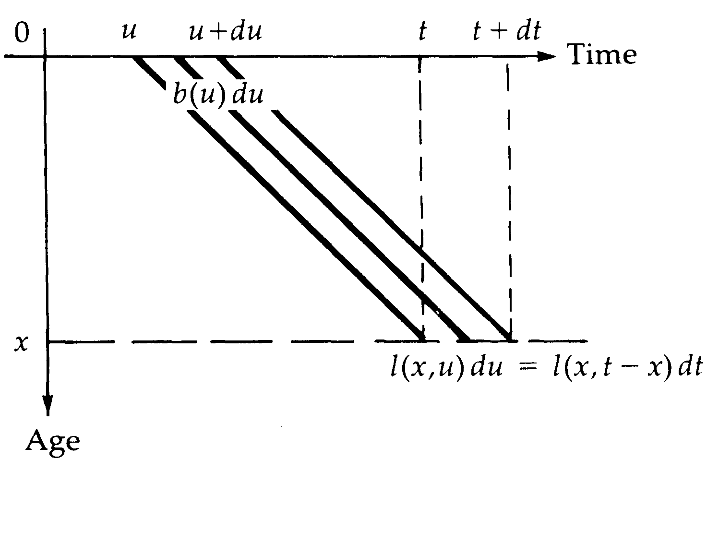
Again, we assume all entries are by birth and all exits are by death. Migration is excluded from the model. Births occur continuously, and $b(u)$ denotes the ***density function for the number of births*** at time u. That is, $b(u)\,du$ is the number of births between times u and $u + du$. We denote by $s(x, u)$ the survival function of those born at time u. This is called a ***generation survival function.*** We define

$$l(x, u) = b(u)\, s(x, u). \tag{19.3.1}$$

The function denoted by $l(x, u)$ is called a ***population density function.***

The interpretation of the function $l(x, u)$ is facilitated by reference to a continuous version of a Lexis diagram. (See Figure 19.3.1.) This figure and the remaining figures in this section are two-dimensional. They are designed to aid in the interpretation of differential terms or to illustrate regions of integration. In each case, a three-dimensional figure, illustrating the function defined on the time-age plane, could have been drawn.

FIGURE 19.3.1
Interpretation of $l(x, u)$



Of the $l(0,u)\,du = b(u)\,du$ births between times u and $u + du$, $l(x,u)\,du$ survive to age x. Let $t = x + u$, then $dt = du$, and this expression can be restated as

$$l(x, t - x)\,dt = \text{(number attaining age } x \text{ between times } t \text{ and } t + dt). \tag{19.3.2}$$

From this it follows that the number attaining age x between times t_0 and t_1 is

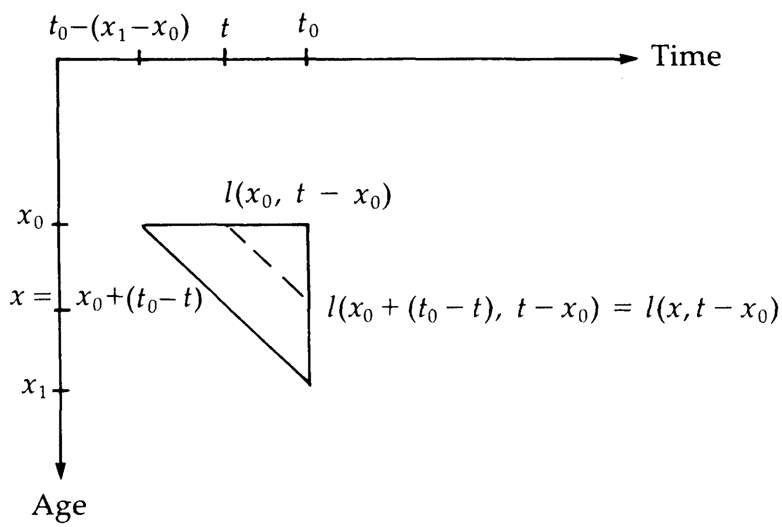
$$\int_{t_0}^{t_1} l(x, t - x)\,dt. \tag{19.3.3}$$

Now we consider a different question. Let $x_0 < x_1$ be two ages and t_0 a given time. How many lives are there between ages x_0 and x_1 at time t_0? In posing this question the word "lives" has been attached to the values of an integral of the function $l(x,u)$ and, as in Chapter 3, no longer denotes a variable that must take on only integer values.

These lives would have attained age x_0 between times $t_0 - (x_1 - x_0)$ and t_0 and then survived to time t_0, as indicated in Figure 19.3.2. The diagonal dashed line traces a typical cohort of lives that will be between ages x_0 and x_1 at time t_0.

FIGURE 19.3.2

Number of Lives between Ages x_0 and x_1 at Time t_0



Thus the number we seek is

$$\int_{t_0-(x_1-x_0)}^{t_0} l(x_0, t - x_0)\,\frac{s(x_0 + t_0 - t,\ t - x_0)}{s(x_0, t - x_0)}\,dt. \tag{19.3.4}$$

In evaluating (19.3.4) we make use of (19.3.1) to write the integrand as $b(t - x_0) \times s(x_0 + t_0 - t, t - x_0) = l(x_0 + t_0 - t, t - x_0)$. If we let $x = x_0 + (t_0 - t)$, we can transform (19.3.4) into

$$-\int_{x_1}^{x_0} l(x, t_0 - x)\, dx = \int_{x_0}^{x_1} l(x, t_0 - x)\, dx. \tag{19.3.5}$$

From (19.3.5) we can make the following statement:

$$l(x, t_0 - x)\, dx = (\text{number of lives between ages } x \text{ and } x + dx \text{ at time } t_0). \tag{19.3.6}$$

Therefore, the population density function has two interpretations. The first is given by (19.3.2) and (19.3.3) and relates to the number of lives attaining age x between times t and $t + dt$. The second is given by (19.3.5) and (19.3.6) and relates to the number of lives between ages x and $x + dx$ at time t_0. The two interpretations correspond to slicing a Lexis diagram for the population with the lines t and $t + dt$ in the first interpretation, and slicing the diagram with lines x and $x + dx$ in the second interpretation.

In order to incorporate deaths into our model we let

$$\mu(x, u) = -\frac{1}{s(x, u)} \frac{\partial}{\partial x} s(x, u) = -\frac{1}{l(x, u)} \frac{\partial}{\partial x} l(x, u) \tag{19.3.7}$$

denote the ***generation force of mortality*** at age x for those born at time u. Figure 19.3.3 provides three interpretations following from this definition. These can be verified by making the indicated linear transformation to the bivariate population density function times the generation force of mortality and confirming that the Jacobian of the transformation is 1.

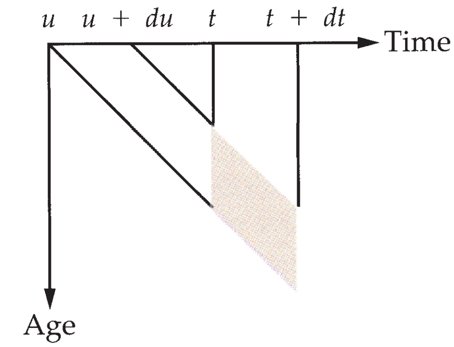
The total number of deaths in a given region of the time-age plane, as depicted in a Lexis diagram, is obtained by integrating one of the expressions in Figure 19.3.3 over the given region. The solution requires the calculation of a double integral.

There is an alternative method called the ***in-and-out method,*** which often provides an easy way to obtain the required number of deaths. This alternative method involves determining the numbers of lives entering and leaving the region. The difference between these two numbers is the number of deaths. In most situations, the in-and-out method requires evaluation of only two single integrals.

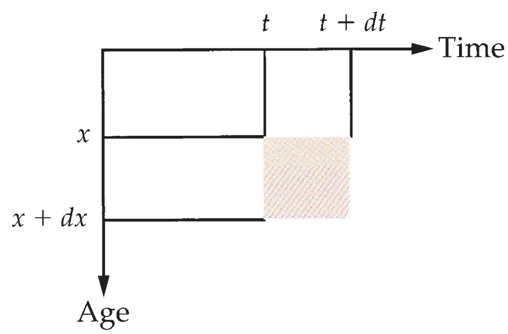
FIGURE 19.3.3

Interpretations of Population Density Times Generation Force of Mortality

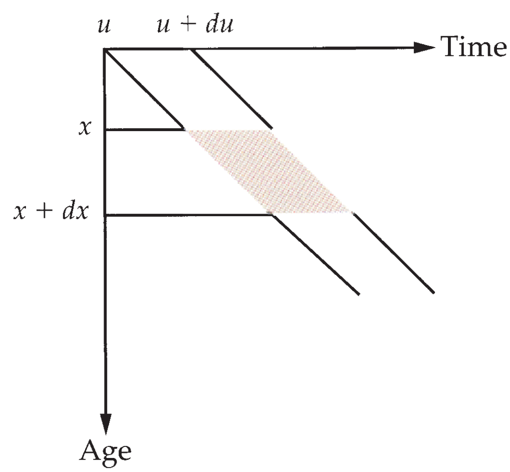
A. $l(t - u, u)\mu(t - u, u)\, du\, dt$ = number of deaths between times t and $t + dt$ among those born between u and $u + du$.



B. Substitute $u = t - x$, and we have $l(x, t - x)\mu(x, t - x)\, dt\, dx$ = number of deaths between ages x and $x + dx$ at times between t and $t + dt$.



C. Substitute $x = t - u$, and we have $l(x, u)\, \mu(x, u)\, du\, dx$ = number of deaths between ages x and $x + dx$ of those born between u and $u + du$.



Example 19.3.1

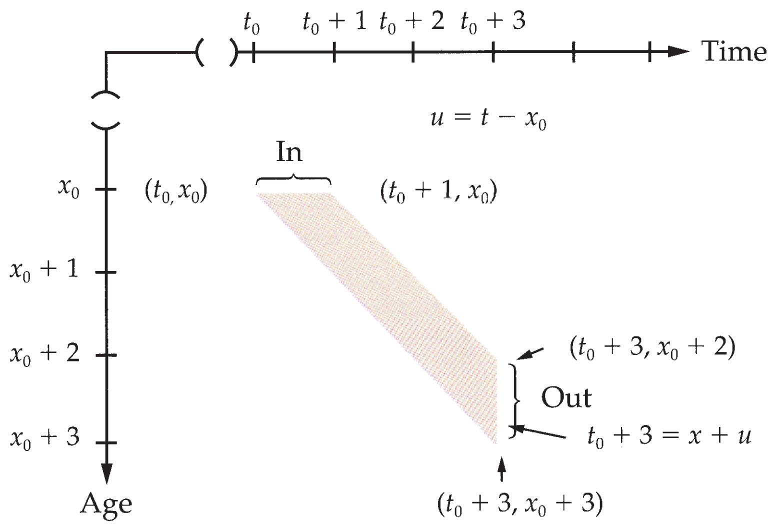
How many lives will attain age x_0 between times t_0 and $t_0 + 1$ and die before time $t_0 + 3$?

Solution:

We must derive an expression for the number of deaths in the trapezoid illustrated in Figure 19.3.4.

FIGURE 19.3.4

Region Where Deaths Are Counted, Example 19.3.1



Double integral method: Using the interpretation of Figure 19.3.3C, we have for the required number of deaths

$$\int_{t_0-x_0}^{t_0+1-x_0} \int_{x_0}^{t_0+3-u} l(x,u)\ \mu(x,u)\, dx\, du.$$

Using (19.3.7), we have for the number of deaths

$$\begin{aligned}
&\int_{t_0-x_0}^{t_0+1-x_0} \int_{x_0}^{t_0+3-u} \left[-\frac{\partial l(x,u)}{\partial x} \right] dx\, du \\
&= \int_{t_0-x_0}^{t_0+1-x_0} [l(x_0,u) - l(t_0 + 3 - u, u)]\, du \\
&= \int_{t_0-x_0}^{t_0+1-x_0} l(x_0,u)\, du - \int_{t_0-x_0}^{t_0+1-x_0} l(t_0 + 3 - u, u)\, du.
\end{aligned}$$

We let $y = u + x_0$ in the first integral and $w = t_0 + 3 - u$ in the second to obtain for the number of deaths

$$\int_{t_0}^{t_0+1} l(x_0, y - x_0)\,dy - \int_{x_0+2}^{x_0+3} l(w, t_0 + 3 - w)\,dw.$$

In-and-out method: To obtain the required number of deaths we take the difference between the number of ins who attain age x_0 between times t_0 and $t_0 + 1$ and the number of outs who are alive between ages $x_0 + 2$ and $x_0 + 3$ at time $t_0 + 3$. Using (19.3.3) and (19.3.5), we have

$$\int_{t_0}^{t_0+1} l(x_0, y - x_0)\,dy - \int_{x_0+2}^{x_0+3} l(w, t_0 + 3 - w)\,dw,$$

which agrees with the result obtained using double integration. ▼

Example 19.3.2

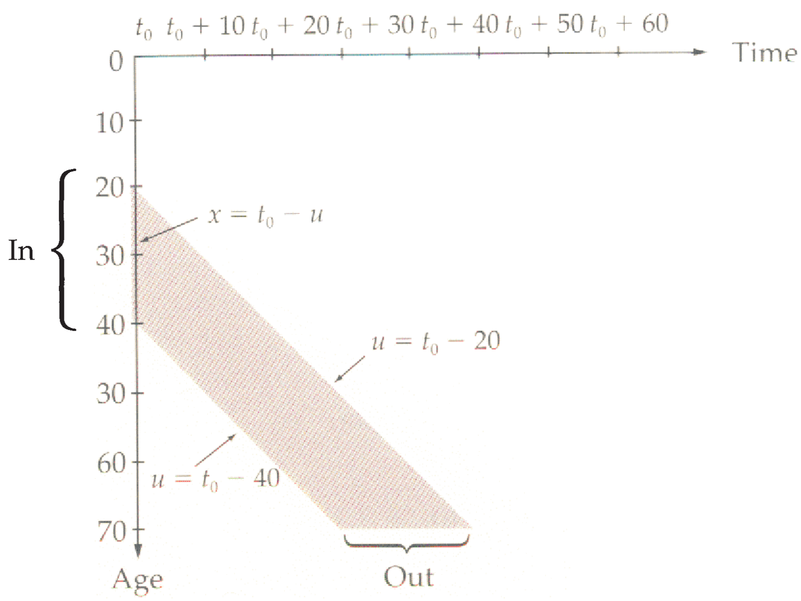
Determine the number of those between ages 20 and 40 at time t_0 who will die before reaching age 70.

Solution:

We are asked to derive an expression for the number of deaths in the trapezoid illustrated in Figure 19.3.5.

FIGURE 19.3.5

Region Where Deaths Are Counted, Example 19.3.2



Double integral method: Using the interpretation in Figure 19.3.3C, the required number of deaths is given by

$$\int_{t_0-40}^{t_0-20} \int_{t_0-u}^{70} l(x, u)\ \mu(x, u)\, dx\, du$$

$$= \int_{t_0-40}^{t_0-20} \int_{t_0-u}^{70} \left[-\frac{\partial l(x, u)}{\partial x} \right] dx\, du$$

$$= \int_{t_0-40}^{t_0-20} [l(t_0 - u, u) - l(70, u)]\, du$$

$$= \int_{t_0-40}^{t_0-20} l(t_0 - u, u)\, du - \int_{t_0-400}^{t_0-20} l(70, u)\, du.$$

We let $y = t_0 - u$ in the first integral and $w = u + 70$ in the second integral to obtain the required number of deaths

$$\int_{20}^{40} l(y, t_0 - y)\, dy - \int_{t_0+30}^{t_0+50} l(70, w - 70)\, dw.$$

In-and-out method: We use (19.3.5) for the ins and (19.3.3) for the outs to obtain

$$\int_{20}^{40} l(x, t_0 - x)\, dx - \int_{t_0+30}^{t_0+50} l(70, t - 70)\, dt,$$

which agrees with the result using double integrals. ▼

19.4 Stationary and Stable Populations

Here we study two important special cases of the model described in Section 19.3. If $l(x, u)$ is independent of u, we call the result a ***stationary population.*** For a stationary population, (19.3.1) becomes

$$l(x, u) = b\, s(x) \tag{19.4.1}$$

where b is the constant density of births and $s(x)$ is a survival function that does not depend on the time of birth. For human populations, b is typically expressed as a number of births per year, and the age variable is measured in years. In accord with (3.3.1), we rewrite (19.4.1) as

$$l(x, u) = b\, s(x) = l_x \tag{19.4.2}$$

where b plays the role of the radix l_0.

For a stationary population we can write (19.3.5) as

$$\int_{x_0}^{x_1} l_x\, dx = T_{x_0} - T_{x_1}$$

and obtain the number of lives in the stationary population between ages x_0 and x_1 at any time t, expressed in terms of the function T_x introduced in (3.5.16) in connection with the analysis of a survivorship group. In addition, the interpretation of $l_x\, \mu(x)$ given by Figure 19.3.3B leads to

$$\int_{x_0}^{x_1} l_x \mu(x)\, dx = l_{x_0} - l_{x_1}$$

as the density of deaths between ages x_0 and x_1 at any time t. In particular, the density of deaths at age x_0 and greater equals the density of the number of lives attaining age x_0 at any time t, the interpretation provided by (19.3.2). These facts illustrate the aptness of the name stationary population.

If the population density function is of the form

$$l(x, u) = e^{Ru} b\, s(x) = e^{Ru} l_x \tag{19.4.3}$$

where $b > 0$ and R are constant and $s(x)$ is a survival function that is independent of the time of birth, the resulting population is called a ***stable population***. The density of births at time u in a stable population is $e^{Ru} b = e^{Ru} l_0$. If $R = 0$, a stable population is a stationary population.

Using (19.3.5) we see that the total population at time t, denoted by $N(t)$, for a stable population is given by

$$N(t) = \int_0^{\infty} l(x, t - x)\, dx = e^{Rt} \int_0^{\infty} e^{-Rx} l_x\, dx. \tag{19.4.4}$$

Therefore, if $R > 0$, the population is growing exponentially, and if $R < 0$, the population is decreasing exponentially.

Again using (19.3.5) we see that the fraction of the total stable population that lies between ages x_0 and x_1 at time t is

$$\frac{\int_{x_0}^{x_1} l(x, t - x)\, dx}{\int_0^{\infty} l(x, t - x)\, dx} = \frac{\int_{x_0}^{x_1} e^{-Rx} l_x\, dx}{\int_0^{\infty} e^{-Rx} l_x\, dx}, \tag{19.4.5}$$

which is independent of t. Thus, while the size of a stable population may change over time, its relative age distribution is constant.

For a stable population, we can express the number of members between ages x_0 and x_1, using (19.3.5), as

$$\begin{aligned}\int_{x_0}^{x_1} l(x, t - x)\, dx &= \int_{x_0}^{x_1} e^{R(t-x)} l_x\, dx \\ &= e^{R(t-x_0)} l_{x_0} \bar{a}_{x_0:\overline{x_1 - x_0}|\delta=R}.\end{aligned} \tag{19.4.6}$$

In the limit as $x_1 \to \infty$, the number of members alive above age x_0 at time t in a stable population may be written as $e^{R(t-x_0)} l_{x_0} \bar{a}_{x_0 @ \delta=R}$.

For a stable population, the force of mortality as given by (19.3.7) becomes, by reference to (19.4.3),

$$\mu(x, u) = -\frac{1}{l(x, u)} \frac{\partial}{\partial x} l(x, u) = \mu(x).$$

We can express the density of deaths at time t for a stable population between ages x_0 and x_1 as

$$\int_{x_0}^{x_1} e^{R(t-x)}\, l_x\, \mu(x)\, dx = l_{x_0}\, e^{R(t-x_0)} \bar{A}^{\,1}_{x_0:\overline{x_1-x_0}|\delta=R}, \tag{19.4.7}$$

and the density of deaths at time t above age x_0 is $e^{R(t-x_0)} l_{x_0} \bar{A}_{x_0}$.

These facts about stable populations can be used, in connection with an identity from Chapter 5, to confirm a property of stable populations:

$$\begin{aligned}
\begin{pmatrix}\text{rate of population}\\ \text{change at time } t\\ \text{above age } x_0\end{pmatrix} &= \frac{\begin{pmatrix}\text{density of those}\\ \text{reaching age } x_0\\ \text{at time } t\end{pmatrix} - \begin{pmatrix}\text{density of deaths}\\ \text{above age } x_0\\ \text{at time } t\end{pmatrix}}{\begin{pmatrix}\text{number above}\\ \text{age } x_0 \text{ at time } t\end{pmatrix}} \\
&= \frac{e^{R(t-x_0)}\, l_{x_0} - R^{(t-x_0)}\, l_{x_0}\, \bar{A}_{x_0}}{e^{R(t-x_0)}\, l_{x_0}\, \bar{a}_{x_0}} \\
&= \frac{1 - \bar{A}_{x_0}}{\bar{a}_{x_0}} = R.
\end{aligned}$$

The final step follows since the functions are calculated at force of interest R.

Example 19.4.1

For a stationary population the complete life expectancy at age 0, derived from the survival function, can be obtained by dividing the number in the population at time t by the birth density. That is,

$$\mathring{e}_0 = \int_0^\infty s(x)\, dx = \int_0^\infty \frac{l_x}{l_0}\, dx = \frac{T_0}{l_0}.$$

What is the result of performing a similar calculation with a stable population?

Solution:

$$\frac{N(t)}{e^{Rt}\, l_0} = \frac{\int_0^\infty l(x, t-x)\, dx}{e^{Rt} l_0} = \frac{\int_0^\infty e^{R(t-x)}\, l_x\, dx}{e^{Rt}\, l_0} = \bar{a}_0 \text{ at } \delta = R.$$

If $R > 0$, $\bar{a}_0 < \mathring{e}_0$, and if $R < 0$, $\bar{a}_0 > \mathring{e}_0$, and if $R = 0$, the stationary population result is obtained. This example demonstrates how the life expectancies cannot be observed directly from stable populations unless $R = 0$. ▼

19.5 Actuarial Applications

The conditions for a stable or stationary population are seldom realized because of changes in either the survival function or the density of births. However, these models are useful in studying alternative plans for funding life insurance or retirement income systems. By a funding plan we mean a budgeting plan for accumulating the funds necessary to provide the insurance or annuity benefits.

In this section and in Chapter 20 we depart from the models developed in Chapters 4 through 11, 15, and 16. These models were built by starting with a consideration of the operation of a single policy. In this section we study aggregate models for life insurance. In Chapter 20 similar models for pension systems are examined. The models considered are especially relevant to social and group insurance systems that provide benefits on death or retirement to broad groups or populations.

Example 19.5.1

Assume a population density function $l(x,u) = b(u)\,s(x)$ where the survival function is independent of u. Further, assume that in this population each member above age a is insured for a unit benefit under a fully continuous, whole life insurance with annual premiums payable from age a. The premium paid by each member is based on force of interest δ and survival function $s(x)$. Prove that

$$\begin{aligned}\bar{P}(\bar{A}_a)\int_a^\infty l(x,t-x)\,dx &+ \delta\int_a^\infty l(x,t-x)\;{}_{x-a}\bar{V}(\bar{A}_a)\,dx\\ &= \int_a^\infty l(x,t-x)\mu(x)\,dx + \frac{d}{dt}\int_a^\infty l(x,t-x)\;{}_{x-a}\bar{V}(\bar{A}_a)\,dx. \qquad (19.5.1)\end{aligned}$$

Solution:

General reasoning solution: Stated in words, (19.5.1) asserts

(rate of premium income at time t)

+ (rate of investment income at time t)

= (rate of benefit outgo at time t)

+ (rate of change in aggregate reserve at time t).

That is, (19.5.1) can be interpreted as an income allocation equation for a life insurance system covering a population aged a or greater. The left-hand side of (19.5.1) displays the sources of income, premiums, and interest, and the right-hand side displays the allocation of income to death benefits and changes in the aggregate reserve fund.

Analytic solution: We start with (8.6.4), implying for the present case

$$\frac{d}{dx}\,{}_{x-a}\bar{V}(\bar{A}_a) - \mu(x)\;{}_{x-a}\bar{V}(\bar{A}_a) + \mu(x) = \bar{P}(\bar{A}_a) + \delta\;{}_{x-a}\bar{V}(\bar{A}_a). \qquad (19.5.2)$$

We multiply (19.5.2) by $l(x, t-x)$ and integrate between a and the upper limit of survival. These operations yield

$$\begin{aligned}\int_a^\infty l(x,t-x)\,d[{}_{x-a}\bar{V}(\bar{A}_a)] &- \int_a^\infty l(x,t-x)\mu(x)\;{}_{x-a}\bar{V}(\bar{A}_a)\,dx + \int_a^\infty l(x,t-x)\mu(x)\,dx\\ &= \bar{P}(\bar{A}_a)\int_a^\infty l(x,t-x)\,dx + \delta\int_a^\infty l(x,t-x)\;{}_{x-a}\bar{V}(\bar{A}_a)\,dx. \qquad (19.5.3)\end{aligned}$$

The first integral on the left-hand side of (19.5.3) is evaluated using integration by parts. We obtain

$$l(x,t-x)\ {}_{x-a}\bar{V}(\bar{A}_a)\Big|_a^\infty + \int_a^\infty [b'(t-x)\,s(x) + l(x,t-x)\mu(x)]\ {}_{x-a}\bar{V}(\bar{A}_a)\,dx. \quad (19.5.4)$$

In completing the integration by parts it is important to recall that

$$\begin{aligned}\frac{d}{dx}\, l(x,t-x) &= \frac{d}{dx}\, b(t-x)\,s(x) \\ &= -b'(t-x)\,s(x) - b(t-x)\,s(x)\mu(x).\end{aligned}$$

Substituting (19.5.4) into (19.5.3) and rearranging yields (19.5.1). ▼

Example 19.5.2

For the population life insurance system described in Example 19.5.1 assume that funding is on an assessment plan rather than on a whole life plan; that is, the annual assessment rate, denoted by π_t, per member at time t is equal to the rate of outgo per member at time t. Determine π_t.

Solution:

The assessment rate can be determined from

$$\pi_t \int_a^\infty l(x,t-x)\,dx = \int_a^\infty l(x,t-x)\mu(x)\,dx$$

or from

$$\pi_t = \frac{\displaystyle\int_a^\infty l(x,t-x)\mu(x)\,dx}{\displaystyle\int_a^\infty l(x,t-x)\,dx}. \quad (19.5.5)$$

▼

Example 19.5.3

Assume a stable population and rework

a. Example 19.5.1

b. Example 19.5.2.

Solution:

a. We start with (19.5.1), which has already been established for the more general population density function $l(x,u) = b(u)\,s(x)$. For this example, $l(x,u) = e^{Ru}\,b\,s(x)$, and the income allocation equation becomes

$$\begin{aligned}&\bar{P}(\bar{A}_a) \int_a^\infty e^{R(t-x)}\, l_x\,dx + \delta \int_a^\infty e^{R(t-x)}\, l_x\ {}_{x-a}\bar{V}(\bar{A}_a)\,dx \\ &\qquad = \int_a^\infty e^{R(t-x)}\, l_x\,\mu(x)\,dx + R\int_a^\infty e^{R(t-x)}\, l_x\ {}_{x-a}\bar{V}(\bar{A}_a)\,dx. \quad (19.5.6)\end{aligned}$$

The factor e^{Rt} can be canceled from each term of (19.5.6). By an interpretation of (19.5.6) developed in the general reasoning solution of Example 19.5.1, the ratio

of the rate of premium income to the rate of benefit outgo is

$$\frac{\bar{P}(\bar{A}_a)\displaystyle\int_a^\infty e^{-Rx}\, l_x\, dx}{\displaystyle\int_a^\infty e^{-Rx}\, l_x\, \mu(x)\, dx} = \frac{\bar{P}(\bar{A}_a)}{\bar{P}'(\bar{A}'_a)}. \tag{19.5.7}$$

Here $\bar{P}'(\bar{A}'_a)$ is computed at force of interest R.

If $R = 0$, the population is stationary, and the income allocation equation (19.5.6) becomes

$$\bar{P}(\bar{A}_a)\, T_a + \delta \int_a^\infty l_x\ {}_{x-a}\bar{V}(\bar{A}_a)\, dx = l_a, \tag{19.5.8}$$

and the ratio of the rate of premium income to the rate of benefit outgo, (19.5.7), becomes $\bar{P}(\bar{A}_a)\, \mathring{e}_a$.

b. In the stable population, the assessment rate determined in (19.5.5) becomes

$$\pi_t = \frac{\displaystyle\int_a^\infty e^{-Rx}\, l_x\, \mu(x)\, dx}{\displaystyle\int_a^\infty e^{-Rx}\, l_x\, dx} = \bar{P}'(\bar{A}'_a), \tag{19.5.9}$$

which is independent of t. If $R = 0$, that is, the population is stationary, then $\pi_t = 1/\mathring{e}_a$. ▼

Remark:

One aspect of Example 19.5.3 deserves special comment. The rate of premium payment required of each member of the stable population above age a under the whole life and assessment funding methods are, respectively, $\bar{P}(\bar{A}_a)$ and $\bar{P}'(\bar{A}'_a)$. In Exercise 19.21 it is demonstrated that if the force of mortality is increasing, then

$$\bar{P}(\bar{A}_a) > \bar{P}'(\bar{A}'_a) \text{ if } \delta < R$$

$$\bar{P}(\bar{A}_a) = \bar{P}'(\bar{A}'_a) \text{ if } \delta = R$$

$$\bar{P}(\bar{A}_a) < \bar{P}'(\bar{A}'_a) \text{ if } \delta > R.$$

That is, if the force of interest is less than the population growth rate, the required premium rate under the assessment funding method is less than under the whole life funding method. If the force of interest is greater than the population growth rate, the whole life funding method results in a smaller premium rate than does the assessment funding method.

Example 19.5.4

Provide a general reasoning interpretation of the stationary population income allocation equation (19.5.8) rearranged as

$$\int_a^\infty l_x \, {}_{x-a}\bar{V}(\bar{A}_a)\,dx = \frac{l_a - \bar{P}(\bar{A}_a)T_a}{\delta}.$$

Solution:

This rearranged form indicates that the aggregate reserve can be interpreted as the difference between the present values of two perpetuities:

$\frac{l_a}{\delta}$ = (the present value of a continuous perpetuity paying death benefits at an annual rate of l_a)

= (the present value of death benefits to current members) + (the present value of death benefits to future members);

$\frac{\bar{P}(\bar{A}_a)T_a}{\delta}$ = (the present value of a continuous perpetuity paying premiums at an annual rate of $\bar{P}(\bar{A}_a)\,T_a$)

= (the present value of premiums for current members) + (the present value of premiums for future members)

Additional insights are obtained by noting that premiums at rate $\bar{P}(\bar{A}_a)$ will be payable from age a for future members. The present value of their premiums will be equal to the present value of their benefits. Hence, the second component of the interpretations of l_a/δ and $\bar{P}(\bar{A}_a)T_a/\delta$ are offsetting, and

$\frac{l_a}{\delta} - \frac{\bar{P}(\bar{A}_a)T_a}{\delta}$ = (the aggregate reserve for current members)

= (the present value of benefits for current members)

− (the present value of premiums for current members)

$$= \int_a^\infty l_x \, {}_{x-a}\bar{V}(\bar{A}_a)\,dx.$$

▼

Examples 19.5.1, 19.5.3, and 19.5.4 treat life insurance funding, or budgeting, methods for which a fund exists. In these examples the characteristics of the funds, after all members of the population above the entry age a are participants and have been since the entry age a, were examined. When all eligible members are participating and have participated since the entry age a, the system is said to be in a ***mature state.*** Until that time the total fund is subject to growth by a stream of new entrants. In our examples it will take $\omega - a$ years for the fund to reach a mature state.

19.6 Population Dynamics

In this section we return to an examination of the function $b(t)$, the density of births at time t. Our goal is to build a foundation under the development of the continuous model of Section 19.3. In addition, the conditions leading to stable or stationary populations, developed in Section 19.4, are explored.

In developing a mathematical model for the density of births we introduce the ***force of birth function,*** denoted by $\beta(x, u)$. Then, $\beta(x, t-x)\,dt$ represents the number of female children born between times t and $t + dt$ to a woman age x who was herself born at time $t - x$. The force of birth function is an age- and generation-specific instantaneous birthrate for female children.

The total number of female children born between t and $t + dt$ is

$$b_f(t)\,dt = \left[\int_0^\infty l_f(x, t-x)\,\beta(x, t-x)\,dx\right] dt. \tag{19.6.1}$$

In (19.6.1) the subscript f denotes that the function relates to female lives. Total births are obtained by multiplying a constant, (total births) / (female births), which is slightly greater than 2 for most human populations.

If we divide (19.6.1) by dt and substitute for $l_f(x, t-x)$ from (19.3.1), we see that the female birth density function satisfies the integral equation

$$b_f(t) = \int_0^\infty b_f(t-x)\,s_f(x, t-x)\,\beta(x, t-x)\,dx. \tag{19.6.2}$$

An integral equation is a statement about the relationship between functions where the relationship involves an integral. The problem is to find $b_f(t)$ given the functions $s_f(x, t-x)$ and $\beta(x, t-x)$. The function $s_f(x, t-x)\,\beta(x, t-x)$ is called the ***net maternity function*** and in (19.6.3) is denoted by $\phi(x, t-x)$.

For the remainder of this section we assume that the net maternity function does not depend on the year of birth of the mother. That is, $s_f(x, t-x)\beta(x, t-x) = \phi(x)$. With this assumption, the integral equation (19.6.2) becomes

$$b_f(t) = \int_0^\infty b_f(t-x)\,\phi(x)\,dx. \tag{19.6.3}$$

In this section we limit ourselves to verifying that a particular solution of (19.6.3) is

$$b_f(t) = b\,e^{Rt} \tag{19.6.4}$$

where b is a positive constant and R is the unique real solution of the equation

$$H(r) = 1 \tag{19.6.5}$$

where

$$H(r) = \int_0^\infty e^{-rx}\,\phi(x)\,dx.$$

Direct substitution of (19.6.4) into (19.6.3) yields

$$b\,e^{Rt} = \int_0^\infty b\,e^{R(t-x)}\,\phi(x)\,dx,$$

and upon the cancellation of constants this becomes

$$1 = \int_0^\infty e^{-Rx}\,\phi(x)\,dx = H(R).$$

We now show how the statement that $H(r) = 1$ yields a unique real solution can be verified.

Observations:

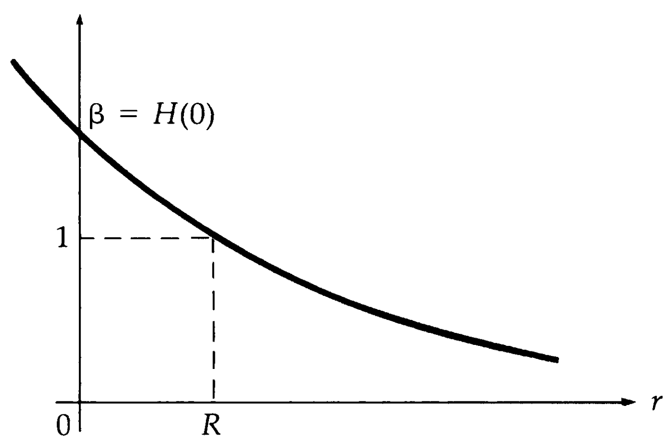
1. $H'(r) = -\int_0^\infty xe^{-rx}\,\phi(x)\,dx < 0$
2. $H(0) = \int_0^\infty \phi(x)\,dx > 0$
3. $\lim_{r\to\infty} H(r) = 0$
4. $\lim_{r\to-\infty} H(r) = \infty.$

These observations are summarized in Figure 19.6.1 together with the fact that

$$H''(r) = \int_0^\infty x^2\,e^{-rx}\,\phi(x)\,dx > 0.$$

FIGURE 19.6.1

Typical *H*(*r*) Function, and Formula (19.6.5)



From Figure 19.6.1 we see that there is a unique real solution R (shown positive, but it could be negative), and the verification is complete.

If $b_f(t) = b\,e^{Rt}$, then $l_f(x, t - x) = b\,e^{R(t-x)}\,s_f(x)$, and the population of females is stable. In the special case where $R = 0$, the population of females is stationary.

To check whether R is positive, zero, or negative we examine the number β where

$$\beta = H(0) = \int_0^\infty \phi(x)\,dx.$$

By studying Figure 19.6.1 we can conclude

- If $\beta > 1$, R is positive, and the population is stable and increasing
- If $\beta = 1$, R is 0, and the population is stationary
- If $\beta < 1$, R is negative, and the population is stable and decreasing.

Since $H(0) = \beta$ can be interpreted as the number of female children produced by each female, it is called the ***net reproduction rate.*** The parameter R is called the ***intrinsic rate of population growth.***

Remarks:

All populations are not stable, as might be inferred from this section. Several aspects of our model may not be in accord with actual experience. Our basic model, given by (19.6.2), is built on the assumption that the survival function and the force of birth do not change over time. In (19.6.3) we restrict it still further by assuming that the net maternity function depends on the age, but not the birth year, of mothers. Public health statistics disclose major changes in survival functions and in the forces of birth over time.

In addition, in solving integral equation (19.6.3), we obtained only the real solution to the equation $H(r) = 1$. Within the complex number field, an infinite number of solutions may be determined in addition to the single real solution. These additional roots of $H(r) = 1$ lead to general solutions of (19.6.3) of the form

$$\sum_j c_j b_f^{(j)}(t)$$

where each $b_f^{(j)}(t)$ is associated with a root of $H(r) = 1$. The complex roots, which occur in conjugate pairs, can serve to put a dampened wave structure into the birth density function.

Population theory is a collection of elegant mathematical ideas. However, there are also some very important statistical problems in estimating its key components, such as the survival function and the force of birth function, from available data. These functions have been observed to shift over time, reflecting the dynamic nature of human society.

As is true of all models of natural phenomena, the mathematical models of populations capture only a small part of the dynamic forces that shape the size and age distribution of real populations. Even if the stable population model is a satisfactory approximation at one time, it cannot be appropriate for the long term. The constant R cannot be greater than zero on a finite planet for a long time horizon. In a like manner, if R is negative for too long a period of time, the stable population faces extinction.

19.7 Notes and References

The foundations of population theory were built, in part, by Lotka who had experience answering life insurance questions. The basic theory is developed in a book by Keyfitz (1968), and applications are found in another book by Keyfitz (1977). Keyfitz and Beekman (1984) wrote a textbook directed toward helping students master demography through a graded set of exercises. Lexis diagrams are named for their originator Wilhelm Lexis (1837–1914), a German statistician, demographer, and economist. Charles Trowbridge (1952, 1955) promoted the use of stationary population models in the study of the characteristics of pension and life insurance funding methods.

Exercises

Section 19.2

19.1. Using the Lexis diagram of Figure 19.2.1 calculate
   a. The average age of employees at time -25
   b. The number of employees who have attained age 50 in the history of this workforce
   c. Of the employees at time -25, the number who have attained or will attain age 50 while in the workforce.

Section 19.3

19.2. Let

$$b(u) = 100\left[1 + \cos\left(\frac{\pi u}{200}\right)\right] \qquad -\infty < u < \infty$$

$$s(x, u) = \cos\left(\frac{\pi x}{200}\right) \qquad 0 < x \le 100.$$

Calculate the number of individuals attaining age 50 between the times 50 and 100.

19.3. Let

$$b(u) = 100(1 - e^{-u/100}) \qquad u > 0$$

$$s(x) = e^{-x/100} \qquad x > 0.$$

Calculate the number of individuals between ages 25 and 50 at time 100.

19.4. Calculate the number of lives who will attain age 25 between times 50 and 51 and die before time 53. Use the functions $b(u)$ and $s(x, u)$ specified in Exercise 19.3. (This is a numerical version of Example 19.3.1.)

19.5. Rework Example 19.3.2 assuming $s(x, u) = s(x)$ and $b(u) = l_0$.

19.6. Exhibit integrals for calculating the number of those between ages 20 and 50 at time 0 who will die at an age less than 80 and before time 50.

Section 19.4

19.7. a. Let $N(t)$ denote the number of members in a stable population at time t and show that $dN(t)/dt = R\,N(t)$.

b. The birthrate at time t is defined as $b(t)/N(t)$. For a stable population show that the birthrate, denoted by $i(t)$, is

$$i(t) = \left[\int_0^\infty e^{-Rx} s(x)\,dx \right]^{-1}.$$

19.8. If $\mu(x) = a\,x$, $a > 0$, and $b(u) = b\,e^{Ru}$, express the size of the total population at time t in terms of $\Phi(z)$, the d.f. for a $N(0,1)$ distribution.

19.9. Assume a stable population and derive an expression for the average age of those between ages a and r at time t. Restate the expression assuming $R = 0$.

19.10. If $\tilde{\mu}(x) = \mu(x) + 0.05/\mathring{e}_x$, show that $\tilde{p}_x = p_x(T_{x+1}/T_x)^{0.05}$.

19.11. Confirm that $s^*(x) = e^{-Rx}s(x)$, $R \geq 0$, is a survival function. Then

a. Exhibit the p.d.f. and d.f. associated with $s^*(x)$

b. Show that the complete expectation of life at age x_0 associated with the survival function $s^*(x)$ is $\bar{a}_{x_0}$ and that the variance of the time-until-death is $2(\bar{I}\bar{a})_{x_0} - \bar{a}_{x_0}^2$. Here the annuity functions are calculated at force of interest R.

19.12. The crude death rate at time t is defined by

$$\frac{\displaystyle\int_0^\infty l(x, t-x)\,\mu(x, t-x)\,dx}{\displaystyle\int_0^\infty l(x, t-x)\,dx}.$$

If the population is stable, show that the crude death rate is equal to $i(t) - R$ where $i(t)$ is the birthrate defined in Exercise 19.7(b).

Section 19.5

19.13. Assume a stationary population with survival function $s(x)$ and that this same survival function is used in the evaluation of actuarial functions. Verify and interpret the equation

$$l_r\,\bar{a}_r + \delta \int_r^\infty l_x\,\bar{a}_x\,dx = T_r\,.$$

19.14. Assume a stable population with survival function $s(x)$ and that this same survival function is used in the evaluation of actuarial functions. Confirm and interpret the following identities:

a. $l(a, t - a)\,\bar{a}_{a:\overline{r-a}|} + \delta \int_a^r l(x, t - x)\,\bar{a}_{x:\overline{r-x}|}\,dx$

$$= \int_a^r l(x, t - x)\,dx + R \int_a^r l(x, t - x)\,\bar{a}_{x:\overline{r-x}|}\,dx.$$

[Hint: Evaluate the derivative of $l(x, t - x)\,\bar{a}_{x:\overline{r-x}|}$ using (5.2.27).]

b. $l(a, t - a)\,\bar{A}_a + \delta \int_a^\infty l(x, t - x)\,\bar{A}_x\,dx$

$$= \int_a^\infty l(x, t - x)\mu(x)\,dx + R \int_a^\infty l(x, t - x)\,\bar{A}_x\,dx.$$

19.15. If $b(u) = 100\,e^{0.01u}$, age $a = 0$, and $s(x) = e^{-x/50}$, calculate the assessment rate, π_t, to fund a whole life insurance program as in Example 19.5.2.

19.16. The old age dependence ratio for a population at time t is

$$f(t) = \frac{\int_{65}^\infty l(x, t - x)\,dx}{\int_{20}^{65} l(x, t - x)\,dx}.$$

For a stable population show that

$$\frac{\partial}{\partial R} \log f(t) = \bar{x}_1 - \bar{x}_2$$

where $\bar{x}_1$ is the average age of those between ages 20 and 65 at time t and $\bar{x}_2$ is the average age of those above age 65 at time t.

Section 19.6

19.17. Given the net maternity function

$$\phi(x) = x^{\alpha-1}e^{-\beta x} \qquad \alpha > 0,\ \beta > 0$$

a. Calculate R
b. Is the population stable or stationary if $\alpha = 2$ and $\beta = 1$?

Miscellaneous

19.18. Assume that the number in a population at time t satisfies the differential equation

$$\frac{dN(t)}{dt} = \frac{c}{a}\{N(t)[a - N(t)]\} \qquad a > 0.$$

Note that as $N(t)$ approaches a, the rate of change in population size approaches 0. Such a model incorporates environmental limits on population growth.

a. Verify that the function $N(T) = a(1 + be^{-ct})^{-1}$, $b > 0$, satisfies the differential equation. This is called the ***logistic function.***

b. If $c > 0$, calculate $\lim_{t \to \infty} N(t)$ and sketch the curve of $N(t)$.

c. Determine the abscissa of the point of inflection of $N(t)$.

19.19. If the force of mortality is strictly increasing, show in turn

a. $s(x)\,s(y) \geq s(x + y),\ x \geq 0,\ y \geq 0$

b. $s(x) \int_0^\infty s(y)\,dy \geq \int_0^\infty s(x + y)\,dy$

c. $s(x) \int_0^\infty s(y)\,dy \geq \int_x^\infty s(w)\,dw$

d. $\int_0^\infty s(y)\,dy \geq \int_x^\infty \dfrac{s(w)}{s(x)}\,dw$

e. $\mathring{e}_0 \geq \mathring{e}_x$.

19.20. In Exercise 19.19, multiply by v^y and show that $\bar{a}_0 \geq \bar{a}_x$.

19.21. a. Show that $\bar{P}(\bar{A}_x)$ can be written as the weighted average of the force of mortality $\mu_x(t)$ where the weight function is

$$w(t, \delta) = \frac{v^t\,{}_tp_x}{\bar{a}_x}.$$

b. Verify that

(i) $\int_0^\infty w(t, \delta)\,dt = 1$

(ii) $\dfrac{\partial}{\partial t} w(t, \delta) \leq 0$

(iii) $\dfrac{\partial}{\partial \delta} w(t, \delta) = \dfrac{v^t\,{}_tp_x[-t\,\bar{a}_x + (\bar{I}\bar{a})_x]}{(\bar{a}_x)^2}$.

c. Verify the following:

(i) $\dfrac{\partial}{\partial \delta} \omega(0, \delta) > 0$

(ii) $\lim_{t \to \infty} \dfrac{\partial}{\partial \delta} \omega(t, \delta) = -\infty$

(iii) For a fixed δ, the only positive root of

$\dfrac{\partial}{\partial \delta} \omega(t, \delta) = 0$ is $\dfrac{(\bar{I}\bar{a})_x}{\bar{a}_x}$.

d. If the force of mortality is strictly increasing, use results (b)(ii) and (b)(iii) to demonstrate that an increase in the force of interest increases the weight attached to small values of the force of mortality and decreases the weight attached to large values of the force of mortality. Therefore, if the force of mortality is strictly increasing, an increase in the force of interest will decrease $\bar{P}(\bar{A}_x)$.

20

THEORY OF PENSION FUNDING

20.1 Introduction

In Section 11.5 we studied the actuarial present values of benefits and contributions with respect to a participant in a pension plan. These actuarial present values for individual participants are necessary inputs into the process of determining the aggregate actuarial present values for the plan. If the pension plan is to provide security to the participants, these aggregated values of future benefits together with the current assets are balanced with the aggregate present values of future contributions. The pattern of aggregate contributions required to balance benefit payments is determined by an ***actuarial cost*** or ***funding method.*** In this chapter we define functions useful in summarizing the status of the funding of a pension plan. These functions are then used to define actuarial cost methods and to explore the properties of these methods.

For this we adopt some of the population theory of Chapter 19. Here the study is similar to that followed in Examples 19.5.1, 19.5.2, 19.5.3, and 19.5.4 for alternative funding or budgeting methods in the examination of a life insurance system for a population.

To integrate the ideas of this chapter with those developed earlier, the reader should keep in mind some basic limitations on the ideas presented here:

(1) To protect the interests of participants and to limit the amount of income on which taxes are deferred (because it is contributed to a pension plan), governments have chosen to regulate actuarial cost methods. These regulations are important in practice, but are not discussed here.

(2) Pension plans frequently provide many types of benefits. In addition to retirement income, death, disability, and withdrawal benefits are common. In many jurisdictions withdrawal benefits are required; these withdrawal benefits are called ***vested benefits*** in pension plans. The determination of the actuarial present value of some of these benefits was covered in Chapter 11. In this chapter, the model used provides only for retirement income benefits. This simplication is adopted so that attention can be focused on the properties of various actuarial cost methods. Although most pension plans provide retirement income at a

rate dependent in some way on income levels before retirement, the model used in this chapter specifies an initial pension benefit rate dependent only on the rate of income payment at retirement. Once again this simplification is made to permit concentration on the actuarial cost methods.

(3) A continuing theme in this work is that actuarial present values require the application of interest factors and probabilities to future contingent payments. In this chapter future payments may depend on a great many uncertain events. However, in accordance with our goal of studying actuarial cost methods, a deterministic view is adopted.

(4) Pension plan contributions are expenses for sponsoring organizations. To promote the comparability of income statements of organizations that sponsor pension plans, accounting practice restricts the actuarial cost methods that are acceptable for financial accounting. The restrictions are not discussed here.

20.2 The Model

We assume a population consisting of members entering at age a, retiring at age r and subject to a survival function, $s(x)$ with $s(a) = 1$. For $a < x < r$, decrement can occur for mortality or other causes, but for $x > r$ mortality is the only cause of decrement. The density of new entrants at age a at time u is given by $n(u)$ and the density of those attaining age x at time t by

$$n(u)\ s(x) \tag{20.2.1}$$

where $u = t - (x - a)$ is the time of entry into the plan. Formula (20.2.1) is related to (19.3.1), except that "births" occur at age a by becoming a plan participant. We assume the survival function does not depend on u.

We also assume that the salary rate for each member age x at time 0 is $w(x)$, $a < x < r$. The function $w(x)$ expresses the individual experience and merit components of salary change. Salary rates also change by a year-of-experience factor reflecting inflation and changes in the productivity of all participants. In this chapter the year-of-experience factor will be $e^{\tau t}$. This factor does not depend on the age of an individual. Thus, the annual salary rate expected at time t by a member age x is given by the formula

$$w(x)\ e^{\tau t} \qquad a < x < r. \tag{20.2.2}$$

This should be compared with the simpler model used in (11.5.1) where salary changes that are functions only of attained age are considered.

Comparing this with (19.3.5), we recognize that the total salary rate at time t for the $n(t - x + a)\ s(x)\ dx$ members between ages x and $x + dx$ is $n(t - x + a)\ s(x)\ w(x)\ \exp(\tau t)$, and the total annual salary rate at time t is

$$W_t = \int_a^r n(t - x + a)\ s(x)\ w(x)\ e^{\tau t}\ dx. \tag{20.2.3}$$

Formula (20.2.3) incorporates a notational convention used in this chapter. A subscripted symbol denotes a quantity for the entire group of covered workers. Thus W_t represents the total payroll payment rate at time t.

The model pension plan considered in this chapter provides retirement annuities payable only after attainment of retirement age r. The initial annual pension rate is a fraction f of the final salary rate. Thus for a member retiring at time t, the projected payment rate is

$$f\, w(r)\, e^{\tau t}. \tag{20.2.4}$$

For a retiree age x at time t, the annual rate of pension payment is projected as

$$f\, w(r)\, e^{\tau(t-x+r)}\, h(x) \qquad x \ge r$$

where $h(x)$ represents an adjustment factor applied to the initial pension payment rate of $f\, w(r)\, e^{\tau(t-x+r)}$ for those who retired $x - r$ years ago. We note that $h(r) = 1$. As an example, $h(x)$ may be the exponential function $\exp[\beta(x - r)]$ where β is a constant rate of increase (possibly related to the expected inflation rate).

The model plan that is at the center of our discussion of actuarial cost methods is a ***defined-benefit plan.*** The plan defines the benefits to be received by retiring participants, and we concentrate on describing actuarial cost methods that produce a stream of contributions and investment income to balance the benefit payments. In ***defined-contribution plans*** the starting point shifts. The contribution made on behalf of each participant is stated, perhaps as a constant or as a fraction of salary. The actuarial problem is then to calculate the benefit level that will produce an actuarial present value equal to the actuarial present value of the contributions. This may be determined at time of retirement by using the accumulated contributions to provide an equivalent retirement annuity, or on a year-by-year basis whereby a deferred retirement annuity is purchased by the contribution of each year.

20.3 Terminal Funding

Under the ***terminal funding method*** pensions are not funded by contributions during active membership. Instead, single contributions are made to the fund at the time of retirement. The required contribution rate, or ***normal cost rate,*** under the terminal funding method at time t, denoted by ${}^{T}P_t$, is the rate at which the actuarial present value of future pensions for members reaching age r is incurred at time t. To determine ${}^{T}P_t$ for the model plan, we assume interest is earned at an annual force of δ, and we denote by $\bar{a}_r^h$ the actuarial present value of a life annuity payable continuously to a life age r with income rate $h(x)$ per year when (r) attains age x. Therefore,

$$\bar{a}_r^h = \int_r^\infty e^{-\delta(x-r)}\, h(x)\, \frac{s(x)}{s(r)}\, dx. \tag{20.3.1}$$

From (19.3.2), we have $n(t - r + a)\, s(r)\, dt$ members attaining age r between times t and $t + dt$, and by (20.2.4) they will collect pensions at an average initial rate of $f\, w(r) \exp(\tau t)$. Therefore,

$$ {}^{T}P_t = f\, w(r)\, e^{\tau t}\, n(t - r + a)\, s(r)\, \bar{a}_r^h. \tag{20.3.2} $$

We will see that ${}^{T}P_t$ is a basic building block for the various functions used to describe the funding operations for the model plan.

To illustrate the theory, we often refer to the ***exponential case*** having the following characteristics:

- $n(u) = ne^{Ru}$. Since we have assumed that the survival function is independent of time, we see from (19.4.3) and the form of $n(u)$ that the size of the population is changing exponentially at rate R, but with a stable age distribution within the population.
- $h(x) = e^{\beta(r-x)}$; that is, pensions are adjusted at a constant annual rate of β.

Before exploring the exponential case, we should understand its limitations. It is clear that conditions for exponential growth or decay cannot exist indefinitely. When the exponential case is approximately realized, the three key economic rates, interest δ, wages τ, and pension adjustment β, are interrelated. For example, if β is related to inflation, it is conventional to assume that $\delta > \beta$ even though there have been periods of unexpected inflation where the reverse holds. If $\beta > \tau$, the consequence would be an improvement in the economic position of retired lives relative to active lives, and therefore $\beta \le \tau$ is usually assumed.

Example 20.3.1

In the exponential case, show that ${}^{T}P_{t+u} = e^{\rho u}\, {}^{T}P_t$ where $\rho = \tau + R$.

Solution:

From the definition of the exponential case we have

$$ n(t + u - r + a) = ne^{R(t+u-r+a)} $$

and from (20.3.1)

$$ \bar{a}_r^h = \int_r^{\infty} e^{-(\delta-\beta)(x-r)} \frac{s(x)}{s(r)}\, dx = \bar{a}_r' $$

where $\bar{a}_r'$ is valued at force of interest $\delta - \beta$. Then, using (20.3.2), we obtain

$$ \begin{aligned} {}^{T}P_{t+u} &= f\, w(r)\, e^{\tau(t+u)}\, n\, s(r)\, e^{R(t+u-r+a)}\, \bar{a}_r' \\ &= e^{(\tau+R)u} f\, w(r)\, e^{\tau t}\, n\, s(r)\, e^{R(t-r+a)}\, \bar{a}_r' = e^{\rho u}\, {}^{T}P_t. \end{aligned} $$

Several terms in this development can be interpreted independently. The rate $\rho = \tau + R$ can be interpreted as a rate of total economic growth or decay. The term $n\, s(r)$ can be interpreted as l_r the number of survivors at age r of n members in a survivorship group at age a governed by the multiple decrement survival function $s(x)$, $a \le x \le r$. ▼

20.4 Basic Functions for Retired Lives

In this section we discuss a number of basic functions defining several main concepts of pension funding as related to the retired group. A prefixed r is used in the notation to indicate the retired group.

20.4.1 Actuarial Present Value of Future Benefits, $(rA)_t$

The $n(t - x + a)\, s(x)\, dx$ members between ages x and $x + dx$ at time t retired $x - r$ years ago with pensions at an initial annual rate of $f\, w(r)\, e^{\tau(t-x+r)}$. For each unit of initial pension of a surviving retiree, there remains the actuarial present value

$$\bar{a}_x^h = \int_x^\infty e^{-\delta(y-x)}\, h(y)\, \frac{s(y)}{s(x)}\, dy \tag{20.4.1}$$

where $s(y)$ is a single decrement survival function based only on mortality. Therefore, from (20.3.2),

$$(rA)_t = \int_r^\infty n(t - x + a)\, s(x)\, f\, w(r)\, e^{\tau(t-x+r)}\, \bar{a}_x^h\, dx. \tag{20.4.2}$$

On substituting from (20.4.1), we can write a double integral form for $(rA)_t$; that is,

$$(rA)_t = \int_r^\infty n(t - x + a)\, f\, w(r)\, e^{\tau(t-x+r)} \left[\int_x^\infty e^{-\delta(y-x)}\, h(y)\, s(y)\, dy \right] dx. \tag{20.4.3}$$

20.4.2 Benefit Payment Rate, B_t

For the retired members there is a new function to consider, B_t, the rate of benefit outgo at time t. In developing (20.4.2) for the actuarial present value of future benefits for retired lives, we saw that pensions for retirees now between ages x and $x + dx$ were paid at the initial rate of $n(t - x + a)\, s(x)\, f\, w(r)\, e^{\tau(t-x+r)}\, dx$. By age x, this rate has been adjusted by the factor $h(x)$. Hence,

$$B_t = \int_r^\infty n(t - x + a)\, s(x)\, f\, w(r)\, e^{\tau(t-x+r)}\, h(x)\, dx. \tag{20.4.4}$$

First, we note that for all differentiable functions, g,

$$\frac{\partial}{\partial t} g(t - x + r) = -\frac{\partial}{\partial x} g(t - x + r).$$

Differentiation of B_t leads to

$$\begin{aligned} \frac{d}{dt} B_t &= \int_r^\infty f\, w(r)\, s(x)\, h(x)\, \frac{\partial}{\partial t} [n(t - x + a)\, e^{\tau(t-x+r)}]\, dx \\ &= -\int_r^\infty f\, w(r)\, s(x)\, h(x)\, \frac{\partial}{\partial x} [n(t - x + a)\, e^{\tau(t-x+r)}]\, dx \end{aligned}$$

$$= -f\, w(r)\, s(x)\, h(x)\, n(t - x + a)\, e^{\tau(t-x+r)} \Bigg|_{x=r}^{x=\infty}$$

$$+ \int_r^\infty f\, w(r)\, n(t - x + a)\, e^{\tau(t-x+r)}\, [s'(x)\, h(x) + s(x)\, h'(x)]\, dx$$

$$= \Bigg[f\, w(r)\, n(t - r + a)\, s(r)\, e^{\tau t}$$

$$- \int_r^\infty f\, w(r)\, n(t - x + a)\, s(x)\, \mu(x) e^{\tau(t-x+r)}\, h(x)\, dx \Bigg]$$

$$+ \int_r^\infty f\, w(r)\, n(t - x + a)\, s(x)\, e^{\tau(t-x+r)}\, h'(x)\, dx. \qquad (20.4.5)$$

The terms within the brackets on the right-hand side of (20.4.5) measure the ***replacement effect.*** The first term is the rate at which the initial pensions for the newly retired members is increasing the benefit payment rate. The second term is the rate at which the benefit payment rate is being reduced by deaths at time t. The term outside the brackets is known as the ***adjustment effect.*** It measures the amount by which the benefit payment rate is being adjusted at time t.

20.4.3 The Allocation Equation

We are now in a position to state a basic formula for retired lives:

$${}^{T}P_t + \delta(rA)_t = B_t + \frac{d}{dt}(rA)_t. \qquad (20.4.6)$$

This equation can be argued from compound interest theory by considering $(rA)_t$ as a fund into which interest and terminal funding costs are paid and from which pensions are paid. The difference between the total rate of incomes and the rate of outgo determines the rate of change of the size of the fund.

The verification of (20.4.6) can be accomplished by differentiating $(rA)_t$ as given by (20.4.3). We have

$$\frac{d}{dt}(rA)_t = \int_r^\infty f\, w(r) \frac{\partial}{\partial t}[n(t - x + a)\, e^{\tau(t-x+r)}] \left[\int_x^\infty e^{-\delta(y-x)}\, h(y)\, s(y)\, dy \right] dx$$

$$= -f\, w(r) \int_r^\infty \left[\int_x^\infty e^{-\delta(y-x)}\, h(y)\, s(y)\, dy \right] \frac{\partial}{\partial x}[n(t - x + a)\, e^{\tau(t-x+r)}]\, dx$$

$$= -f\, w(r) \Bigg\{ n(t - x + a)\, e^{\tau(t-x+r)} \int_x^\infty e^{-\delta(y-x)}\, h(y)\, s(y)\, dy \Bigg|_{x=r}^{x=\infty}$$

$$- \int_r^\infty \left[\delta \int_x^\infty e^{-\delta(y-x)}\, h(y)\, s(y)\, dy - s(x)\, h(x) \right] n(t - x + a)\, e^{\tau(t-x+r)}\, dx \Bigg\}$$

$$= {}^{T}P_t + \delta(rA)_t - B_t$$

where (20.4.4) is used to identify the B_t term.

Example 20.4.1

Show that for the exponential case,

a. $B_{t+u} = e^{\rho u} B_t$, $\rho = \tau + R$ (20.4.7)

b. $(rA)_{t+u} = e^{\rho u} (rA)_t$ (20.4.8)

c. ${}^TP_t + \theta (rA)_t = B_t$, $\theta = \delta - \rho$ (20.4.9)

d. ${}^TP_t < B_t$ if $\theta > 0$
${}^TP_t = B_t$ if $\theta = 0$
${}^TP_t > B_t$ if $\theta < 0$. (20.4.10)

Solution:

a. From (20.4.4),

$$B_{t+u} = \int_r^\infty n e^{R(t+u-x+a)} \, e^{\tau(t+u-x+r)} f\, w(r)\, s(x)\, e^{\beta(x-r)}\, dx$$

$$= e^{(R+\tau)u} \int_r^\infty n e^{R(t-x+a)} \, e^{\tau(t-x+r)} f\, w(r)\, s(x)\, e^{\beta(x-r)}\, dx$$

$$= e^{\rho u} B_t .$$

b. Substituting into (20.4.2) and following the pattern of the solution of part (a) yields the result.

c. Rewriting (20.4.8) as

$$\frac{(rA)_{t+u} - (rA)_t}{u} = \frac{e^{\rho u} - 1}{u} (rA)_t$$

and letting $u \to 0$, we obtain

$$\frac{d}{dt} (rA)_t = \rho\, (rA)_t . \tag{20.4.11}$$

Then substituting (20.4.11) into (20.4.6) yields the result for part (c).

d. The inequalities follow from (20.4.9). This example reveals the critical role played by $\theta = \delta - \tau - R$ in the exponential case. ▼

Example 20.4.2

For the model plan operating in a stationary population with fixed salaries and level pensions, develop and interpret the formula

$$(rA)_t = f\, w(r)\, \frac{T_r - l_r \bar{a}_r}{\delta}. \tag{20.4.12}$$

Solution:

Here $h(x) = 1$, $\tau = 0$, $\theta = \delta$, $B_t = f\, w(r)\, T_r$, ${}^TP_t = f\, w(r)\, l_r\, \bar{a}_r$, and (20.4.12) follows by substituting into (20.4.9). To interpret this result, we note that pensions of $f\, w(r)$ per year are payable continuously to all persons age r or older in the stationary population. This includes pensions for future new retirees who become eligible at the rate of l_r per year. These future pension payments form a perpetuity with present value equal to $f\, w(r)\, l_r\, \bar{a}_r / \delta$. The difference in the present values of these two

perpetuities is the present value of the future pensions to the closed group of participants now age r years or older, $(rA)_t$. ▼

20.5 Accrual of Actuarial Liability

Actuarial cost methods differ by the rate at which prospective pension obligations are recognized during the participants' working lifetimes. The terminal cost method described in Section 20.3 does not recognize the liability until the attainment of retirement age r. To express the accrual of actuarial liability for a pension commencing at age r, we define for a cost method an ***accrual function*** $M(x)$. Here $M(x)$ represents that fraction of the actuarial value of future pensions accrued as an actuarial liability at age x under the actuarial cost method. The function $M(x)$ is a nondecreasing, right-continuous function of the age variable with $0 \le M(x) \le 1$ for all $x \ge a$. Under ***initial funding,*** all the liability for the future pension is recognized when the participant enters at age a; thus $M(x) = 0$ for $x < a$ and $M(x) = 1$ for $x \ge a$. For other actuarial cost methods it will be assumed that $M(a) = 0$. For funding methods requiring accrual or recognition of the total liability by age r, $M(r) = 1$ for $x \ge r$.

The function $M(x)$ can also be defined in terms of a ***pension accrual density function*** denoted by $m(x)$ such that

$$M(x) = \int_a^x m(y)\, dy \qquad x \ge a. \tag{20.5.1}$$

Note the analogy between $M(x)$ and $m(x)$ and the d.f., $F_X(x)$, and p.d.f., $f_X(x)$. In general we assume that $m(x)$ is continuous for $a < x < r$, right continuous at a, and left continuous at r, and that $m(x) = 0$ for $x > r$. In this continuous case it follows from (20.5.1) that

$$m(x) = M'(x). \tag{20.5.2}$$

At points of discontinuity of $M'(x)$, the density $m(x)$ is not defined, and we can assign an arbitrary value to it, for example, the limit from the left or from the right.

The advantage of introducing the accrual function is that we can develop pension theory simultaneously for a whole family of actuarial cost methods rather than separately for each method.

Example 20.5.1

For $M(x) = \bar{a}_{a:\overline{x-a}|} / \bar{a}_{a:\overline{r-a}|}$, $a < x < r$, verify that

a. $M(x)$ has the properties of an accrual function.
b. $M(x)\ {}_{r-x|}\bar{a}_x$ is equal to the reserve at age x on a continuous annual premium deferred life annuity issued at age a and paying a continuous annuity of 1 per year commencing at age r.

Solution:

a.

$$M(x) = \frac{\int_a^x e^{-\delta(y-a)}\, s(y)\, dy}{\int_a^r e^{-\delta(y-a)}\, s(y)\, dy};$$

thus

$$M'(x) = m(x) = \frac{e^{-\delta(x-a)}\, s(x)}{\int_a^r e^{-\delta(y-a)}\, s(y)\, dy}. \qquad (20.5.3)$$

$M(a) = 0$ and $M(r) = 1$ confirm that $M(x)$ has the properties of an accrual function.

b. A retrospective formula gives the reserve at age x as

$$\begin{aligned}
\bar{P}({}_{r-a|}\bar{a}_a)\, \bar{s}_{a:\overline{x-a}|} &= \frac{{}_{r-a|}\bar{a}_a}{\bar{a}_{a:\overline{r-a}|}}\, \bar{s}_{a:\overline{x-a}|} \\
&= \frac{{}_{x-a}E_a\ {}_{r-x|}\bar{a}_x}{\bar{a}_{a:\overline{r-a}|}}\, \frac{\bar{a}_{a:\overline{x-a}|}}{{}_{x-a}E_a} \\
&= {}_{r-x|}\bar{a}_x\, M(x).
\end{aligned}$$

▼

20.6 Basic Functions for Active Lives

In this section we define a number of basic functions related to the funding of pension benefits in the model plan. The functions relate to the active group and are denoted in the symbols with a prefixed a.

20.6.1 Actuarial Present Value of Future Benefits, $(aA)_t$

The $n(t - x + a)\, s(x)$ members between ages x and $x + dx$ at time t will, at the end of $r - x$ years, incur the terminal funding cost of ${}^{T}P_{t+r-x}\, dx$. Hence,

$$(aA)_t = \int_a^r e^{-\delta(r-x)}\, {}^{T}P_{t+r-x}\, dx. \qquad (20.6.1)$$

Example 20.6.1

Show that

$$\frac{d}{dt}(aA)_t = e^{-\delta(r-a)}\, {}^{T}P_{t+r-a} - {}^{T}P_t + \delta(aA)_t \qquad (20.6.2)$$

and interpret the equation.

Solution:

Again, we note that

$$\frac{\partial}{\partial t}\, {}^{T}P_{t+r-x} = -\frac{\partial}{\partial x}\, {}^{T}P_{t+r-x}. \qquad (20.6.3)$$

Thus

$$\begin{aligned}
\frac{d}{dt}(aA)_t &= \int_a^r e^{-\delta(r-x)} \frac{\partial}{\partial t}\, {}^{T}P_{t+r-x}\, dx \\
&= -\int_a^r e^{-\delta(r-x)} \frac{\partial}{\partial x}\, {}^{T}P_{t+r-x}\, dx \\
&= -e^{-\delta(r-x)}\, {}^{T}P_{t+r-x} \Bigg|_{x=a}^{x=r} + \delta \int_a^r {}^{T}P_{t+r-x}\, e^{-\delta(r-x)}\, dx \\
&= {}^{T}P_{t+r-a}\, e^{-\delta(r-a)} - {}^{T}P_t + \delta(aA)_t .
\end{aligned}$$

The rate of change in the actuarial present value of future pensions equals the present value of the future terminal funding rate for new entrants (who will retire $r - a$ years later) less the terminal funding rate for active members retiring now plus the rate of interest income on the actuarial present value at time t. ▼

20.6.2 Normal Cost Rate, P_t

We assume that an actuarial cost method with accrual function $M(x)$ has been selected. We now want to express the normal cost rate for the model plan, that is, to display the function that, for our continuous model, allocates the actuarial present value of future pension benefits to the various times of valuation in a participant's active service.

As in (20.6.1) the future terminal funding cost for members between ages x and $x + dx$ at time t is ${}^{T}P_{t+r-x}\, dx$. In the normal cost function this liability is being recognized at an accrual rate $m(x)$. We have

$$\begin{aligned}
P_t &= \int_a^r e^{-\delta(r-x)}\, {}^{T}P_{t+r-x}\, m(x)\, dx \\
&= e^{\tau t} f\, w(r)\, s(r)\, \bar{a}_r^h \int_a^r e^{-(\delta-\tau)(r-x)}\, n(t - x + a)\, m(x)\, dx. \qquad (20.6.4)
\end{aligned}$$

One can visualize how the normal cost rate, P_t, $u \le t \le u + r - a$, completely funds the pension benefit of a member who enters at age a at time u and retires $r - a$ years later. Consider the participants who enter between u and $u + du$. Their ultimate terminal funding cost rate will be ${}^{T}P_{u+r-a}$. At time t, the density of contributions of this group to the integral defining P_t is

$$e^{-\delta(r-x)}\, {}^{T}P_{t+r-x}\, m(x)$$

where $x = a + t - u$. In the $r - x$ years until retirement this will increase, because of interest earned, to

$${}^{T}P_{t+r-x}\, m(x), \qquad (20.6.5)$$

and this density in terms of u is ${}^{T}P_{u+r-a}\ m(a + t - u)$. Integrating these interest-accumulated contributions, we obtain

$$\int_{u}^{u+r-a} {}^{T}P_{u+r-a}\ m(a + t - u)\ dt = {}^{T}P_{u+r-a},$$

the required terminal funding cost rate.

Example 20.6.2

a. Show that in the exponential case

$$P_t = \exp\{-\delta[r - X(\theta)]\}\ {}^{T}P_{t+r-X(\theta)} \tag{20.6.6}$$

where

$$\theta = \delta - \rho = \delta - \tau - R$$

and

$$e^{\theta X(\theta)} = \int_{a}^{r} m(x)\ e^{\theta x}\ dx. \tag{20.6.7}$$

b. Interpret (20.6.6).

Solution:

a. From (20.6.4) and the solution of Example 20.3.1 we have

$$\begin{aligned}
P_t &= \int_{a}^{r} e^{-\delta(r-x)}\ {}^{T}P_{t+r-x}\ m(x)\ dx \\
&= \int_{a}^{r} e^{-\delta(r-x)}\ {}^{T}P_{t+[X(\theta)-x]+r-X(\theta)}\ m(x)\ dx \\
&= \int_{a}^{r} e^{-\delta(r-x)}\ e^{\rho[X(\theta)-x]}\ {}^{T}P_{t+r-X(\theta)}\ m(x)\ dx \\
&= e^{[-\delta r+\rho X(\theta)]}\ {}^{T}P_{t+r-X(\theta)} \int_{a}^{r} e^{(\delta-\rho)x}\ m(x)\ dx.
\end{aligned}$$

Substitute $\delta - \theta$ for ρ; using (20.6.7), we have

$$P_t = e^{[-\delta r+(\delta-\theta)X(\theta)]}\ {}^{T}P_{t+r-X(\theta)}\ e^{\theta X(\theta)}$$

which reduces to (20.6.6).

b. The annual normal cost rate at time t is sufficient with interest to provide the terminal funding cost $r - X(\theta)$ years later. The number $X(\theta)$ has its existence assured by the mean value theorem for integrals and may be interpreted as an average age of normal cost payment associated with the accrual density function $m(x)$ in the exponential case with $\theta = \delta - \tau - R$. Hence, $X(\theta)$ depends on the interest rate and on salary and population change rates. ▼

20.6.3 Actuarial Accrued Liability, $(aV)_t$

We assume, as in Section 20.6.2, that an actuarial cost method with accrual function $M(x)$ has been chosen. By analogy with (20.6.4) we have that the ***actuarial accrued liability*** for active lives at time t is given by

$$(aV)_t = \int_a^r e^{-\delta(r-x)}\ {}^TP_{t+r-x}\ M(x)\ dx. \tag{20.6.8}$$

In the integral we are applying the concept that a fraction $M(x)$ of the actuarial present value of the future pension has accrued as an actuarial liability by age x.

If we rewrite (20.6.4) in the form

$$P_t = \int_a^r e^{-\delta(r-x)}\ {}^TP_{t+r-x}\ dM(x)$$

and integrate by parts, we obtain, by use of (20.6.3),

$$\begin{aligned} P_t &= e^{-\delta(r-x)}\ {}^TP_{t+r-x}\ M(x) \Big|_{x=a}^{x=r} - \delta \int_a^r M(x)\ e^{-\delta(r-x)}\ {}^TP_{t+r-x}\ dx \\ &\quad + \int_a^r M(x)\ e^{-\delta(r-x)} \frac{\partial}{\partial t}\ {}^TP_{t+r-x}\ dx \\ &= {}^TP_t - \delta(aV)_t + \frac{d}{dt}(aV)_t \end{aligned}$$

or

$$P_t + \delta(aV)_t = {}^TP_t + \frac{d}{dt}(aV)_t. \tag{20.6.9}$$

Equation (20.6.9) can be interpreted from the viewpoint of compound interest theory. We consider the actuarial accrued liability, $(aV)_t$, as a fund into which normal costs, at rate P_t, are paid and from which terminal funding costs, at rate ${}^TP_t$, are transferred when active members retire. The left-hand side of (20.6.9) is the income rate to the fund from normal costs and interest. The right-hand side represents the allocation of this income rate to the terminal funding rate and rate change in the fund size.

Example 20.6.3

Show that in the exponential case

a. $P_{t+u} = e^{\rho u} P_t$, $\rho = \tau + R$ (20.6.10)

b. $(aV)_{t+u} = e^{\rho u} (aV)_t$ (20.6.11)

c. $P_t + \theta(aV)_t = {}^TP_t$, $\theta = \delta - \rho$ (20.6.12)

d. $P_t < {}^TP_t$ if $\theta > 0$
   $P_t = {}^TP_t$ if $\theta = 0$
   $P_t > {}^TP_t$ if $\theta < 0$. (20.6.13)

Solution:

a. In Example 20.3.1 we saw that ${}^{T}P_{t+u} = e^{\rho u}\, {}^{T}P_{t}$. Then, substituting into (20.6.4), we obtain

$$\begin{aligned} P_{t+u} &= \int_a^r e^{-\delta(r-x)}\, {}^{T}P_{t+u+r-x}\, m(x)\, dx \\ &= e^{\rho u} \int_a^r e^{-\delta(r-x)}\, {}^{T}P_{t+r-x}\, m(x)\, dx \\ &= e^{\rho u}\, P_t. \end{aligned}$$

b. The solution starts with the definition of $(aV)_t$ in (20.6.8) and follows the same steps as in part (a).

c. Rewriting (20.6.11) as

$$\frac{(aV)_{t+u} - (aV)_t}{u} = \frac{e^{\rho u} - 1}{u} (aV)_t$$

and letting $u \longrightarrow 0$, we obtain

$$\frac{d}{dt} (aV)_t = \rho\, (aV)_t. \tag{20.6.14}$$

Substituting (20.6.14) into (20.6.9) yields (20.6.12).

d. The inequalities follow from (20.6.12). Again, we see the critical role that $\theta = \delta - \tau - R$ plays in the exponential case. ▼

In line with our assumption that $M(x) = 1$ for $x \geq r$, there is no future normal cost in respect to the closed group of retirees at time t. Therefore, the actuarial accrued liability, $(rV)_t$, for retired members equals the actuarial present value of their future pensions; that is,

$$(rV)_t = (rA)_t. \tag{20.6.15}$$

This has the further effect that we then have a differential equation for the actuarial accrued liability for retired lives, which is

$${}^{T}P_t + \delta (rV)_t = B_t + \frac{d}{dt} (rV)_t. \tag{20.6.16}$$

20.6.4 Actuarial Present Value of Future Normal Costs, $(Pa)_t$

In Section 20.6.1 we noted that $n(t - x + a)\, s(x)$ members between ages x and $x + dx$ at time t will have a terminal funding cost of ${}^{T}P_{t+r-x}\, dx$ when they retire $r - x$ years later. As these members pass from age y to $y + dy$, $x \leq y < r$, the normal cost $e^{-\delta(r-y)}\, {}^{T}P_{t+r-x}\, dx\, m(y)\, dy$ will be payable. The present value of this normal cost is

$$e^{-\delta(r-x)}\, {}^{T}P_{t+r-x}\, dx\, m(y)\, dy, \tag{20.6.17}$$

and the present value of future normal costs, denoted by $(Pa)_t$, for all active members is

$$(Pa)_t = \int_a^r e^{-\delta(r-x)}\, {}^T P_{t+r-x} \int_x^r m(y)\, dy\, dx \tag{20.6.18}$$

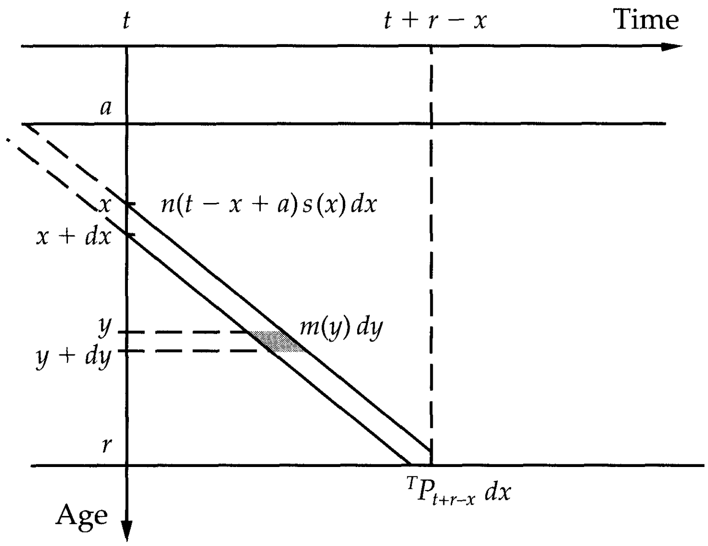
or

$$= e^{\tau t} f\, w(r)\, s(r)\, \bar{a}_r^h \int_a^r e^{-(\delta-\tau)(r-x)}\, n(t - x + a)\, [1 - M(x)]\, dx. \tag{20.6.19}$$

Figure 20.6.1 illustrates the ideas in the development of $(Pa)_t$. Expression (20.6.17) represents the present value at time t of the cost element in the shaded area. In (20.6.18), the inside integral represents the addition of elements along the diagonal, and the outer integral the present value of future normal costs at time t for all ages.

FIGURE 20.6.1

Formulation of $(Pa)_t$



It follows from (20.6.18), (20.6.1), and (20.6.8) that

$$(Pa)_t = (aA)_t - (aV)_t$$

or

$$(aV)_t = (aA)_t - (Pa)_t. \tag{20.6.20}$$

Formula (20.6.20) expresses the same concept as the prospective reserve formulas of Chapter 7 and is frequently used to define $(aV)_t$; that is,

(the actuarial liability at time t for active members) = (the actuarial present value of future pensions for active members) − (the actuarial present value of future normal costs).

By analogy with concepts from Chapter 7, $V = A - Pa$ or $A = V + Pa$, one can argue that the actuarial present value of future pensions for active lives is balanced by the actuarial accrued liability for active lives and the actuarial present value of future normal costs; that is,

$$(aA)_t = (aV)_t + (Pa)_t. \tag{20.6.21}$$

The split between the two terms on the right-hand side of (20.6.21) is determined by the actuarial cost method selected as reflected in the accrual function $M(x)$.

Example 20.6.4

a. Consider two accrual density functions, $M_I(x)$ and $M_{II}(x)$. Show that if $D(x) = M_I(x) - M_{II}(x)$ is such that $D'(a) > 0$ and $D'(x) = 0$ has exactly one solution, $a < x < r$, then $(aV)_{It} > (aV)_{IIt}$.

b. If

$$M_I(x) = \frac{\bar{a}_{a:\overline{x-a|}}}{\bar{a}_{a:\overline{r-a|}}}$$

and

$$M_{II}(x) = \frac{x - a}{r - a},$$

show that

$$(aV)_{It} > (aV)_{IIt}.$$

Solution:

a. By properties of the accrual function, $D(a) = D(r) = 0$. We are given that $D'(a) > 0$ and $D'(x) = 0$ for exactly one value of x, $a < x < r$; hence, $D(x) > 0$ for $a < x < r$. Thus

$$(aV)_{It} - (aV)_{IIt} = \int_a^r e^{-\delta(r-x)}\, {}^T p_{t+r-x}\, D(x)\, dx$$

is greater than 0, and the inequality follows.

b. We have

$$D'(x) = \frac{e^{-\delta(x-a)}\, s(x)}{\int_a^r e^{-\delta(y-a)}\, s(y)\, dy} - \frac{1}{r - a}.$$

Further, if $\delta > 1$, $e^{-\delta(y-a)}\, s(y) < 1$, and thus

$$\int_a^r e^{-\delta(y-a)}\, s(y)\, dy < \int_a^r dy = r - a.$$

Therefore

$$D'(a) = \frac{1}{\int_a^r e^{-\delta(y-a)}\, s(y)\, dy} - \frac{1}{r - a} > 0.$$

By a similar argument, $D'(r) < 0$. Since both $e^{-\delta(x-a)}$ and $s(x)$ are decreasing but positive functions of x, $D''(x) < 0$ and thus $D'(x) = 0$ for exactly one value of x, $a < x < r$. Then $(aV)_{It} > (aV)_{IIt}$ follows from part (a).

In addition, by (20.6.21), we have

$$(aA)_t = (aV)_{It} + (Pa)_{It} = (aV)_{IIt} + (Pa)_{IIt}$$

so that

$$(Pa)_{IIt} > (Pa)_{It}. \qquad \blacktriangledown$$

20.7 Individual Actuarial Cost Methods

The general actuarial cost method defined by the accrual function $M(x)$ or its derivative, the accrual density function, is an individual cost method in the sense that $m(x)$ and $M(x)$ can be applied to yield the normal cost rate and the actuarial accrued liability for each participant. The total normal cost rate and actuarial accrued liability for active lives in the plan may be determined by adding the components attributed to each participant.

The individual pension funding functions, for an annuity starting at age r with a unit initial benefit rate for an active life age x, $a \le x \le r$, are defined as follows:

The actuarial present value of the benefit is given by

$$(aA)(x) = e^{-\delta(r-x)} \frac{s(r)}{s(x)} \bar{a}_r^h. \tag{20.7.1}$$

The normal cost rate is given by

$$P(x) = (aA)(x)\, m(x), \tag{20.7.2}$$

and the accrued actuarial liability is given by

$$(aV)(x) = (aA)(x)\, M(x). \tag{20.7.3}$$

The actuarial present value of future normal costs is defined by

$$(Pa)(x) = (aA)(x) - (aV)(x) = (aA)(x)\,[1 - M(x)]. \tag{20.7.4}$$

Note that these functions for a unit benefit for (x), instead of aggregate plan functions as of time t. Exercise 20.18 develops the details of the relations between these functions and the basic functions relating to the entire group studied in Section 20.6.

In ***accrued benefit cost methods*** $M(x)$ is directly related to the accrued benefit that a participant has acquired at age x under provisions of the plan. We look at two possibilities. If the projected benefit accrues uniformly during active service,

$$m(x) = \frac{1}{(r - a)}. \tag{20.7.5}$$

If the accrual of the benefit is in proportion to total salary where there is an exponential time trend affecting all salaries,

$$m(x) = k\, w(x)\, e^{\tau x} = \frac{w(x)\, e^{\tau x}}{\displaystyle\int_a^r w(y)\, e^{\tau y}\, dy}. \tag{20.7.6}$$

As a special case if the accrual of benefits is in proportion to total salary with no time trend in salary, so that $\tau = 0$, then

$$m(x) = k\, w(x) = \frac{w(x)}{\displaystyle\int_a^r w(y)\, dy}. \tag{20.7.7}$$

For ***entry-age actuarial cost methods,*** the projected benefit is funded by a level contribution from entry age to retirement. Again, we look at two possibilities. If we define the normal cost rate, $P(x)$, given by (20.7.2) to be a constant, we have $P(x) = k = (aA)(x)\, m(x)$, so that

$$m(x) = \frac{k}{(aA)(x)} = \frac{k\, s(x)}{e^{-\delta(r-x)}\, s(r)\, \bar{a}_r^h} = k_1\, s(x)\, e^{-\delta x}.$$

Now, since $m(x)$ must integrate to 1 between a and r,

$$m(x) = \frac{s(x)\, e^{-\delta x}}{\displaystyle\int_a^r s(y)\, e^{-\delta y}\, dy}. \tag{20.7.8}$$

If, on the other hand, the contribution rate is a level fraction, π, of the salary where there is an exponential trend affecting all salaries, we have

$$P(x) = \pi\, w(x)\, e^{\tau x} = m(x)\, (aA)(x)$$

so

$$m(x) = \frac{e^{-\delta x}\, s(x)\, e^{\tau x}\, w(x)}{\displaystyle\int_a^r e^{-\delta y}\, s(y)\, e^{\tau y}\, w(y)\, dy}. \tag{20.7.9}$$

The actuarial accrued liability for an individual age x, that is, the difference between the actuarial present values of benefits and of future contributions, is

$$\begin{aligned}(aV)(x) &= e^{-\delta(r-x)} \frac{s(r)}{s(x)}\, \bar{a}_r^h - \pi \int_x^r e^{-\delta(y-x)} \frac{s(y)}{s(x)}\, w(y)\, dy \\ &= e^{-\delta(r-x)} \frac{s(r)}{s(x)}\, \bar{a}_r^h \left\{ 1 - \left[\frac{\displaystyle\int_x^r e^{-\delta y}\, s(y)\, w(y)\, dy}{\displaystyle\int_a^r e^{-\delta y}\, s(y)\, w(y)\, dy} \right] \right\} \\ &= (aA)(x)\, M(x).\end{aligned}$$

This confirms our choice of $m(x)$ function above in (20.7.9).

By a similar process it can be shown that if no exponential trend in salaries by time is assumed, the pension accrual density function is

$$m(x) = \frac{s(x)\, w(x)\, e^{-\delta x}}{\int_a^r s(y)\, w(y)\, e^{-\delta y}\, dy}. \tag{20.7.10}$$

Note that the fraction of projected benefits accrued as an actuarial liability in entry-age actuarial cost methods differs from the definition of the accrued benefits in most plans. A definition of accrued benefits is required for regulatory purposes and to communicate to participants about the benefits they are accruing. The distinction here is similar to the distinction between the reserve and the nonforfeiture benefit in ordinary insurance.

Also note that the contribution rate, paid by a plan sponsor that is following an individual actuarial cost method, will usually differ from the total normal cost rate specified by that method. There are two general reasons for this. First, at the inception of a plan or at times when a plan is amended, actuarial accrued liabilities for prior service may be changed. Second, the actuarial assumptions will not be realized exactly, thereby generating funding gains or losses. The decisions on how to adjust the contribution rate to fund these changes in the actuarial accrued liability, or to adjust for gains or losses, are important ones that are subject to regulation. The particular adjustments chosen are not compelled by the choice of individual actuarial cost method.

20.8 Group Actuarial Cost Methods

In this section we consider group or aggregate actuarial cost methods for which contributions are determined on a collective basis and not as a sum of contributions for individual participants. For the purpose of defining aggregate actuarial cost methods, we need three additional functions:

1. $(aF)_t$, the fund allocated to active members at time t
2. $(aC)_t$, the annual contribution rate at time t with respect to active participants
3. $(aU)_t$, the unfunded actuarial accrued liability with respect to active participants at time t.

Thus,

$$(aU)_t = (aV)_t - (aF)_t. \tag{20.8.1}$$

The fund for active members at time t can be described by the differential equation

$$\frac{d}{dt}(aF)_t = (aC)_t + \delta\,(aF)_t - {}^{T}P_t \tag{20.8.2}$$

with the initial value $(aF)_0$. The right-hand member of (20.8.2) indicates the two sources of income to the fund and the one source of outgo, that is, the transfer of the terminal funding cost to a fund for retired members.

In relation to an actuarial cost method, as determined by an accrual function, which implies a normal cost rate P_t [see (20.6.4)] and the unfunded accrued liability

of active members $(aU)_t$ [see (20.6.8) and (20.8.1)], a natural form of contribution rate is

$$(aC)_t = P_t + \lambda(t)\,(aU)_t. \tag{20.8.3}$$

In (20.8.3) $\lambda(t)$ defines the process for amortizing $(aU)_t$.

Equation (20.8.3) points out a characteristic of aggregate actuarial cost methods not yet stated. These methods define a contribution rate, $(aC)_t$, that depends on the level of funding, that is, on the magnitude of $(aU)_t$. Here the adjustments required by plan changes or by gains and losses can be made automatically by following the actuarial cost method since the value of $(aU)_t$ will reflect such changes and gains and losses.

We consider one such amortization process, one in which

$$\lambda(t) = \frac{1}{\bar{a}_{P_t}} \tag{20.8.4}$$

where

$$\bar{a}_{P_t} = \frac{(Pa)_t}{P_t}. \tag{20.8.5}$$

Thus $(Pa)_t = P_t\,\bar{a}_{P_t}$, so $\bar{a}_{P_t}$ is the value of a unit temporary annuity such that this temporary annuity with a level income rate at the current normal cost rate, P_t, equals the actuarial present value of future normal costs for the current active members, $(Pa)_t$. In this case the notation has been selected to suggest the motivating idea.

Formula (20.8.3) can be rewritten for this particular choice of $\lambda(t)$ as

$$\begin{aligned}(aC)_t &= P_t + \frac{(aV)_t - (aF)_t}{\bar{a}_{P_t}} \\ &= \frac{(Pa)_t + (aV)_t - (aF)_t}{\bar{a}_{P_t}} \\ &= \frac{(aA)_t - (aF)_t}{\bar{a}_{P_t}} \end{aligned} \tag{20.8.6}$$

making use of (20.6.20). Thus, with $\lambda(t)$ given by (20.8.4), we have

$$(aC)_t\,\bar{a}_{P_t} = (aA)_t - (aF)_t. \tag{20.8.7}$$

The interpretation of (20.8.7) is that a temporary annuity at the rate of $(aC)_t$ is equivalent to the actuarial present value of future benefits for active members less the fund for them.

The formula governing the progress of the fund, (20.8.2), becomes, for $\lambda(t)$ given by (20.8.4),

$$\frac{d}{dt}(aF)_t = P_t + \frac{(aU)_t}{\bar{a}_{P_t}} + \delta\,(aF)_t - {}^TP_t. \tag{20.8.8}$$

We can write (20.6.9) as

$$\frac{d}{dt}(aV)_t = P_t + \delta\,(aV)_t - {}^TP_t. \tag{20.8.9}$$

By subtracting (20.8.8) from (20.8.9), we get

$$\frac{d}{dt}(aU)_t = -\frac{(aU)_t}{\bar{a}_{P_t}} + \delta\,(aU)_t. \tag{20.8.10}$$

The differential equation (20.8.10) may be solved by replacing t by u, integrating with respect to u from 0 to t, and taking exponentials to obtain

$$(aU)_t = (aU)_0 \exp\left[-\int_0^t \left(\frac{1}{\bar{a}_{P_u}} - \delta\right) du\right]. \tag{20.8.11}$$

Upon substituting (20.8.1), we obtain

$$(aF)_t = (aV)_t - [(aV)_0 - (aF)_0] \exp\left[-\int_0^t \left(\frac{1}{\bar{a}_{P_u}} - \delta\right) du\right]. \tag{20.8.12}$$

Provided that a_{P_u} is smaller than $\bar{a}_{\overline{\infty}|} = 1/\delta$ so that $1/\bar{a}_{P_u} - \delta \geq \epsilon > 0$,

$$\exp\left[-\int_0^t \left(\frac{1}{\bar{a}_{P_u}} - \delta\right) du\right] \to 0$$

as $t \to \infty$, therefore $(aF)_t \to (aV)_t$.

Here the aggregate cost method with $\lambda(t) = 1/\bar{a}_{P_t}$ is asymptotically equivalent to the individual cost method defined by the accrual function used to evaluate $(aV)_t$ and P_t. There may be many accrual functions that produce functions such that $(Pa)_t/P_t$ is sufficiently small to assure the convergence of $(aF)_t$ to $(aV)_t$. Each of these accrual functions could produce a different pattern of contributions and a different ultimate fund. For completeness, when referring to an aggregate actuarial cost method, always specify the accrual function used. The aggregate cost method with entry-age accrual is particularly important in practice.

Clearly there are many possible choices for the function $\lambda(t)$ in determining the rate of amortization of $(aU)_0$. If the goal is the completion of amortization by the end of n years from some initial time 0, one choice for $\lambda(t)$ is

$$\lambda(t) = \frac{1}{\bar{a}_{\overline{n-t}|}} \qquad 0 < t < n.$$

Then, corresponding to (20.8.11), we obtain

$$\begin{aligned}(aU)_t &= (aU)_0 \exp\left[-\int_0^t \left(\frac{1}{\bar{a}_{\overline{n-u}|}} - \delta\right) du\right] \\ &= (aU)_0 \exp\left(-\int_0^t \frac{1}{\bar{s}_{\overline{n-u}|}}\, du\right). \end{aligned} \tag{20.8.13}$$

It can be shown (Exercise 20.21) that

$$-\int_0^t \frac{1}{\bar{s}_{\overline{n-u}|}}\, du = \log\left(\frac{\bar{s}_{\overline{n}|} - \bar{s}_{\overline{t}|}}{\bar{s}_{\overline{n}|}}\right).$$

Therefore (20.8.13) becomes

$$(aU)_t = (aU)_0 \frac{\bar{s}_{\overline{n}|} - \bar{s}_{\overline{t}|}}{\bar{s}_{\overline{n}|}},$$

and it can be shown that

$$(aC)_t = P_t + \frac{(aU)_0}{\bar{a}_{\overline{n}|}} \qquad 0 \le t \le n.$$

At time n, the funding goal will be achieved with $(aU)_n = 0$, and $(aC)_t$ will drop to P_t.

In practice, a common amortization pattern choice is to define $\lambda(t)$ as the reciprocal of an average annuity value for the future wages of the active lives. We define this to be $\bar{a}_{W_t} = (Wa)_t / W_t$, where $(Wa)_t$ is given by

$$(Wa)_t = \int_a^r n(t - x + a)\, s(x)\, w(x) \left[\int_x^r e^{-\delta(y-x)} \frac{s(y)}{s(x)} \frac{w(y)}{w(x)} e^{\tau(t+y-x)}\, dy\right] dx$$

$$= \int_a^r n(t - x + a) \left[\int_x^r e^{-\delta(y-x)}\, s(y)\, w(y)\, e^{\tau(t+y-x)}\, dy\right] dx. \tag{20.8.14}$$

Combining (20.2.3) and (20.8.14), we have

$$\bar{a}_{W_t} = \frac{\displaystyle\int_a^r n(t - x + a) \int_x^r e^{-(\delta-\tau)(y-x)}\, s(y)\, w(y)\, dy\, dx}{\displaystyle\int_a^r n(t - x + a)\, s(x)\, w(x)\, dx}. \tag{20.8.15}$$

You will be asked in Exercise 20.23 to verify that the above ratio is the same as $\bar{a}_{P_t}$ as defined by (20.8.5) for the entry-age actuarial cost method using a level percentage of salary as the normal cost pattern. Thus, the choice of $\lambda(t) = 1/\bar{a}_{W_t}$ yields a costing pattern that is asymptotically equivalent to the individual entry-age actuarial cost method with normal costs equal to a level percentage of salary using $m(x)$ given in (20.7.9).

Example 20.8.1

Assume a stationary population, that is, $n(a) = l_a$, $\tau = 0$, and $h(x) = 1$, with the accrual function associated with the level amount entry-age actuarial cost method, $M(x) = \bar{a}_{a:\overline{x-a}|} / \bar{a}_{a:\overline{r-a}|}$ (see Example 20.5.1).

a. Display λ.
b. Calculate $(aC)_0$ if $(aF)_0 = 0$.

Solution:

a. Formula (20.3.2) gives, for the stationary case,

$${}^{T}P_t = f\, w(r)\, l_r\, \bar{a}_r$$

for all t. Thus from (20.6.19),

$$(Pa)_t = \int_a^r e^{-\delta(r-x)} f\, w(r)\, l_r\, \bar{a}_r \left(1 - \frac{\bar{a}_{a:\overline{x-a}|}}{\bar{a}_{a:\overline{r-a}|}}\right) dx.$$

Now, $M'(x) = m(x) = {}_{x-a}E_a / \bar{a}_{a:\overline{r-a}|}$, so by (20.6.4)

$$P_t = \int_a^r e^{-\delta(r-x)} f\, w(r)\, l_r\, \bar{a}_r \frac{{}_{x-a}E_a}{\bar{a}_{a:\overline{r-a}|}} dx.$$

Thus, (20.8.5) gives

$$\bar{a}_{P_t} = \frac{(Pa)_t}{P_t} = \frac{\int_a^r e^{\delta x}\, {}_{x-a}E_a\, \bar{a}_{x:\overline{r-x}|} dx}{\int_a^r e^{\delta x}\, {}_{x-a}E_a\, dx}$$

$$= \frac{\int_a^r l_x\, \bar{a}_{x:\overline{r-x}|}\, dx}{\int_a^r l_x\, dx},$$

and $\lambda = 1/\bar{a}_{P_t}$.

b. Substituting into (20.8.6) from (20.6.1),

$$(aC)_0 = \frac{\int_a^r e^{-\delta(r-x)} f\, w(r)\, l_r\, \bar{a}_r\, dx}{\bar{a}_{P_t}}$$

$$= \frac{f\, w(r)\, l_r\, \bar{a}_r\, \bar{a}_{\overline{r-a}|} \int_a^r l_x\, dx}{\int_a^r l_x\, \bar{a}_{x:\overline{r-x}|}\, dx}.$$

▼

20.9 Basic Functions for Active and Retired Members Combined

In Sections 20.4 and 20.6 we develop separate basic functions for retired and active members; this is a useful division for many purposes. The administrative system, the actuarial valuation problems, and even the investment policy may be different for the two groups. However, for other purposes it is useful to consider basic functions for the combined group of active and retired members.

The basic actuarial functions for the combined group are the sums of those for the retired members given in Section 20.4 and those for the active members given in Section 20.6. These are summarized in Table 20.9.1.

TABLE 20.9.1

Actuarial Functions for the Active, Retired, and Combined Member Groups of a Pension Plan

Function	Actives	Retireds	Combined
Actuarial present value at time t of future pensions[a,b]	$(aA)_t$	$(rA)_t$	$A_t = (aA)_t + (rA)_t$
Normal cost rate[c]	P_t	0	P_t
Actuarial accrued liability[d,e]	$(aV)_t$	$(rV)_t$	$V_t = (aV)_t + (rV)_t$
Actuarial present value of future normal costs[f]	$(Pa)_t$	0	$(Pa)_t$

[a] $(aA)_t$ is given in (20.6.1)
[b] $(rA)_t$ is given in (20.4.2)
[c] P_t is given in (20.6.4)
[d] $(aV)_t$ is given in (20.6.8)
[e] $(rV)_t$ is given in (20.6.15)
[f] $(Pa)_t$ is given in (20.6.18)

We can use the income allocation equations for active members (20.6.9) and for retired members (20.4.6) to obtain such an equation for the combined group. Thus,

$$P_t + \delta\, V_t = B_t + \frac{d}{dt}\, V_t. \tag{20.9.1}$$

In this equation normal cost and interest income into the fund are allocated to pension benefit payments and change in the actuarial accrued liability.

To obtain formulas for the combined group under aggregate funding we assume that pensions for retired members are fully funded so that $(rV)_t = (rF)_t$. Then, (20.8.1) may be rewritten as the unfunded actuarial liability for all members as

$$\begin{aligned} U_t &= V_t - F_t \\ &= (aV)_t + (rV)_t - (aV)_t - (rF)_t \\ &= (aU)_t. \end{aligned} \tag{20.9.2}$$

Further, since no contribution is required for the retired members, the contribution rate C_t for all members equals $(aC)_t$, the contribution rate for active members. In this case (20.8.3) may be rewritten as

$$C_t = P_t + \lambda(t)\, U_t. \tag{20.9.3}$$

If $\lambda(t) = 1/\bar{a}_{P_t}$, the contribution rate becomes

$$\begin{aligned} C_t &= \frac{P_t\, \bar{a}_{P_t} + V_t - F_t}{\bar{a}_{P_t}} \\ &= \frac{A_t - F_t}{\bar{a}_{P_t}}. \end{aligned} \tag{20.9.4}$$

Thus when $(rF)_t = (rV)_t$, the results of the aggregate cost method defined for active

members at t by (20.8.6) are equivalent to the results defined for all members by (20.9.4).

20.10 Notes and References

Many of the rudiments of the theory of pension funding appeared in a government publication known as the Bulletin on 23P. Charles Trowbridge (1952, 1963) did much to create a mathematical theory of pension funding. The more elaborate model used in this chapter was developed for a series of papers by Bowers, Hickman, and Nesbitt (1976, 1979). The stress on separate functions for active and retired lives is due to Kischuk (1976).

Several authors have studied the problems created for pension funding by inflationary influences on salaries, interest rates, and benefits. Papers by Allison and Winklevoss (1975) and Myers (1960) are in this class. John Trowbridge (1977) provides many observations on changes in pension funding in different nations in response to inflation.

Exercises

Section 20.2

20.1. In a stationary population with level salaries at rate w, what is the payroll function W_t?

Section 20.3

20.2. In the exponential case, what is the payroll function W_t?

20.3. Assume that the initial annual rate of retirement income for a life retiring at time t is given by

$$\frac{f}{b}\int_{t-b}^{t} w(r - t + y)\, e^{\tau y}\, dy \qquad 0 < b < r - a.$$

Other aspects of the model plan remain unchanged. For this benefit definition, based on a final average formula,

a. Show that the initial benefit rate at time t is given by

$$\frac{f}{b}\int_{0}^{b} w(r - z)\, e^{\tau(t-z)}\, dz$$

b. Display a formula for the terminal funding cost rate at time t

c. Rework Example 20.3.1.

20.4. The initial annual rate of retirement income for a life retiring at time t is given by $c(r - a)\, w\, e^{\tau t}$. Other aspects of the model plan remain unchanged.

For this initial benefit rate, based on the product of years of service and final salary level,

a. Display a formula for the terminal funding cost rate at time t

b. Rework Example 20.3.1.

20.5. If $s(x) = e^{-\mu(x-a)}$, $a \le x \le r$, display ${}^{T}P_t$ in the exponential case.

20.6. Consider an initially immature model with a stationary active population arising from the following assumptions:
$a = 25$; $r = 65$; $n(t) = 0$ for $t < -40$ and $n(t) = 75$ for $t > -40$; $s(x) = (100 - x)/75$ for $25 < x < 100$; $\delta = 0.06$; $w(x) = 525/(100 - x)$; $\tau = 0.02$; $f = 0.6$; $h(x) = 1$.

a. Find $\bar{a}_x^h$ and in particular $\bar{a}_{65}^h$.

b. Find ${}^{T}P_t$.

Section 20.4

20.7. In the exponential case, show that $B_t = {}^{T}P_t\ (\bar{a}_r'^h / \bar{a}_r^h)$ where

$$\bar{a}_r'^h = \int_r^{\infty} e^{-(\rho-\beta)(x-r)} \frac{s(x)}{s(r)}\, dx.$$

20.8. For the initially immature model with the stationary active population of Exercise 20.6:

a. Find $(rA)_t$ by evaluating expression (20.4.3) for $t > 35$

b. Find B_t for $t > 35$

c. Verify the allocation equation (20.4.6) for $t > 35$.

Section 20.5

20.9. What is $M(x)$ in the case of terminal funding?

Section 20.6

20.10. Using the assumptions of Exercise 20.5 with $m(x) = 1/(r - a)$, determine P_t.

20.11. a. Show that

$$(aA)_t = \int_t^{t+r-a} e^{-\delta(y-t)}\ {}^{T}P_y\, dy.$$

b. Differentiate the expression in part (a) to obtain an alternative solution to Example 20.6.1.

20.12. If

$$e^{\theta X(\theta)} = \int_a^r e^{\theta x}\, m(x)\, dx$$

and

$$\mu = \int_a^r x\, m(x)\, dx,$$

show that

a. $X(\theta) > \mu$ if $\theta > 0$

b. $X(\theta) < \mu$ if $\theta < 0$

[Hint: Use Jensen's inequality (1.3.2) or (1.3.3)]

c. limit $X(\theta) = \mu$ as $\theta \to 0$.

[Hint: Think of $e^{\theta X(\theta)} = E[e^{\theta X}]$ as a m.g.f.]

20.13. In the exponential case, show that

a. $P_t = {}^T P_t \exp\{-\theta[r - X(\theta)]\}$

b. $(aV)_t = {}^T P_t\, \bar{a}_{\overline{r-X(\theta)}|\,\theta} = P_t\, \bar{s}_{\overline{r-X(\theta)}|\,\theta}$.

20.14. a. What do the formulas in Exercise 20.13 become if the model plan operates in a stationary population with $\tau = 0$?

b. What do the formulas in Exercise 20.13 become if $\theta = \delta - \rho = 0$?

20.15. a. Derive a normal cost rate to be applied to all salaries of those who enter at time u. Other aspects of the model plan are unchanged.

b. Using the result in (a), display the corresponding pension accrual density function for those who enter at time u.

20.16. a. For the model plan show that

$$P_t = f\, w(r)\, s(r)\, \bar{a}_r^h \int_a^r e^{-\delta(r-x)}\, e^{\tau(t+r-x)}\, n(t - x + a)\, m(x)\, dx.$$

b. If $\tau = 0$, $n(t) = l_a$, show that

$$P_t = f\, w(r) \int_a^r l_x\, {}_{r-x}E_x\, \bar{a}_r^h\, m(x)\, dx$$

where ${}_{r-x}E_x$ is based on the survival function $s(x)$ and force of interest δ.

20.17. If $n(t) = l_a$ and

$$m(x) = \frac{w(x)\, e^{\tau x}}{\int_a^r w(y)\, e^{\tau y}\, dy}$$

show that

$$P_{t+u} = e^{\tau u}\, P_t.$$

Section 20.7

20.18. Assume that projected initial benefit rate at retirement for a life age x at time t is $f\, w(r)\, e^{\tau(t+r-x)}$ and the number of lives age x to $x + dx$ at time t is $n(t - x + a)\, s(x)\, dx$.

a. Verify that $(aA)_t$ as given by (20.6.1) is equal to

$$\int_a^r f\, w(r)\, e^{\tau(t+r-x)}\, n(t - x + a)\, s(x)\, (aA)(x)\, dx.$$

b. Verify that P_t as given by (20.6.4) is equal to

$$\int_a^r f\, w(r)\, e^{\tau(t+r-x)}\, n(t - x + a)\, s(x)\, P(x)\, dx.$$

c. Verify that $(aV)_t$ as given by (20.6.8) is equal to

$$\int_a^r f\, w(r)\, e^{\tau(t+r-x)}\, n(t - x + a)\, s(x)\, (aV)(x)\, dx.$$

d. Verify that $(Pa)_t$ as given by (20.6.18) is equal to

$$\int_a^r f\, w(r)\, e^{\tau(t+r-x)}\, n(t - x + a)\, s(x)\, (Pa)(x)\, dx.$$

20.19. Verify (20.7.9).

20.20. For the initially immature model with a stationary active population of Exercise 20.6,

a. Find $(aA)_t$.

b. Find $M(x)$ and $m(x)$ for the level accrual of benefits. For this actuarial cost method find

   i. $(aV)_t$ and

   ii. P_t.

   iii. Verify the allocation equation (20.6.9).

c. Find $M(x)$ and $m(x)$ for costs as a level percentage of projected wages between ages 25 and 65. For this actuarial cost method find

   i. $(aV)_t$ and

   ii. P_t.

   iii. Verify the allocation equation (20.6.9).

Section 20.8

20.21. Verify that

a.
$$-\int_0^t \frac{1}{\bar{s}_{\overline{n-y}|}}\, dy = \log\left(\frac{\bar{s}_{\overline{n}|} - \bar{s}_{\overline{t}|}}{\bar{s}_{\overline{n}|}}\right)$$

b.
$$-\int_0^t \frac{1}{\bar{a}_{\overline{n-y}|}}\, dy = \log\left(\frac{\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|}}{\bar{a}_{\overline{n}|}}\right).$$

20.22. a. Obtain a simplified formula for $\bar{a}_{P_t}$ in the exponential case.

b. What does $\bar{a}_{P_t}$ become in the exponential case if $\theta = \delta - \rho = 0$?

20.23. Verify that the ratio $\bar{a}_{W_t}$, given by (20.8.15), is equal to the ratio $\bar{a}_{P_t}$, given by (20.8.5), when applied to the pension accrual density function given by (20.7.9).

20.24. For the initially immature model with the stationary active population of Exercise 20.6,

a. Find W_t

b. Find $(Wa)_t$, the present value of future projected wages for the active work force at time t

c. Verify that $\bar{a}_{W_t} = (Wa)_t / W_t$, the average annuity value for future wages, equals the average annuity value for entry-age normal costing (Exercise 20.20(c)) defined by

$$\bar{a}_{P_t} = \frac{(Pa)_t}{P_t} = \frac{(aA)_t - (aV)_t}{P_t}.$$

Miscellaneous Exercises

20.25. For the initially immature model with the stationary active population of Exercise 20.6, display and solve the differential equation for the size of the fund for active lives assuming a 15-year amortization of $(aV)_0$ using

a. The actuarial cost method of Exercise 20.20(b).

b. The actuarial cost method of Exercise 20.20(c).

[Hint: For $0 < t < 15$, modify (20.8.2) so that $(aC)_t$ equals the normal cost plus the amortization payment rate.]

21

INTEREST AS A RANDOM VARIABLE

21.1 Introduction

The developments in Chapters 3 through 11 and 15 through 18 were built on the basic assumptions that time until decrement and cause or type of decrement are random variables and their joint distribution is known. When interest earnings were introduced into the models for long-term financial operations, their effect was captured by interest rates that were assumed to be deterministic and usually constant. An examination of a set of observations of interest rates confirms that this assumption is unrealistic. Table 21.1.1 illustrates this point.

TABLE 21.1.1

Average Yield to Maturity for 30-Year U.S. Treasury Bonds in January of Year Indicated*

Year	Yield†	Year	Yield†
1978	8.18%	1987	7.39
1979	8.94	1988	8.83
1980	10.60	1989	8.93
1981	12.14	1990	8.26
1982	14.22	1991	8.27
1983	10.63	1992	7.58
1984	11.75	1993	7.34
1985	11.45	1994	6.29
1986	9.40	1995	7.85
		1996	6.05

*Source: "Economic Statistics for Employee Benefit Actuaries," April 1996, Schaumburg, Ill.: Society of Actuaries.
†Bond equivalent yield.

21.1.1 Incorporating Variability of Interest

There are several methods for incorporating the variability of interest rates into actuarial models.

1. Preset interest rate scenarios. These scenarios are sequences of future interest rates, indexed by time, that will be used with other assumptions and the equivalence principle in premium and reserve calculations. In more comprehensive models each scenario would specify other variables such as expenses and withdrawal rates in a way to be consistent with the corresponding interest rates. Furthermore, in these latter models, the interest rates themselves, both within a scenario as well as across scenarios, might be constructed to satisfy certain economic conditions.
   a. The scenarios can be specified without modeling of past data and designed simply to measure the adequacy of premiums and reserves over different paths of plausible future economic conditions. This would be a type of sensitivity analysis and is the subject of Section 21.2.1.
   b. The scenarios can be determined after a systematic review of alternative macroeconomic projections and a personal probability attached to each scenario by the actuary. This approach is the subject of Section 21.2.2.
2. Stochastic models most often based on an analysis of past data. This method is typically data centered, and both the selection of the model and the estimation of the model parameters are influenced by past observations. Data from some segments of the capital markets support the hypothesis that annual interest rates can be modeled as independent and identically distributed random variables. Other data may support models in which annual interest rates are dependent random variables. Each of these classes of models can be divided into those in which it is assumed that relevant economic information is captured in observed interest rates and those in which interest rates are modeled as depending on other economic variables that are incorporated into the model. These data-based stochastic models are studied in Sections 21.3 and 21.4.
3. Stochastic models that depend on assumed characteristics of capital markets. Within financial economics, elaborate theories of the operations of capital markets have been developed. A consequence of one of these theories is that a consensus forecast of future financial conditions is provided by current security prices and their relationships. For example, the relationship among yield rates and the corresponding maturity dates, a yield curve, contains information that can be used in building stochastic models for future interest rates and, perhaps, other economic variables. These ideas are introduced in Section 21.5.

In this chapter elements of each of these methods for making provision for the variability of interest rates are introduced. The application of these methods requires knowledge of macroeconomics, applied statistics, and financial economics, respectively. The three methods are developed in this chapter in the order in which they are listed, which is also the order in which they entered actuarial literature. Because of the diversity of the prerequisite ideas, the sections of this chapter cannot provide a complete background for any of the methods.

In earlier chapters, in which time and cause of decrement were assumed to be random variables, attention was devoted to risk management tools for moderating the unfortunate financial consequences of experience that deviated from that expected. These tools for an insurer included acquiring capital, buying reinsurance, increasing contingency loadings to premiums, and increasing the size of the insured group. In Section 21.6 some tools for managing interest rate risk are discussed.

21.1.2 Notation and Preliminaries

We use the symbol I_k to denote the random variable interpreted as the effective interest rate in the k-th transaction period; that is, I_k is the ***one-period*** interest rate for that k-th period: 1 at the end of the period has a present value of $1/(1 + I_k)$ at the beginning of the period. In most applications considered in this chapter, the transaction period will be a policy year. In most earlier developments it was assumed that I_k, $k = 1, 2, \ldots$, has a single point or degenerate distribution such that $\Pr(I_k = i) = 1$, $k = 1, 2, \ldots$.

An immediate consequence of assuming that the effective (one-period) interest rate is a random variable can be derived from Jensen's inequality, (1.3.3), where $u''(x) > 0$. The restated inequality is

$$E[u(X)] \geq u(E[X]).$$

If $u(x) = (1 + x)^{-1}$ and $X = I_k$, we have $u''(x) > 0$, $-1 < x$, and

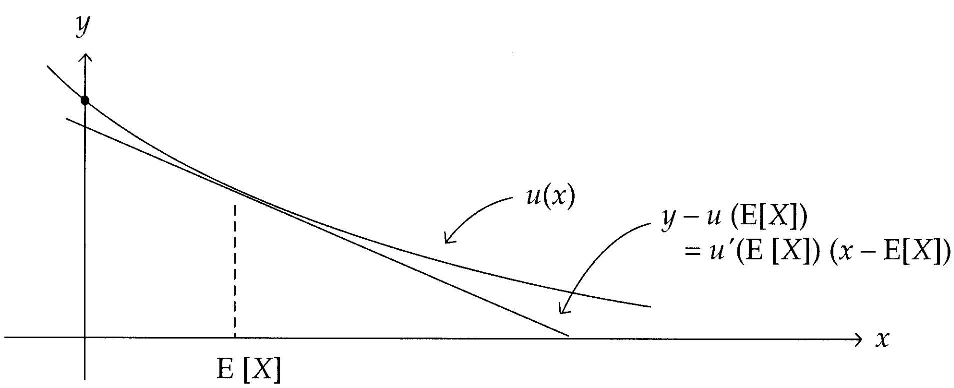
$$E[(1 + I_k)^{-1}] \geq [1 + E(I_k)]^{-1}, \tag{21.1.1}$$

with equality holding only if I_k has a single point distribution.

Figure 21.1.1, a revision of Figure 1.3.1, is a graphical representation of the verification of (21.1.1).

FIGURE 21.1.1

Demonstration of Jensen's Inequality, $u'(x) < 0$, $u''(x) > 0$



Inequality (21.1.1) can be converted to a statement about the effect of a random interest rate on the actuarial present value of 1 unit paid at the end of one period. The actuarial present value cannot be less than the present value of the payment at the expected interest rate.

21.2 Scenarios

A scenario is an outline of a projected sequence of events. In accordance with this definition, the determination of premiums and reserves in life insurance requires scenarios of future demographic and economic events. In previous chapters there has been a single scenario of future interest rates, and usually the rates have been identical. In this section, the idea of creating and using a number of scenarios of interest rates is developed.

21.2.1 Deterministic Scenarios

In this chapter a deterministic interest rate scenario is taken to be a sequence of future one-period interest rates $(i_1, i_2, i_3, \ldots)$ that has been determined by the actuary for use in an actuarial calculation. The elements of the sequence are selected as representing plausible prospective interest rates in accordance with the actuary's view of possible future economic environments. One scenario can be sufficient if the actuary is certain about future investment returns as a result of past investment decisions or special knowledge. As an alternative, several scenarios can be specified so that the sensitivity of actuarial present values to changes in the economic environment can be studied.

If several scenarios are used, we index the scenarios by $j = 1, 2, \ldots, m$, where m is the number of scenarios; that is, $({}_ji_1, {}_ji_2, {}_ji_3, \ldots)$ denotes interest scenario j. In addition, discount factors and annuity values that are specific to scenario j are denoted as follows:

$$ {}_jv^0 = 1, \quad {}_jv^k = \prod_{r=1}^{k} (1 + {}_ji_r)^{-1}, \quad \text{and } {}_j\ddot{a}_{\overline{n}|} = \sum_{k=0}^{n-1} {}_jv^k. $$

Using this notation, and ideas from Chapters 4, 5, and 6, it is natural to define

$$ {}_jA_x = \mathrm{E}[{}_jv^{K+1}] = \sum_{k=0}^{\infty} {}_jv^{k+1} \, {}_kp_x \, q_{x+k}, \tag{21.2.1a} $$

$$ {}_j\ddot{a}_x = \mathrm{E}[{}_j\ddot{a}_{\overline{K+1}|}] = \sum_{k=0}^{\infty} {}_j\ddot{a}_{\overline{k+1}|} \, {}_kp_x \, q_{x+k}, \tag{21.2.1b} $$

where K is the random variable defined as the number of complete future life years of a life age x. Applying the equivalence principle, we have

$$ {}_jP_x = \frac{{}_jA_x}{{}_j\ddot{a}_x}. \tag{21.2.1c} $$

The symbol ${}_jP_x$ used in (21.2.1c) is not part of IAN and runs the risk of being confused with the symbol used in Chapter 6 to denote the benefit premium for a limited payment life insurance. Despite the possible confusion, the symbol is used in this section to promote consistency with the symbols used to denote actuarial present values that are specific to scenario j.

Developing variances for the loss variables implicit in (21.2.1a, b, and c) requires care. The simplifications available with constant interest rates cannot be employed, and we must remember that the variances recognize only the random nature of time until death for a given interest rate scenario.

We have

$$\mathrm{Var}({}_j v^{K+1}) = \sum_{k=0}^{\infty} {}_j v^{2(k+1)}\, {}_k p_x\, q_{x+k} - ({}_j A_x)^2 = {}_j^2 A_x - ({}_j A_x)^2,$$

$$\mathrm{Var}({}_j \ddot{a}_{\overline{K+1}|}) = \sum_{k=0}^{\infty} ({}_j \ddot{a}_{\overline{k+1}|})^2\, {}_k p_x\, q_{x+k} - ({}_j \ddot{a}_x)^2$$

$$= \sum_{k=0}^{\infty} \left(\sum_{r=0}^{k} {}_j v^r \right)^2 {}_k p_x\, q_{x+k} - ({}_j \ddot{a}_x)^2,$$

and

$$\mathrm{Var}({}_j v^{K+1} - {}_j P_x\, {}_j \ddot{a}_{\overline{K+1}|}) = \sum_{k=0}^{\infty} ({}_j v^{k+1} - {}_j P_x\, {}_j \ddot{a}_{\overline{k+1}|})^2\, {}_k p_x\, q_{x+k}.$$

Example 21.2.1

Four interest rate scenarios are defined.

j	${}_j i_1$	${}_j i_2$	${}_j i_3$	${}_j i_4$
1	0.06	0.06	0.06	0.06
2	0.06	0.03	0.03	0.03
3	0.06	0.09	0.09	0.09
4	0.06	0.03	0.09	0.03

The discrete distribution of the curtate future lifetime of (x) is given in the following table.

k	${}_k p_x\, q_{x+k}$
0	0.1
1	0.2
2	0.3
3	0.4

Calculate ${}_j v^k$, ${}_j \ddot{a}_{\overline{k}|}$ for $j = 1, 2, 3, 4$ and $k = 0, 1, 2, 3$, and ${}_j A_x$, ${}_j \ddot{a}_x$, ${}_j P_x$ for $j = 1, 2, 3, 4$.

Solution:

	$_jv^{k+1}$				$_j\ddot{a}_{\overline{k+1}\rceil}$			
j	$k=0$	$k=1$	$k=2$	$k=3$	$k=0$	$k=1$	$k=2$	$k=3$
1	0.9434	0.8900	0.8396	0.7921	1.0000	1.9434	2.8334	3.6730
2	0.9434	0.9159	0.8892	0.8633	1.0000	1.9434	2.8593	3.7486
3	0.9434	0.8655	0.7940	0.7285	1.0000	1.9434	2.8089	3.6029
4	0.9434	0.9159	0.8403	0.8158	1.0000	1.9434	2.8593	3.6996

j	$_jA_x$	$_j\ddot{a}_x$	$_jP_x$
1	0.8411	2.8079	0.2995
2	0.8896	2.8459	0.3126
3	0.7970	2.7725	0.2875
4	0.8559	2.8263	0.3028

▼

Example 21.2.2

Using the assumptions of Example 21.2.1, calculate $\sqrt{\mathrm{Var}(_jv^{K+1})}$ and $\sqrt{\mathrm{Var}(_j\ddot{a}_{\overline{K+1}\rceil})}$, $j = 1, 2, 3, 4$.

Solution:

	$_jv^{2(k+1)}$				$(_j\ddot{a}_{\overline{k+1}\rceil})^2$			
j	$k=0$	$k=1$	$k=2$	$k=3$	$k=0$	$k=1$	$k=2$	$k=3$
1	0.8900	0.7921	0.7049	0.6274	1.0000	3.7768	8.0282	13.4910
2	0.8900	0.8389	0.7908	0.7454	1.0000	3.7768	8.1757	14.0517
3	0.8900	0.7491	0.6305	0.5307	1.0000	3.7768	7.8899	12.9811
4	0.8900	0.8389	0.7061	0.6656	1.0000	3.7768	8.1757	13.6871

	$\sqrt{\mathrm{Var}(_jv^{K+1})} = \sqrt{\mathrm{E}[_jv^{2(K+1)}] - (\mathrm{E}[_jv^{K+1}])^2}$	$\sqrt{\mathrm{Var}(_j\ddot{a}_{\overline{K+1}\rceil})} = \sqrt{\mathrm{E}[(_j\ddot{a}_{\overline{K+1}\rceil})^2] - (\mathrm{E}[_j\ddot{a}_{\overline{K+1}\rceil}])^2}$
1	$\sqrt{0.7099 - (0.8411)^2} = 0.0490$	$\sqrt{8.6602 - (2.8079)^2} = 0.8808$
2	$\sqrt{0.7921 - (0.8896)^2} = 0.0265$	$\sqrt{8.9286 - (2.8459)^2} = 0.9108$
3	$\sqrt{0.6402 - (0.7970)^2} = 0.0704$	$\sqrt{8.4147 - (2.7725)^2} = 0.8532$
4	$\sqrt{0.7348 - (0.8559)^2} = 0.0469$	$\sqrt{8.7828 - (2.8263)^2} = 0.8915$

▼

21.2.2 Random Scenarios: Deterministic Interest Rates

In building an interest rate model assume that the actuary has formulated m plausible interest scenarios. As a next step, a probability distribution on the m scenarios could be specified using methods for eliciting personal probabilities. The symbol $p(j)$ denotes the probability of scenario j. It is not the p.f. of claim amount as in Chapter 12.

The probability assignments should reflect the actuary's view of future investment returns. The probability elicitation process is closely related to the utility function elicitation process illustrated in Chapter 1.

Actuarial present values are defined using the joint distribution of the curtate future lifetime years (K) and interest scenario (J). The symbol J will be used in this chapter to denote the random variable interpreted as the index on the interest rate scenario. It is not the cause of decrement as in Chapter 10. We assume that K and J are independent. The asterisk presubscript has been added to actuarial present-value symbols to indicate that the expectation has been taken with respect to K and J:

$$
\begin{aligned}
{}_*A_x &= \mathrm{E}_J\, \mathrm{E}_{K|J}\, [{}_J v^{K+1}] \;=\; \mathrm{E}_J\, [{}_J A_x] \\
&= \sum_{j=1}^{m} {}_j A_x\, p(j), \qquad (21.2.2a) \\
{}_*\ddot{a}_x &= \mathrm{E}_J\, \mathrm{E}_{K|J}\, [{}_J \ddot{a}_{\overline{K+1}|}] = \mathrm{E}_J\, [{}_J \ddot{a}_x] \\
&= \sum_{j=1}^{m} {}_j \ddot{a}_x\, p(j). \qquad (21.2.2b)
\end{aligned}
$$

Continuing the same notational convention for the loss variable and premium for a fully discrete annual level premium whole life insurance, we have

$$L = {}_J v^{K+1} - {}_*P_x\, {}_J \ddot{a}_{\overline{K+1}|}.$$

Using the equivalence principle, we have

$$\mathrm{E}_J\, \mathrm{E}_{K|J}\, [L] = 0,$$

or

$${}_*P_x = \frac{{}_*A_x}{{}_*\ddot{a}_x},$$

and the variance of L is given by

$$\mathrm{Var}(L) = \sum_{j=1}^{m} \left[\sum_{k=0}^{\infty} ({}_j v^{k+1} - {}_*P_x\, {}_j \ddot{a}_{\overline{k+1}|})^2\, {}_k p_x\, q_{x+k} \right] p(j).$$

Example 21.2.3

Show that ${}_*A_x$, as given in (21.2.2a), can also be derived from $\mathrm{E}_K\, \mathrm{E}_{J|K}\, [{}_J v^{K+1}]$.

Solution:

Because of independence, levels of interest rates do not affect the distribution of mortality:

$$\mathrm{E}_K\, \mathrm{E}_{J|K}\, [{}_J v^{K+1}] = \mathrm{E}_K \left[\sum_{j=1}^{m} {}_j v^{K+1}\, p(j) \right]$$

$$= \sum_{k=0}^{\infty} \left[\sum_{j=1}^{m} {}_j v^{K+1}\, p(j) \right] {}_k p_x\, q_{x+k}$$

$$= \sum_{j=1}^{m} {}_j A_x\, p(j) = {}_* A_x. \qquad \blacktriangledown$$

Now assume that there is a set of n lives each age x. Each member of the set has been issued an identical fully discrete annual level premium whole life insurance, and the curtate future lifetime random variables K_i, $i = 1, 2, \ldots$ are identically distributed. In addition K_i, $i = 1, 2, \ldots,$ and J are mutually independent. The total loss random variable, if the benefit premium is determined by the equivalence principle, for the set of n insured is

$$\sum_{i=1}^{n} ({}_J v^{K_i+1} - {}_* P_x\, {}_J \ddot{a}_{\overline{K_i+1}|}).$$

The variance of total losses is given by

$$\mathrm{Var}\left[\sum_{i=1}^{n} ({}_J v^{K_i+1} - {}_* P_x\, {}_J \ddot{a}_{\overline{K_i+1}|}) \right]$$

$$= \sum_{i=1}^{n} \mathrm{Var}({}_J v^{K_i+1} - {}_* P_x\, {}_J \ddot{a}_{\overline{K_i+1}|})$$

$$+ n(n-1)\mathrm{Cov}({}_J v^{K_1+1} - {}_* P_x\, {}_J \ddot{a}_{\overline{K_1+1}|},\ {}_J v^{K_2+1} - {}_* P_x\, {}_J \ddot{a}_{\overline{K_2+1}|})$$

$$= n\, \mathrm{E}[({}_J v^{K+1} - {}_* P_x\, {}_J \ddot{a}_{\overline{K+1}|})^2]$$

$$+ n(n-1)\, \mathrm{E}[({}_J v^{K_1+1} - {}_* P_x\, {}_J \ddot{a}_{\overline{K_1+1}|})({}_J v^{K_2+1} - {}_* P_x\, {}_J \ddot{a}_{\overline{K_2+1}|})]$$

$$= n\, \mathrm{E}_J\, \mathrm{E}_{K|J}\, [({}_J v^{K+1} - {}_* P_x\, {}_J \ddot{a}_{\overline{K+1}|})^2]$$

$$+ n(n-1)\, \mathrm{E}_J\, \mathrm{E}_{K|J}\, [({}_J v^{K_1+1} - {}_* P_x\, {}_J \ddot{a}_{\overline{K_1+1}|})({}_J v^{K_2+1} - {}_* P_x\, {}_J \ddot{a}_{\overline{K_2+1}|})]$$

$$= n\, \mathrm{Var}({}_J v^{K+1} - {}_* P_x\, {}_J \ddot{a}_{\overline{K+1}|}) + n(n-1) \left[\sum_{j=1}^{m} ({}_j A_x - {}_* P_x\, {}_j \ddot{a}_x)^2\, p(j) \right]$$

$$= n \sum_{j=1}^{m} \left[\sum_{k=0}^{\infty} ({}_j v^{k+1} - {}_* P_x\, \ddot{a}_{\overline{k+1}|})^2\, {}_k p_x\, q_{x+k} + (n-1)\, ({}_j A_x - {}_* P_x\, {}_j \ddot{a}_x)^2 \right] p(j).$$

If we are interested in the average loss, rather than total loss, in our risk portfolio of n identical policies, we have

$$\mathrm{Var}\left[\frac{\sum_{i=1}^{n} ({}_J v^{K_i+1} - {}_* P_x\, {}_J \ddot{a}_{\overline{K_i+1}|})}{n} \right] =$$

$$\frac{\text{Var}({}_Jv^{K+1} - {}_*P_x\, {}_J\ddot{a}_{\overline{K+1}|})}{n} + \left(1 - \frac{1}{n}\right) \sum_{j=1}^{m} ({}_jA_x - {}_*P_x\, {}_j\ddot{a}_x)^2\, p(j). \qquad (21.2.3)$$

As $n \to \infty$, the first term of (21.2.3) approaches zero, whereas the second term remains positive. The n components of the sum of loss variables that make up total losses are not independent. Increasing the number of independent insureds, in this more comprehensive model with its random interest scenarios, does not make the variance of average loss approach zero as it did when the future lifetime random variables were assumed to be mutually independent and the interest scenario was deterministic. This can be appreciated by observing that the summation in the second term of (21.2.3) becomes zero when there is a single deterministic interest scenario. The fact that all losses are subject to the same randomly determined interest scenario has recognized a risk management problem.

Example 21.2.4

Use the assumptions of Examples 21.2.1 and 21.2.2, assume that the actuary has made the following probability assignments to the interest rate scenarios, $p(1) = 0.5$, $p(2) = 0.2$, $p(3) = 0.2$, and $p(4) = 0.1$ and compute ${}_*P_x$ and $\text{Var}({}_Jv^{K+1} - {}_*P_x\, \ddot{a}_{\overline{K+1}|})$.

Solution:

Using the equivalence principle we have

$$\text{E}_J\, \text{E}_{K|J}\, [{}_Jv^{K+1} - {}_*P_x\, {}_J\ddot{a}_{\overline{K+1}|}] = 0,$$

$$({}_1A_x - {}_*P_x\, {}_1\ddot{a}_x)p(1) + ({}_2A_x - {}_*P_x\, {}_2\ddot{a}_x)p(2)$$

$$+ ({}_3A_x - {}_*P_x\, {}_3\ddot{a}_x)p(3) + ({}_4A_x - {}_*P_x\, {}_4\ddot{a}_x)p(4) = 0,$$

$${}_*P_x = \frac{{}_*A_x}{{}_*\ddot{a}_x} = \frac{0.8435}{2.8103} = 0.3001,$$

and

$$\text{Var}({}_Jv^{K+1} - {}_*P_x\, {}_J\ddot{a}_{\overline{K+1}|}) = \text{E}_J\, \text{E}_{K|J}\, [({}_Jv^{K+1} - 0.3001\, {}_J\ddot{a}_{\overline{K+1}|})^2]$$

$$= (0.0987)(0.5) + (0.0912)(0.2) + (0.1078)(0.2) + (0.0983)(0.1) = 0.0990. \quad \blacktriangledown$$

21.3 Independent Interest Rates

In Section 21.2.2 random interest scenarios or paths were introduced. The probability assignments were made using the economic knowledge of the actuary. We consider now a stochastic model for interest rates in which the selection of the model and the estimates of the model parameters have been influenced by data. Suppose that the actuary has decided to model the forces of interest and has adopted the model

$$\log(1 + I_k) = \delta + \epsilon_k \quad k = 1, 2, 3, \ldots \tag{21.3.1}$$

where δ is a non-negative constant and the ϵ_k are independent and identically distributed random variables with $N(0, \sigma^2)$ distributions. This model can be viewed as a long-term mean force of interest subject to random shocks. Because of the assumptions made about the distribution of the random shock terms, negative forces of interest are possible. Some actuaries view this possibility as invalidating the model in (21.3.1). Other actuaries adopt the model because it seems natural to model forces of interest, and negative values are observed in investment operations. Then the random variables $\log(1 + I_k)$ have identical $N(\delta, \sigma^2)$ distributions, and the $(1 + I_k)$ random variables have lognormal distributions.

The lognormal distribution was introduced in Table 14.2.1 as a claim amount distribution. From that table we recall that

$$E[1 + I_k] = \exp\left(\delta + \frac{\sigma^2}{2}\right) \geq 1$$

and

$$\mathrm{Var}(1 + I_k) = (e^{\sigma^2} - 1)\exp(2\delta + \sigma^2) \geq 0.$$

The logarithm of the random variable version of the deterministic interest accumulation function $(1 + i)^n$ is the random variable

$$\log \prod_{k=1}^{n} (1 + I_k) = \sum_{k=1}^{n} \log(1 + I_k).$$

Using (21.3.1), this random variable has a $N(n\delta, n\sigma^2)$ distribution. As a consequence, the interest accumulation function has a lognormal distribution with

$$E\left[\prod_{k=1}^{n} (1 + I_k)\right] = e^{n(\delta+\sigma^2/2)}$$

and

$$\mathrm{Var}\left[\prod_{k=1}^{n} (1 + I_k)\right] = (e^{\sigma^2} - 1)e^{n(2\delta+\sigma^2)}.$$

It is instructive to observe that if $\sigma^2 = 0$, the expected interest accumulation is $e^{n\delta}$, and its variance is zero.

The logarithm of the discount factor

$$\log(1 + I_k)^{-1} = -\log(1 + I_k) = -\delta - \epsilon_k$$

has a $N(-\delta, \sigma^2)$ distribution, and $(1 + I_k)^{-1}$ has a lognormal distribution with

$$E[(1 + I_k)^{-1}] = e^{-(\delta-\sigma^2/2)} > 0$$

and

$$\mathrm{Var}[(1 + I_k)^{-1}] = (e^{\sigma^2} - 1)(e^{-2\delta+\sigma^2}) \geq 0.$$

We define the discount function as the random variable

$$\tilde{v}_n = \prod_{k=1}^{n} (1 + I_k)^{-1}$$

and

$$\tilde{v}_0 = 1.$$

The choice of the symbol $\tilde{v}_n$ is motivated by the use of v_n for the discount factor in Section 4.3. The tilde distinguishes a random variable, $\tilde{v}_n$. Then $\log \tilde{v}_n = -\Sigma_{k=1}^{n} \log (1 + I_k)$ has a $N(-n\delta, n\sigma^2)$ distribution, and $\tilde{v}_n$ has a lognormal distribution with

$$\mathrm{E}[\tilde{v}_n] = e^{-n(\delta-\sigma^2/2)}$$

and

$$\mathrm{Var}(\tilde{v}_n) = (e^{n\sigma^2} - 1)(e^{n(-2\delta+\sigma^2)}). \tag{21.3.2}$$

Once again, if $\sigma^2 = 0$, the deterministic results are recaptured.

We will assume that I_k, $k = 1, 2, 3, \ldots$ and K, curtate future lifetime, are mutually independent and consider comprehensive actuarial models. The actuarial present values of a unit benefit life insurance is

$$\begin{aligned} {}_*A_x &= \mathrm{E}[\tilde{v}_{K+1}] \\ &= \mathrm{E}_{\tilde{v}}\, \mathrm{E}_{K|\tilde{v}}\, [\tilde{v}_{K+1}] \\ &= \mathrm{E}_{\tilde{v}} \left[\sum_{k=0}^{\infty} \tilde{v}_{K+1}\ {}_kp_x\ q_{x+k} \right] \\ &= \sum_{k=0}^{\infty} e^{-(\delta-\sigma^2/2)(k+1)}\ {}_kp_x\ q_{x+k}. \end{aligned} \tag{21.3.3}$$

The actuarial present value ${}_*A_x$ is calculated at deterministic force of interest $\delta - (\sigma^2/2)$. To measure risk, we determine

$$\begin{aligned} \mathrm{Var}(\tilde{v}_{K+1}) &= \mathrm{E}[(\tilde{v}_{K+1})^2] - ({}_*A_x)^2 \\ &= \mathrm{E}_{\tilde{v}}\, \mathrm{E}_{K|\tilde{v}}\, [(\tilde{v}_{K+1})^2] - ({}_*A_x)^2 \\ &= \mathrm{E}_{\tilde{v}} \left[\sum_{k=0}^{\infty} (\tilde{v}_{k+1})^2\ {}_kp_x\ q_{x+k} \right] - ({}_*A_x)^2. \end{aligned}$$

Since $\tilde{v}_{k+1}$ has a lognormal distribution with parameters $-(k + 1)\delta$ and $(k + 1)\sigma^2$, we can calculate $\mathrm{E}[(\tilde{v}_{k+1})^2]$, the second moment about the origin, by

$$\begin{aligned} \mathrm{E}[(\tilde{v}_{k+1})^2] &= \mathrm{Var}(\tilde{v}_{k+1}) + (\mathrm{E}[\tilde{v}_{k+1}])^2 \\ &= (e^{(k+1)\sigma^2} - 1)(e^{(k+1)(-2\delta+\sigma^2)}) + e^{-2(k+1)(\delta-\sigma^2/2)} \\ &= e^{-(k+1)[2(\delta-\sigma^2)]}, \end{aligned} \tag{21.3.4}$$

and then

$$\mathrm{Var}(\tilde{v}_{K+1}) = \sum_{k=0}^{\infty} e^{-(k+1)[2(\delta-\sigma^2)]} \, {}_kp_x \, q_{x+k} - ({}_*A_x)^2.$$

We can see that when $\sigma^2 = 0$, the results of Chapter 4 are reproduced.

The formulas for the actuarial present values of annuities require more extensive developments. In the development, we continue to assume that I_k, $k = 1, 2, \ldots,$ and K are mutually independent.

We define

$$\ddot{a}_{\overline{K}|\tilde{v}} = \sum_{s=0}^{K-1} \tilde{v}_s;$$

then

$$\begin{aligned} \mathrm{E}_{\tilde{v}|K}\,[\ddot{a}_{\overline{K+1}|\tilde{v}}] &= \sum_{s=0}^{K} e^{s(-\delta+\sigma^2/2)} \\ &= \ddot{a}_{\overline{K+1}|\delta-\sigma^2/2}, \end{aligned}$$

and

$$\begin{aligned} {}_*\ddot{a}_x &= \mathrm{E}[\ddot{a}_{\overline{K+1}|\tilde{v}}] = \mathrm{E}_K\,\mathrm{E}_{\tilde{v}|K}\,[\ddot{a}_{\overline{K+1}|\tilde{v}}] \\ &= \mathrm{E}_K\,[\ddot{a}_{\overline{K+1}|\delta-\sigma^2/2}] \\ &= \sum_{k=0}^{\infty} (\ddot{a}_{\overline{k+1}|\delta-\sigma^2/2}) \, {}_kp_x \, q_{x+k}. \end{aligned} \tag{21.3.5}$$

To evaluate $\mathrm{Var}(\ddot{a}_{\overline{K+1}|\tilde{v}})$ we start with

$$\begin{aligned} \mathrm{E}_{\tilde{v}}\left[\left(\sum_{s=0}^{k} \tilde{v}_s\right)^2\right] &= \mathrm{E}_{\tilde{v}}\,[(1 + \tilde{v}_1 + \tilde{v}_2 + \cdots + \tilde{v}_k)^2] \\ &= \mathrm{E}_{\tilde{v}}\left[\sum_{s=0}^{k} (\tilde{v}_s)^2 + 2\sum_{s=0}^{k-1}\sum_{r=s+1}^{k} \tilde{v}_s\,\tilde{v}_r\right] \\ &= \sum_{s=0}^{k} e^{-s[2(\delta-\sigma^2)]} + 2\sum_{s=0}^{k-1}\sum_{r=s+1}^{k} \mathrm{E}[\tilde{v}_s\,\tilde{v}_r]. \end{aligned}$$

Terms of the form $\mathrm{E}[(\tilde{v}_s)^2]$ were evaluated using (21.3.4).

We now examine the terms in the double summation and find that

$$\begin{aligned} \mathrm{E}[\tilde{v}_s\,\tilde{v}_r] &= \mathrm{E}[(1 + I_1)^{-2} \cdots (1 + I_s)^{-2}(1 + I_{s+1})^{-1} \cdots (1 + I_r)^{-1}] \quad s < r \\ &= e^{-s[2(\delta-\sigma^2)]-(r-s)(\delta-\sigma^2/2)}. \end{aligned} \tag{21.3.6}$$

The independence of the random variables I_k, $k = 1, 2, \ldots$ and (21.3.2) and (21.3.4) have been used to complete (21.3.6). As a result, we have

$$\mathrm{E}_{\tilde{v}}\left[\left(\sum_{s=0}^{k} \tilde{v}_s\right)^2\right] = \sum_{s=0}^{k} e^{-s[(2(\delta-\sigma^2)]} + 2\sum_{s=0}^{k-1}\sum_{r=s+1}^{k} e^{-s[2(\delta-\sigma^2)]-(r-s)(\delta-\sigma^2/2)}. \tag{21.3.7}$$

The double summation in (21.3.7) can be simplified by interchanging the order of summation and using the formula for the summation of a geometric series. We have

$$2 \sum_{r=1}^{k} e^{-r(\delta-\sigma^2/2)} \sum_{s=0}^{r-1} e^{-s[\delta-(3\sigma^2/2)]} = 2 \sum_{r=1}^{k} e^{-r(\delta-\sigma^2/2)} \left(\frac{1 - e^{-r[\delta-(3\sigma^2/2)]}}{1 - e^{-[\delta-(3\sigma^2/2)]}} \right). \tag{21.3.8}$$

Note that the summation in (21.3.8) can start with $r = 0$ because the summand is then zero.

Combining the intermediate results in (21.3.7) and (21.3.8), we have

$$\begin{aligned} \mathrm{Var}(\ddot{a}_{\overline{K+1}|\tilde{v}}) &= \mathrm{E}_K\, \mathrm{E}_{\tilde{v}|K}\, [(\ddot{a}_{\overline{K+1}|\tilde{v}})^2] - ({}_*\ddot{a}_x)^2 \\ &= \mathrm{E}_K \left[\sum_{s=0}^{K} e^{-s[2(\delta-\sigma^2)]} + 2 \sum_{r=0}^{K} \left(\frac{e^{-r(\delta-\sigma^2/2)} - e^{-r(2\delta-2\sigma^2)}}{1 - e^{-[\delta-(3\sigma^2/2)]}} \right) \right] - ({}_*\ddot{a}_x)^2 \\ &= {}^{\alpha}\ddot{a}_x + 2\, \frac{{}_*\ddot{a}_x - {}^{\alpha}\ddot{a}_x}{1 - e^{-[\delta-(3\sigma^2/2)]}} - ({}_*\ddot{a}_x)^2 \end{aligned} \tag{21.3.9}$$

where ${}^{\alpha}\ddot{a}_x$ is evaluated at force of interest $2(\delta - \sigma^2)$.

If $\sigma^2 = 0$, (21.3.9) reduces to

$${}^2\ddot{a}_x + \frac{2(\ddot{a}_x - {}^2a_x)}{d} - (\ddot{a}_x)^2,$$

where ${}^2\ddot{a}_x$ is valued at force of interest 2δ and $\ddot{a}_x$ is valued at force of interest δ.

This result may be compared with (5.3.8) where

$$\begin{aligned} \mathrm{Var}(\ddot{a}_{\overline{K+1}|}) &= \frac{{}^2A_x - (A_x)^2}{d^2} \\ &= \frac{1 - (2d - d^2)\, {}^2\ddot{a}_x - 1 + 2\, d\ddot{a}_x - d^2\, \ddot{a}_x^2}{d^2} \\ &= {}^2\ddot{a}_x + 2\, \frac{(\ddot{a}_x - {}^2\ddot{a}_x)}{d} - (\ddot{a}_x)^2, \end{aligned}$$

confirming again that when $\sigma^2 = 0$, deterministic interest results are recovered.

Example 21.3.1

Assume that $\log(1 + I_k) = \delta + \epsilon_k$, $k = 1, 2, 3 \ldots$ where $\delta = 0.06$, and the random shock terms have a $N(0, 0.0001)$ distribution. The curtate future lifetime random variable has the discrete distribution shown in Example 21.2.1. Calculate (a) $\mathrm{E}[\tilde{v}_{K+1}]$, (b) $\mathrm{Var}(\tilde{v}_{K+1})$, (c) $\mathrm{E}[\ddot{a}_{\overline{K+1}|\tilde{v}}]$, and (d) $\mathrm{Var}(\ddot{a}_{\overline{K+1}|\tilde{v}})$. Assume that K and I_k, $k = 1, 2, 3 \ldots$, are independent.

Solution:

Formula numbers refer to those displayed in this section.

a. $\mathrm{E}[\tilde{v}_{K+1}] = \sum_{k=0}^{3} e^{-(0.06-0.00005)(k+1)} \; {}_kp_x \, q_{x+k}$

$= (0.9418)(0.1) + (0.8870)(0.2) + (0.8354)(0.3) + (0.7868)(0.4)$

$= 0.8369.$ by (21.3.3)

b. $\mathrm{Var}(\tilde{v}_{K+1}) = \sum_{k=0}^{3} e^{-[2(0.06-0.0001)](k+1)} \; {}_kp_x \, q_{x+k} - ({}_*A_x)^2$

$= (0.8871)(0.1) + (0.7869)(0.2) + (0.6981)(0.3) + (0.6193)(0.4) - (0.8369)^2$

$= 0.0028.$ by (21.3.4)

c. $\mathrm{E}[\ddot{a}_{\overline{K+1}|\tilde{v}}] = \sum_{k=0}^{3} \sum_{s=0}^{k} e^{-s(0.06-0.00005)} \; {}_kp_x \, q_{x+k}$

$= (1)(0.1) + (1.9418)(0.2) + (2.8288)(0.3) + (3.6642)(0.4)$

$= 2.8027.$

d. An input into the calculation of $\mathrm{Var}(\ddot{a}_{\overline{K+1}|\tilde{v}})$ is ${}^{\alpha}\ddot{a}_x$, valued at force of interest $2(\delta - \sigma^2) = 0.1198$:

$${}^{\alpha}\ddot{a}_x = (1)(0.1) + (1.8871)(0.2) + (2.6740)(0.3) + (3.3721)(0.4)$$

$$= 2.6285,$$

$$\mathrm{Var}(\ddot{a}_{\overline{K+1}|\tilde{v}}) = 2.6285 + 2\,\frac{(2.8027 - 2.6285)}{1 - e^{-0.05985}} - (2.8027)^2$$

$$= 8.6257 - 7.8551 = 0.7705.$$ by (21.3.9)

▼

Example 21.3.2

Adopt the assumptions about the distributions of $\log(1 + I_K)$ and K used in Example 21.3.1. Display the d.f. of $\tilde{v}_{K+1}$.

Solution:

$$\Pr(\tilde{v}_{K+1} \le y) = \mathrm{E}_K \Pr(\tilde{v}_{K+1} \le y | K = k)$$

$$= \sum_{k=0}^{3} \Pr(\tilde{v}_{K+1} \le y) \; {}_kp_x \, q_{x+k}.$$

We now use the fact that $\log \tilde{v}_n$ has a $N(-n\delta, n\sigma^2)$ distribution:

$$\Pr(\tilde{v}_{K+1} \le y) = \sum_{k=0}^{3} \Pr(\log \tilde{v}_{K+1} \le \log y) \; {}_kp_x \, q_{x+k}$$

$$= \sum_{k=0}^{3} \Pr\left[\frac{\log \tilde{v}_{K+1} + (k+1)\delta}{\sqrt{(k+1)\sigma^2}} \le \frac{\log y + (k+1)\delta}{\sqrt{(k+1)\sigma^2}}\right] {}_kp_x \, q_{x+k}$$

$$= \sum_{k=0}^{3} \Phi \left[\frac{\log y + (k+1)\delta}{\sqrt{(k+1)\sigma^2}} \right] {}_kp_x \, q_{x+k}$$

$$= \sum_{k=0}^{3} \Phi \left[\frac{\log y + (k+1)(0.06)}{\sqrt{(k+1)(0.01)}} \right] {}_kp_x \, q_{x+k},$$

where $\Phi(w)$ is the d.f. of a random variable with a $N(0, 1)$ distribution. To illustrate, let $y = \mathrm{E}[\tilde{v}_{K+1}] = {}_*A_x = 0.8369$ and calculate $\Pr(\tilde{v}_{K+1} \le 0.8369) = \Phi(-11.8051)(0.1) + \Phi(-4.1048)(0.2) + \Phi(0.1125)(0.3) + \Phi(3.0945)(0.4) = 0.5630$. Because the median of the distribution of $\tilde{v}_{K+1}$ is less than the mean, we have evidence that the distribution is skewed to the right. ▼

21.4 Dependent Interest Rates

Within financial economics there has been a continuing discussion about whether effective rates of interest within various classes of investments can be modeled as independent and identically distributed random variables. If the actuary accepts the evidence supporting the independent and identically distributed hypothesis, the methods developed in Section 21.3 are available. The actuary could alter these methods. For example, the distribution of the ϵ_k random shock terms might be assumed to be other than the $N(0, \sigma^2)$ distribution.

If the actuary rejects the hypothesis that effective rates are independent and identically distributed, two options are open. The first option is to develop a multivariate model that does not change as time passes. Such models are called stationary models. A simple model within this class will be developed in Section 21.4.1.

The second option is to adopt a model that incorporates the possibility of structural shifts in the investment environment. We do not discuss this type of model.

21.4.1 Moving Average Model

The developments of this section are limited to the model

$$\log(1 + I_k) = \delta + \epsilon_k - \theta\epsilon_{k-1} \quad k = 1, 2, 3, \ldots \tag{21.4.1}$$

where $\delta > 0$, and ϵ_k, $k = 1, 2, \ldots$, are random variables that are mutually independent and each has a $N(0, \sigma^2)$ distribution. In addition, $|\theta| \le 1$ and ϵ_0 is known. If $\theta = 0$, the model reduces to (21.3.1). This model is called a ***moving average model of order one***, abbreviated MA(1).

The rationale for the model is that the force of interest has a long-term mean, denoted by δ, but random economic shocks create deviations from the mean. The shock for period k, ϵ_k, has a delayed and moderated impact on the force of interest in period $k + 1$ of size $-\theta\epsilon_k$.

We define $\tilde{v}_n$ as in Section 21.3:

$$\tilde{v}_n = \prod_{k=1}^{n} (1 + I_k)^{-1} = e^{-\Sigma_1^n(\delta+\epsilon_k-\theta\epsilon_{k-1})};$$

then

$$\begin{aligned} \log \tilde{v}_n &= -\sum_{k=1}^{n} (\delta + \epsilon_k - \theta\epsilon_{k-1}) \\ &= -n\delta + \epsilon_n - \theta\epsilon_0 + (1 - \theta) \sum_{k=1}^{n-1} \epsilon_k, \end{aligned}$$

and

$$\mathrm{E}[\tilde{v}_n] = \mathrm{E}[e^{-[n\delta+\epsilon_n-\theta\epsilon_0+(1-\theta)\Sigma_{k=1}^{n-1}\epsilon_k]}].$$

We have assumed that the shock terms ϵ_k, $k = 1, 2, \ldots n$, are mutually independent and each has a $N(0, \sigma^2)$ distribution. Therefore, recalling the m.g.f. of a $N(0, \sigma^2)$ distribution, we have

$$\begin{aligned} \mathrm{E}[e^{t\epsilon_k}] &= e^{\,t^2\sigma^2/2} \quad k = 1, 2, 3, \ldots. \\ &= M(t). \end{aligned}$$

This result enables us to write

$$\begin{aligned} \mathrm{E}[\tilde{v}_n] &= e^{-n\delta}\, M(-1)\, e^{\theta\epsilon_0}\, M(\theta - 1)^{n-1} \\ &= C_1 e^{-n\delta'} \quad n = 1, 2, 3, \ldots, \end{aligned} \tag{21.4.2}$$

where $C_1 = M(-1)\, e^{\theta\epsilon_0}\, M(\theta - 1)^{-1}$ and $\delta' = \delta - \log M(\theta - 1)$. Note that, as in Section 21.3, we define $\tilde{v}_0 = 1$ and $\mathrm{E}[\tilde{v}_0] = 1$, not C_1. If $\theta = 0$, then $\delta' = \delta - \log M(-1)$, $C_1 = 1$, and $\mathrm{E}[\tilde{v}_n] = e^{-n[\delta-\log M(-1)]} = e^{-n(\delta-\sigma^2/2)}$, which agrees with (21.3.2), for the lognormal independent model.

With these preliminary results, we can calculate actuarial present values. The initial development will follow that used in (21.3.3):

$$\begin{aligned} {}_*A_x &= \mathrm{E}[\tilde{v}_{K+1}] = \mathrm{E}_{\tilde{v}}\, \mathrm{E}_{K|\tilde{v}}\, [\tilde{v}_{K+1}] \\ &= \mathrm{E}_{\tilde{v}} \left[\sum_{k=0}^{\infty} \tilde{v}_{k+1}\, {}_kp_x\, q_{x+k} \right] \\ &= C_1 \sum_{k=0}^{\infty} e^{-(k+1)\delta'}\, {}_kp_x\, q_{x+k}. \end{aligned} \tag{21.4.3}$$

Similarly,

$$\begin{aligned} {}_*\ddot{a}_x &= \mathrm{E}[\ddot{a}_{\overline{K+1}|\tilde{v}}] = \mathrm{E}_{\tilde{v}}\, \mathrm{E}_{K|\tilde{v}}\, [\ddot{a}_{\overline{K+1}|\tilde{v}}] \\ &= \mathrm{E}_{\tilde{v}} \left[\sum_{k=0}^{\infty} \left(1 + \sum_{s=1}^{k} \tilde{v}_s \right) {}_kp_x\, q_{x+k} \right] \end{aligned}$$

$$= \sum_{k=0}^{\infty} \left(1 + \sum_{s=1}^{k} C_1\, e^{-s\delta'}\right) {}_kp_x\, q_{x+k}$$

$$= \sum_{k=0}^{\infty} (1 + C_1 a_{\overline{k}|\delta'})\, {}_kp_x\, q_{x+k}. \tag{21.4.4}$$

To continue the outline used in Section 21.3, we must develop formulas for $\mathrm{E}[\tilde{v}_r\, \tilde{v}_s]$, $s < r$:

$$\begin{aligned} \mathrm{E}[(\tilde{v}_n)^2] &= \mathrm{E}[e^{-2[n\delta+\epsilon_n-\theta\epsilon_0+(1-\theta)\Sigma_{k=1}^{n-1}\epsilon_k]}] \\ &= e^{-2n\delta}\, \mathrm{E}[e^{-2\epsilon_n}]\, \mathrm{E}[e^{2\theta\epsilon_0}]\, \mathrm{E}[e^{-2(1-\theta)\epsilon}]^{n-1} \\ &= e^{-2n\delta}\, M(-2)\, e^{2\theta\epsilon_0}\, M(2\theta - 2)^{n-1}. \end{aligned}$$

We use abbreviated notation

$$\mathrm{E}[(\tilde{v}_n)^2] = C_2\, e^{-\delta'' n}, \tag{21.4.5}$$

where $\delta'' = 2\delta - \log M(2\theta - 2)$ and

$$C_2 = \frac{M(-2)\, e^{2\theta\epsilon_0}}{M(2\theta - 2)}.$$

The term $\mathrm{E}[\tilde{v}_r\, \tilde{v}_s]$ appears in expressions for $\mathrm{Var}(\ddot{a}_{\overline{K+1}|\tilde{v}})$. We note that $\mathrm{E}[\tilde{v}_r\, \tilde{v}_0] = \mathrm{E}[\tilde{v}_r]$. If $r > s \geq 1$, we have

$$\begin{aligned} \mathrm{E}[\tilde{v}_r\, \tilde{v}_s] &= \mathrm{E}[e^{-[r\delta+\epsilon_r-\theta\epsilon_0+(1-\theta)\Sigma_{j=1}^{r-1}\epsilon_j]} \times e^{-[s\delta+\epsilon_s-\theta\epsilon_0+(1-\theta)\Sigma_{j=1}^{s-1}\epsilon_j]}] \\ &= e^{-(r+s)\delta}\, \mathrm{E}[e^{-\epsilon_r}]\mathrm{E}[e^{(\theta-2)\epsilon_s}]\mathrm{E}[e^{2\theta\epsilon_0}]\mathrm{E}[e^{-2(1-\theta)\Sigma_{j=1}^{s-1}\epsilon_j}]\mathrm{E}[e^{-(1-\theta)\Sigma_{j=s+1}^{r-1}\epsilon_j}] \\ &= e^{-(r+s)\delta}\, M(-1)M(\theta - 2)e^{2\theta\epsilon_0}M[2(\theta - 1)]^{s-1}M(\theta - 1)^{r-s-1} \\ &= C_3\, e^{-\delta'' s}\, e^{-\delta'(r-s)} \end{aligned}$$

where δ' and δ'' are as defined previously and

$$C_3 = M(-1)\, M(\theta - 2)e^{2\theta\epsilon_0}\{M[2(\theta - 1)]^{-1}\, M(\theta - 1)^{-1}\}.$$

Using these building blocks, we turn to the development of formulas for the variances of present-value random variables in which the force of interest has a MA(1) model:

$$\begin{aligned} \mathrm{Var}(\tilde{v}_{K+1}) &= \mathrm{E}[(\tilde{v}_{K+1})^2] - ({}_*A_x)^2 \\ &= \sum_{k=0}^{\infty} C_2 e^{-\delta''(k+1)}\, {}_kp_x\, q_{x+k} - ({}_*A_x)^2, \end{aligned} \tag{21.4.6}$$

where (21.4.3) and (21.4.5) have been used in the development.

For $\mathrm{Var}(\ddot{a}_{\overline{K+1}|\tilde{v}})$, we start with

$$\begin{aligned} \mathrm{E}_{\tilde{v}}\left[\left(\sum_{s=0}^{k} \tilde{v}_s\right)^2\right] &= \mathrm{E}_{\tilde{v}}\left[\left(1 + \sum_{s=1}^{k} \tilde{v}_y^2\right) + 2\sum_{s=0}^{k-1}\sum_{r=s+1}^{k} \tilde{v}_r\, \tilde{v}_s\right] \\ &= \left(1 + \sum_{s=1}^{k} C_2\, e^{-\delta'' s}\right) + 2\left(\sum_{r=1}^{k} C_1 e^{-r\delta'} + \sum_{s=1}^{k-1}\sum_{r=s+1}^{k} C_3\, e^{-\delta'' s-\delta'(r-s)}\right). \end{aligned} \tag{21.4.7}$$

Therefore,

$$\mathrm{Var}(\ddot{a}_{\overline{K+1}|\tilde{v}}) = \sum_{k=0}^{\infty} \mathrm{E}_{\tilde{v}} \left[\left(1 + \sum_{s=1}^{k} \tilde{v}_s \right)^2 \right] {}_kp_x \, q_{x+k} - ({}_*\ddot{a}_x)^2, \tag{21.4.8}$$

where components come from (21.4.7) and (21.4.4).

Example 21.4.1

Assume that $\log(1 + I_k)$ is given by (21.4.1) with $\delta = 0.06$, $\mathrm{Var}(\epsilon_k) = 0.0001$, $\theta = -0.8$, and $\epsilon_0 = 0$. Calculate (a) δ', (b) C_1, (c) δ'', (d) C_2, and (e) C_3.

Solution:

a. $\delta' = \delta - \log M(\theta - 1)$

$= 0.06 - \log(e^{(-0.8-1)^2(0.0001)/2})$

$= 0.05984.$

b. $C_1 = \dfrac{M(-1)\, e^{\theta\epsilon_0}}{M(\theta - 1)} = e^{(0.0001)/2}\, e^{-(1.8)^2(0.0001)/2}$

$= 0.99989.$

c. $\delta'' = 2\delta - \log M[2(\theta - 1)] = 0.12 - \log(e^{[2(-1.8)]^2(0.0001)/2})$

$= 0.119352.$

d. $C_2 = \dfrac{M(-2)e^{2\theta\epsilon_0}}{M(2\theta - 2)} = \dfrac{e^{(-2)^2(0.0001)/2}}{e^{(-3.6)^2(0.0001)/2}}$

$= 0.99955.$

e. $C_3 = \dfrac{M(-1)\, M(\theta - 2)\, e^{2\theta\epsilon_0}}{M(2\theta - 2)\, M(\theta - 1)}$

$= \dfrac{(e^{(0.0001)/2})(e^{(-2.8)^2(0.0001)/2})}{(e^{(-3.6)^2(0.0001)/2})(e^{(-1.8)^2(0.0001)/2})}$

$= 0.99963.$ ▼

Example 21.4.2

Assume the interest rate model of Example 21.4.1 and the survival distribution of Example 21.2.1. Determine (a) $\mathrm{E}[\tilde{v}_{K+1}]$, (b) $\mathrm{Var}(\tilde{v}_{K+1})$, (c) $\mathrm{E}[\ddot{a}_{\overline{K+1}|\tilde{v}}]$, and (d) $\mathrm{Var}(\ddot{a}_{\overline{K+1}|\tilde{v}})$.

Solution:

Formula numbers refer to those displayed in this section.

a. $E[\tilde{v}_{K+1}] = E_K \, E_{\tilde{v}|K} \, [\tilde{v}_{K+1}]$ by (21.4.3)

$$= \sum_{k=0}^{\infty} C_1 \, e^{-(k+1)\delta'} \, {}_kp_x \, q_{x+k}$$

$$= (0.99989) \sum_{k=0}^{3} e^{-(k+1)(0.05984)} \, {}_kp_x \, q_{x+k}$$

$$= (0.99989)[(0.94192)(0.1) + (0.88720)(0.2) + (0.83567)(0.3) + (0.78713)(0.4)] = 0.83710.$$

b. $\mathrm{Var}(\tilde{v}_{K+1}) = \sum_{k=0}^{3} C_2 \, e^{-\delta''(k+1)} \, {}_kp_x \, q_{x+k} - ({}_*A_x)^2$ by (21.4.6)

$$= (0.99955)[(0.88721)(0.1) + (0.78714)(0.2) + (0.69836)(0.3) + (0.61959)(0.4)] - (0.83710)^2$$

$$= 0.0024.$$

c. $E[\ddot{a}_{\overline{K+1}|\tilde{v}}] = E_{\tilde{v}} \, E_{K|\tilde{v}} \, [\ddot{a}_{\overline{K+1}|\tilde{v}}]$ by (21.4.4)

$$= \sum_{k=0}^{3} \left(1 + \sum_{s=1}^{k} C_1 \, e^{-s\delta'}\right) {}_kp_x \, q_{x+k}$$

$$= (1)(0.1) + (1.94181)(0.2) + (2.82892)(0.3) + (3.66450)(0.4)$$

$$= 2.80284.$$

d. As a preliminary, we compute $E_{\tilde{v}} \, [(1 + \Sigma_{s=1}^{k} \, \tilde{v}_s)^2]$ for $k = 0, 1, 2, 3$ using (21.4.7).

The computations are summarized in the following table.

k	$E_{\tilde{v}} \, [(1 + \Sigma_{s=1}^{k} \, \tilde{v}_s)^2]$	
0	1	= 1
1	$[1 + C_2 \, e^{-\delta''} + 2C_1 \, e^{-\delta'}]$	= 3.77043
2	$[1 + C_2(e^{-\delta''} + e^{-2\delta''}) + 2(C_1 \, e^{-\delta'} + C_1 e^{-2\delta'} + C_3 \, e^{-\delta''-\delta'})]$	= 8.00214
3	$[1 + C_2(e^{-\delta''} + e^{-2\delta''} + e^{-3\delta''}) + 2 \, (C_1 e^{-\delta'} + e^{-2\delta'} + e^{-3\delta'}) + C_3(e^{-\delta''-\delta'} + e^{-\delta''-2\delta'}) + C_3(e^{-2\delta''-\delta''})]$	= 13.42729

$$\mathrm{Var}[\ddot{a}_{\overline{K+1}|\tilde{v}}] = (1)(0.1) + (3.77043)(0.2) + (8.00214)(0.3) + (13.42729)(0.4) - (2.80284)^2 = 0.7699.$$

▼

21.4.2 Implementation

Sections 21.3 and 21.4.1 illustrate that the development of formulas for the moments of present-value random variables, when interest rates are assumed to be random variables, can involve several steps. Other statistical models for $X_k = \log(1 + I_k)$, in the same class of models, such as

a. Autoregressive of order one AR(1)

$$(X_k - \delta) = \phi(X_{k-1} - \delta) + \epsilon_k \tag{21.4.9a}$$

b. AR(1) and MA(1)

$$(X_k - \delta) - \phi(X_{k-1} - \delta) = \epsilon_k - \theta\epsilon_{k-1} \tag{21.4.9b}$$

c. AR(1) on first differences

$$(X_k - X_{k-1}) - \phi(X_{k-1} - X_{k-2}) = \epsilon_k \tag{21.4.9c}$$

could be the subject of similar developments. The selection of an appropriate model for the force of interest and the estimation of the parameter are topics in statistics.

The problem of displaying the d.f. of present-value random variables remains. The technique used in Sections 4.2, 5.2, 6.2, and 7.2 are not feasible when K and $\tilde{v}_k$, the discrete future lifetime random variable and the discount factors, have a joint distribution.

In earlier chapters, where only future lifetimes were random variables, approximations were developed for the distribution of losses from a portfolio of risks. In these developments it was assumed that the future lifetime random variables are mutually independent. Typically these developments depended on a central limit theorem type of result to justify using an approximating normal distribution. When each of the component present-value random variables is a function of the same random interest process, the present-value random variables are no longer independent. This was illustrated in (21.2.3). Consequently, the distribution of total losses from a portfolio of present-value random variables cannot be routinely approximated using a normal distribution when interest rates are also random variables.

There are simulation-based approaches to the three problems of estimating the moments of present-value random variables, approximating the d.f. of a present value, and approximating the d.f. of the present value of losses from a portfolio of insurance risks. With a source of realizations of a random variable with a $N(0, \sigma^2)$ distribution, sample paths of $(I_1, I_2, \ldots)$ for models such as (21.3.1), (21.4.1), and (21.4.6a, b, and c) could be generated. If $\{I_k\}$, denoting the sequence of random future effective interest rates, and K, completed life years, are assumed to be independent, an empirical d.f. that would approximate the d.f. of an individual loss variable could be developed in a routine fashion.

To illustrate, suppose 100 sequences of future interest rates are generated using the MA(1) model of (21.4.1). For each of these sequences a realization of the random

variable K, completed life years, using the survival function that has been assumed would be determined. These results could be used to compute 100 sample values of $\tilde{v}_{K+1}$. These values can be treated as a sample derived from the joint distribution of $\{I_k\}$ and K. The mean and variance of these 100 simulation sample values would be estimates of the mean and variance of the distribution of $\tilde{v}_{K+1}$. The empirical d.f. would estimate the d.f. of $\tilde{v}_{K+1}$.

The simulation process can also be used to approximate the d.f. of the present value of total losses from a portfolio with n individual risks. In this instance there is a set of completed life year random variables, K_i: $i = 1, 2, \ldots, n$. If these random variables are assumed to be independent, a set of realizations for each K_i random variable would be combined with a randomly generated interest scenario to produce a sample outcome of present value of total losses.

It should be clear why simulation, using computer-generated realizations of present-value random variables that may be functions of several random variables, has been widely used to construct empirical d.f.'s. These applications have made simulation an important tool in actuarial science.

If there is evidence that the random variables time-until-decrement and cause-of-decrement are not independent of $\{I_k\}$, then the generation of realizations of present-value random variables becomes more complicated. For example, the act and time of withdrawal from a life insurance or pension plan may not be independent of $\{I_k\}$.

21.5 Financial Economics Models

From the models developed in Section 21.3 and 21.4, one could be selected, and the parameters estimated, using data from the investment operations of the financial system being modeled. Critics of this procedure assert that it ignores important information available in current capital markets.

To illustrate the variability over time of yields to maturity of one type of security, the average yields to maturity of 30-year U.S. Treasury bonds were displayed in Figure 21.1.1. The changes in bond prices and yields reflect variations in the bond markets' assessment of future economic events. There are, of course, many other investments that might have displayed different patterns of yields over the same period. Economic news does not affect the yields of all securities in an identical fashion. Even Treasury securities with different maturities may exhibit various time series of yield rates.

21.5.1 Information in Prices and Maturities

To extract the information about the relationship between interest rates and maturities, free of confounding factors such as default and call (early maturity at the

option of the borrower) risk, it is usual to analyze securities issued by central governments. In the United States this means Treasury obligations. Other bonds are analyzed by comparing them with Treasury bonds.

To illustrate the methods used in summarizing the relationship between interest rates and maturity dates, a review of basic ideas in the mathematics of finance is required. We consider ***pure discount bonds*** that pay 1 unit at maturity and are traded in a market with no transaction costs. These bonds are not subject to default risk. The number s denotes the current time, and discount bonds are available with maturities at times $s, s + 1 \dots$. The prices of one bond at time s that matures t periods in the future is denoted by $P(s, s + t)$. We assume that

$$P(s, s) = 1,$$

$$\lim_{t \to \infty} P(s, t) = 0,$$

and if $u > t$,

$$P(s, t) > P(s, u).$$

The third assumption is equivalent to attaching a higher value to the earlier of two equal payments. The yield rate for t unit periods is denoted by $i(s, s + t)$ and is defined by

$$P(s, s + t) = [1 + i(s, s + t)]^{-t}. \tag{21.5.1}$$

The number $i(s, s + t)$ is called the t period ***spot rate*** at time s. The name derives from the fact that such rates can be determined from the current market and relate to a single payment at a particular future time. Spot rates $i(s, s + t)$ viewed as a function of t are called the ***term structure*** of interest rates at time s.

Forward rates are an alternative way of studying the relationship between time to maturity and interest rates. As the name suggests, forward rates are the interest rates that would be used for contracts concluded currently that cover transactions in future periods. It will be required that these rates be consistent with the set of spot rates observed in the current market. The consistency that will be required is that ***no-arbitrage*** opportunities will be present in the forward rates. An arbitrage opportunity exists in a capital market if there are two investment strategies available for the same investment period such that one strategy will result with certainty in greater wealth at the end of the period than the alternative strategy. To illustrate the no-arbitrage requirement, suppose an investor pays 1 for a discount bond maturing at time $s + u$ for amount $[1 + i(s, s + u)]^u$. Alternatively, the investor could buy a t period, $t < u$, bond and at time $s + t$ invest the maturity value in a second discount bond that will mature at time $s + u$ for amount

$$[1 + i(s, s + t)]^t \, [1 + j(s, s + t, s + u)]^{u-t},$$

where $j(s, s + t, s + u)$ is the forward rate at time s for a future transaction with cash flows at times $s + t$ and $s + u$. If no arbitrage opportunities exist, the two ultimate wealth amounts must be equal and

$$[1 + i(s, s + u)]^u = [1 + i(s, s + t)]^t [1 + j(s, s + t, s + u)]^{u-t} \qquad (21.5.2)$$

or

$$[1 + j(s, s + t, s + u)]^{u-t} = \frac{[1 + i(s, s + u)]^u}{[1 + i(s, s + t)]^t} \quad 0 \le t \le u.$$

The special case when $u = t + 1$ yields

$$[1 + j(s, s + t, s + t + 1)] = \frac{[1 + i(s, s + t + 1)]^{t+1}}{[1 + i(s, s + t)]^t},$$

and when $t = 0$,

$$j(s, s, s + 1) = i(s, s + 1). \qquad (21.5.3)$$

Repeated applications of (21.5.3) starting with $t = 0$ yield

$$[1 + i(s, s + t)]^t = [1 + j(s, s, s + 1)][1 + j(s, s + 1, s + 2)] \cdots [1 + j(s, s + t - 1, s + t)]. \qquad (21.5.4)$$

The current price of a bond paying coupons of amount c at the end of each of n periods and then paying a maturity value of F can be expressed in a consistent way using prices of discount bonds and spot or future rates as follows:

Using prices of discount bonds,

$$c \sum_{k=1}^{n} P(s, s + k) + F\, P(s, s + n). \qquad (21.5.5a)$$

Using spot rates,

$$c \sum_{k=1}^{n} [1 + i(s, s + k)]^{-k} + F[1 + i(s, s + n)]^{-n}. \qquad (21.5.5b)$$

Using forward rates,

$$c \sum_{k=1}^{n} \prod_{w=0}^{k-1} [1 + j(s, s + w, s + w + 1)]^{-1} + F \prod_{w=0}^{n-1} [1 + j(s, s + w, s + w + 1)]^{-1}. \qquad (21.5.5c)$$

The equality of these three formulas for a bond price rests on the no-arbitrage assumption.

The time interval measuring the time of future coupon payments is not always 1 year. U.S. Treasury bonds typically have semiannual coupons. The bonds that were the subject of Table 21.1.1 would have $n = (30)(2) = 60$.

Yet another way of measuring the relationship between interest rates and time of maturity can be derived from (21.5.4b). The set of spot rates at time s, $i(s, s + k)$, $k = 0, 1, 2, \ldots$ is given, and the ***par yield***, denoted by $y(s, s + m)$, is a set of artificial coupon payments determined from

$$1 = y(s, s + m) \sum_{k=1}^{m} [1 + i(s, s + k)]^{-k} + [1 + i(s, s + m)]^{-m} \quad m = 1, 2, \ldots, n. \tag{21.5.6}$$

The par yield can be described as the coupon rate on a bond trading at its maturity value in a market with a known set of spot rates and no-arbitrage opportunities. The graph of $y(s, s + n)$ as a function of n is the ***yield curve*** at time s. Par yields are most often expressed as ***bond-equivalent yields***, that is, semiannual nominal rates.

FIGURE 21.5.1

Prototype Yield Curves

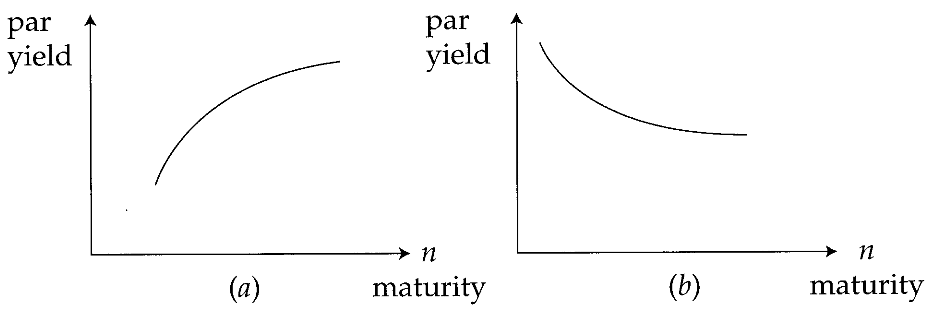


Figure 21.5.1(a) with its positive slope is a typical yield curve. Figure 21.5.1(b) with its negative slope is called an ***inverted yield curve***. An inverted yield curve might result from the action of national monetary authorities to keep short-term interest rates high to retard price inflation. The capital markets' expectation is that inflation will be of short duration and long-term rates are not affected.

Example 21.5.1

Determine equivalent forward rates and par yield rates given the following set of spot rates.

t	$i(s, s + t)$
1	0.06
2	0.065
3	0.070

Solution:

Using (21.5.2),

$$1 + j(s, s + t, s + t + 1) = \frac{[1 + i(s, s + t + 1)]^{t+1}}{[1 + i(s, s + t)]^{t}}.$$

t	$j(s, s + t, s + t + 1)$
0	0.06
1	0.07002
2	0.08007

Using (21.5.5),

$$y(s, s + t) = \frac{1 - [1 + i(s, s + t)]^{-t}}{\sum_{k=1}^{t} [1 + i(s, s + k)]^{-k}}.$$

t	$y(s, s + t)$
1	0.06
2	0.06484
3	0.06955

Note that all three rates have a positive slope when viewed as a function of t. ▼

21.5.2 Stochastic Models

In this section we illustrate a method for generating sequences of future par yield rates. The randomly generated sequences can be inputs into simulations of the financial operations of a portfolio of insurance or pension contracts. In practice, par yield rates rather than spot or forward rates are modeled because investment experts are more familiar with them. It is therefore easier for these experts to select a model and specify the parameters.

We assume that the progress of the logarithm of par yield rates is determined by a modified autoregressive model of order one:

$$\log[Y(t, n)] = \lambda(t, n) + (1 - \phi_n) \log [Y(t - 1, n)] + \sigma_n \, \epsilon_{t,n}, \; t = 0, 1, 2, 3, \ldots . \tag{21.5.7}$$

The terms in (21.5.7) are defined as follows:

$Y(t, n)$ = the random par yield at time t for a bond maturing in n periods. It is assumed that the present time is $t = 0$.

$\lambda(t, n)$ = a drift parameter appropriate for period t for bonds with n periods until maturity. If $\lambda(t, n) = \lambda_n$ (21.5.7) would become a standard autoregressive model of order one [AR(1)]. The drift parameter $\lambda(t, n)$ can be adjusted for each t so that the no-arbitrage constraint or other constraint from financial economics can be satisfied.

$(1 - \phi_n)$ = an autoregressive parameter appropriate for bonds with maturity in n periods. The parameter ϕ_n determines the rate at which previous perturbations decay. For that reason, the parameter is called the rate of mean reversion in finance. In order to have a stationary time series model, $|1 - \phi_n| < 1$.

σ_n = the standard deviation of the random shocks.

$\epsilon_{t,n}$ = a random variable with a $N(0, 1)$ distribution for bonds with n periods to maturity for period t. The random variables ϵ_{t_1,n_1} and ϵ_{t_2,n_2} are independent if $t_1 \neq t_2$ and have correlation ρ_{n_1,n_2} if $t_1 = t_2$. If $n_1 = n_2$, $\rho_{n_1,n_2} = 1$. We are assuming that contemporaneous shocks are correlated.

Remark:

The model described by (21.5.7) is only one of a wide class of stochastic models that could be adopted. The selection of the model and estimation of the parameters would be influenced by the theory of financial economics and statistical data analysis. Formula (21.5.7) also illustrates another option in modeling interest rates. The models specified by (21.3.1), (21.4.1), and (21.4.6a, b, and c) were written for use with log $(1 + I_k)$, the random force of interest for period k. Formula (21.5.7) involves the logarithmic transformation of $Y(t, n)$. The transformation can be motivated as a device to stabilize the variance of the observations or to keep realizations of $Y(t, n)$ non-negative. This non-negative constraint is regarded by some actuaries as important. Exercise 21.9 illustrates the random walk model, which can be viewed as a special case of (21.5.7) with $1 - \phi = 1$ and $\lambda = 0$. The model developed in Exercise 21.9 for the rate of interest shares a property of (21.5.7) in that the random rate of interest is non-negative.

Models such as (21.5.7) can be used to estimate future yield curves. If (21.5.7) were adopted, the estimation of yield curves would require the selection of a set of maturity times, n, and corresponding parameter values λ_n, ϕ_n, σ_n for each value of n and correlation coefficients ρ_{n_1,n_2} for each pair of maturity times selected. The number of values of n would be kept small so that the number of parameters that would have to be specified would be manageable. Then using sets of randomly determined values of $\epsilon_{t,n}$, the outline of future yield curves at various values of t could be determined by simulating future values of $Y(t, n)$ using (21.5.7) for the selected key maturity times.

If the investment portfolio of interest to the actuary holds bonds with a constant maturity, (21.5.7) could be used directly to produce randomly generated sequences of future interest rates for use in simulation studies.

As yet, no adjustments have been made in the model parameters to conform to the actuary's judgment about the long-term mean rate or to force compliance with a market consistency requirement such as no-arbitrage. The parameter $\lambda(t, n)$ is available to incorporate such information.

For example, the following development provides a tool to the actuary for incorporating information about the long-term mean of the par yield rate for securities with fixed maturity n:

Let $\lambda(t, n) = \phi_n \log \mu_n$, which is independent of t; we can rewrite (21.5.7) as

$$\log[Y(t, n)] = \phi_n \log \mu_n + (1 - \phi_n) \log[Y(t - 1, n)] + \sigma_n \epsilon_{t,n}$$

$$t = 1, 2, 3, \ldots . \tag{21.5.8}$$

To facilitate subsequent steps, let

$$Z_t = \log[Y(t, n)],$$

$$\theta = \phi_n \log \mu_n,$$

$$\Psi = (1 - \phi_n) \quad 0 < \Psi < 1,$$

and (21.5.8) becomes

$$Z_t = \theta + \Psi Z_{t-1} + \sigma_n \epsilon_{t,n}, \tag{21.5.9}$$

a standard AR(1) model with drift parameter θ. Multiply successive equations for Z_{t-j} by Ψ^j to obtain

$$\begin{aligned} Z_t - \Psi Z_{t-1} &= \theta + \sigma_n \epsilon_{t,n}, \\ \Psi Z_{t-1} - \Psi^2 Z_{t-2} &= \Psi\, \theta + \Psi\, \sigma_n \epsilon_{t-1,n}, \\ \Psi^2 Z_{t-2} - \Psi^3 Z_{t-3} &= \Psi^2 \theta + \Psi^2 \sigma_n \epsilon_{t-2,n}. \\ \vdots \qquad & \qquad \vdots \end{aligned}$$

Adding this sequence of equations and assuming that t is large, we have

$$Z_t \doteq \frac{\theta}{1 - \Psi} + \sigma_n \sum_{j=0}^{t} \epsilon_{t-j,n} \Psi^j.$$

Exponentiating this approximation yields

$$e^{Z_t} \doteq \exp\left(\frac{\theta}{1 - \Psi} + \sigma_n \sum_{j=0}^{t} \epsilon_{t-j,n} \Psi^j\right),$$

$$Y(t, n) \doteq \mu_n \exp\left(\sigma_n \sum_{j=0}^{t} \epsilon_{t-j,n} \Psi^j\right).$$

The random variable $\sigma_n \sum_{j=0}^{t} \epsilon_{t,n} \Psi^j$ in the exponent has a normal distribution with mean zero and variance for large values of t that is approximately $\sigma_n^2/(1 - \Psi^2)$.

Using these facts about the distribution of the random component, and recalling the m.g.f. for random variables with normal distributions, we have for large values of t

$$\mathrm{E}[Y(t, n)] \doteq \mu_n \exp\left\{\frac{\sigma_n^2}{2[1 - (1 - \phi_n)^2]}\right\}. \tag{21.5.10}$$

If data or the actuary's judgment supports the existence of a long-term mean for the par yield on bonds that mature in n periods, the selected values of $\lambda(t, n) = \phi_n$, $\log \mu_n$, and σ_n^2 should be consistent with (21.5.10).

In addition to subjecting model parameters to consistency checks relative to estimates of the long-term mean, adjustments to the parameter $\lambda(t, n)$ can be made to be consistent with the no-arbitrage condition. In an efficient capital market, the observed rates $[i(0, n_1), i(0, n_2), \ldots, i(0, n_m)]$ for m different maturities would not present significant arbitrage opportunities. Such opportunities would present riskless strategies to increase wealth and would disappear as traders, seeking to exploit such opportunities, would reduce them. The estimated future yield curves determined by averaging many simulations may exhibit arbitrage opportunities. Such opportunities would constitute inconsistent behavior in the capital markets and to retain such opportunities in an estimated future yield curve would similarly be inconsistent.

To develop a method for adjusting an entire estimated yield curves to preclude arbitrage opportunities would involve several ideas beyond the scope of this development. Consequently, only bonds that mature in one period, $n = 1$, are considered. We adopt a simplification of (21.5.8) and assume that the goal is to produce interest scenarios each of length H periods; the model is

$$\log [Y(t, 1)] = \mu + \sigma_1 \epsilon_{t,1} \quad t = 1, 2, \ldots, H. \tag{21.5.11}$$

Generating m sequences of the form $(e_{1,1}, e_{2,1}, \ldots, e_{H,1})$ composed of realizations of the random variables $\epsilon_{t,1}$, would produce m sequences of possible future one-period rates $(e^{\mu+\sigma_1 e_{1,1}}, e^{\mu+\sigma_1 e_{2,1}}, \ldots, e^{\mu+\sigma_1 e_{H,1}})_j$, $j = 1, 2, \ldots, m$.

To minimize the opportunity for arbitrage at time 1, the value of the mean parameter μ in the model will be adjusted so that expected rates available at time 1 will not present such an opportunity. To achieve this objective each of the m values of $e^{\mu+\sigma_1 e_{1,1}}$ will be multiplied by $e^{\lambda_1-\mu}$ where λ_1 is determined from

$$\sum_{j=1}^{m} \frac{[1 + y(0, 1)]^{-1} (1 + e^{\lambda_1+\sigma_1 e_{1,1}})_j^{-1}}{m} = P(0, 2).$$

As before, $P(0, 2)$ is the observed price of a two-period discount bond. The symbol $(1 + e^{\lambda_1+\sigma_1 e_{1,1}})_j^{-1}$ denotes that the variable $e_{1,1}$ comes from scenario j generated from (21.5.11).

The no-arbitrage adjustment for other years in the H-year planning period will be determined by solving successively for λ_{h-1}:

$$\sum_{j=1}^{m} \frac{[1 + y(0, 1)]^{-1}(1 + e^{\lambda_1+\sigma_1 e_{1,1}})_j^{-1} \cdots (1 + e^{\lambda_{h-1}+\sigma_1 e_{h-1,1}})_j^{-1}}{m} = P(0, h),$$

where $h = 2, 3, \ldots, H$. The resulting m sequences of the form $[y(0, 1), e^{\lambda_1+\sigma_1 e_{1,1}}, \ldots, e^{\lambda_{H-1}+\sigma_1 e_{H-1,1}}]_j$ can be used as stochastically generated scenarios of future interest rates where each scenario is assigned weight $1/m$.

Remark:

The no-arbitrage assumption about capital markets has been a theme in this section. Empirical investigations indicate that although arbitrage opportunities exist, they do not persist for long because of the activity of traders.

21.6 Management of Interest Risk

Constructing actuarial models for financial security systems provides a rational basis for pricing the promises made, for reporting on the financial status of the system, and for managing the risks inherent in the operations of these systems. Chapters 12 through 14 dealt in part with ideas for managing short-term risks. Chapters 4 through 11 dealt in part with managing the adverse consequences to financial security systems attributable to the random nature of time and cause of decrement.

In this section ideas for controlling the adverse consequences of changes in interest rates are discussed. In Section 21.6.1 special notation is introduced, and a simplified set of rules for managing interest rate risk within a deterministic environment is developed. In Section 21.6.2 a rather general set of conditions for the time and amount of asset cash flows to minimize interest rate risk within a random model is developed.

21.6.1 Immunization

Our model consists of the following:

$$\text{Reserve or Liabilities} = L(i) = n(A_{x+t} - P_x\, \ddot{a}_{x+t})$$

$$\text{Assets} = A(i) = \sum_{j=0}^{\infty} v^j\, a(j)$$

$$\text{Surplus} = S(i) = A(i) - L(i)$$

where

n = number of identical whole life policies in the model.

t = number of years since the n surviving policies were issued. Assets, liabilities, and surplus are measured at this time.

P_x = benefit premium at issue, but not necessarily appropriate at time t for a realistic appraisal of liabilities. This premium is used in cash flow calculations under the simplifying assumption that expense loadings match expenses.

$\{a(j)\}$ = a sequence of cash flows, coupons, dividends, and maturity values from existing assets, paid at the end of future policy years. In this deterministic environment these amounts are assumed to be certain, not subject to default.

$i =$ valuation rate appropriate at time t. It is assumed, unrealistically, that i does not depend on the timing of future cash flows and that any immediate changes in i will not change the flat yield curve implicit in our assumptions about i.

Clearly this simplified single decrement model ignores withdrawal benefits, expenses, and their provision in contract premiums. Both asset and liability cash flows are assumed to be known and independent of the interest rate i. Some of these unrealistic features can be changed within a more comprehensive model. Other adjustments, such as making asset and liability cash flows depend in a realistic fashion on the valuation interest rate, are more complex.

Within our simple model, we might choose to think of the sequence of future asset cash flows $\{a(j)\}$ as a control variable. Management might elect to enter capital markets to achieve a $\{a(j)\}$ such that

$$\frac{dS(i)}{di} = \frac{d}{di}[A(i) - L(i)] = 0 \tag{21.6.1a}$$

and

$$\frac{dS^2(i)}{di^2} = \frac{d^2}{di^2}[A(i) - L(i)] > 0. \tag{21.6.1b}$$

These two conditions characterize a minimum value of $S(i)$. If a sequence of asset cash flows $\{a(j)\}$ could be found that would satisfy (21.6.1a) and (21.6.1b), any change in the valuation interest rate would lead to an increased surplus. The valuation interest rate is derived from current economic conditions at duration R and is independent of the interest rate used at duration zero in calculating P_x.

From the first derivative (21.6.1a), we have

$$\frac{d}{di} S(i) = v \left\{ -\sum_{j=1}^{\infty} v^j\, a(j) + n \left[\sum_{k=0}^{\infty} (k+1) v^{k+1}\, p_{x+t}\, q_{x+t+k} - P_x \sum_{k=1}^{\infty} k\, v^k\, {}_kp_{x+t} \right] \right\} = 0$$

or

$$\sum_{j=1}^{\infty} v^j\, a(j) = n[(IA)_{x+t} - P_x\, (Ia)_{x+t}]. \tag{21.6.2}$$

Combining the requirements on the first and second derivatives for a minimum value of $S(i)$, (21.6.1a) and (21.6.1b), we have

$$\sum_{j=1}^{\infty} j^2\, v^j\, a(j) > n \left[\sum_{k=0}^{\infty} (k+1)^2\, v_k^{k+1}\, p_{x+t}\, q_{x+k+t} - P_x \sum_{k=1}^{\infty} k^2\, {}_kp_x \right]. \tag{21.6.3}$$

The first derivative condition for the selection of $\{a(j)\}$ may be attainable. It is unlikely that the second derivative condition can be achieved in the capital markets.

If the second derivative condition could be realized and the assumptions of the model were valid, especially the flat yield curve used for pricing assets and valuing liabilities, it would constitute an arbitrage opportunity, for it would be an oppor-

tunity to increase surplus with no new investments or increase in risk on the basis of any change in valuation interest rate. In an efficient market such an opportunity would be nonexistent or fleeting.

The investment selection rules implicit in (21.6.2) and (21.6.3) have been called the immunization rules because their implementation would "immunize" or protect the value of $S(i)$ from changes in i.

21.6.2 General Stochastic Model

We now extend the ideas and their related symbols used in Section 21.6.1 to introduce stochastic elements. We have

$$S(\tilde{v}, \tilde{K}) = A(\tilde{v}) - L(\tilde{v}, \tilde{K}) \tag{21.6.4}$$

where $\tilde{v}$ is a sequence $\{\tilde{v}_j\}$ of random discount factors and $\tilde{K} = (K_1, K_2, \ldots, K_n)$ is a vector of n independent time-until-death random variables.

Unrealistically it will be assumed that the asset cash flow sequence $\{a(j)\}$ is deterministic. Callability and default, among other asset flow option-like characteristics, are not considered.

Using (2.2.11) we have

$$\mathrm{Var}[S(\tilde{v}, \tilde{K})] = \mathrm{Var}[\mathrm{E}[S(\tilde{v}, \tilde{K})|\tilde{v}]] + \mathrm{E}[\mathrm{Var}[S(\tilde{v}, \tilde{K})|\tilde{v}]].$$

Because we assume that asset cash flows are deterministic in the whole life illustration of Section 21.6.1, we have

$$\mathrm{E}[S(\tilde{v}, \tilde{K})|\tilde{v}] = \sum_{j=0}^{\infty} v_j\, a(j) - n \left[\sum_{k=0}^{\infty} (v_{k+1} - P_x\, \ddot{a}_{\overline{k+1}|\tilde{v}})\; {}_kp_x\, q_{x+k} \right]$$

and

$$\mathrm{Var}[S(\tilde{v}, \tilde{K})|\tilde{v}] = \mathrm{Var}[L(\tilde{v}, \tilde{K})|\tilde{v}].$$

Therefore,

$$\begin{aligned} \mathrm{Var}[S(\tilde{v}, \tilde{K})] = {} & \mathrm{Var}_{\tilde{v}}\, [A(\tilde{v}) - \mathrm{E}[L(\tilde{v}, \tilde{K})|\tilde{v}]] \\ & + \mathrm{E}_{\tilde{v}}\, [\mathrm{Var}[L(\tilde{v}, \tilde{K})|\tilde{v}]]. \end{aligned} \tag{21.6.5}$$

We can rewrite (21.6.5) as

$$\begin{aligned} \mathrm{Var}[S(\tilde{v}, \tilde{K})] = {} & \mathrm{Var}_{\tilde{v}}\, [A(\tilde{v})] + \mathrm{Var}_{\tilde{v}}\, [\mathrm{E}[L(\tilde{v}, \tilde{K}]|\tilde{v}] \\ & - 2\, \mathrm{Cov}_{\tilde{v}}[A(\tilde{v}), \mathrm{E}[L(\tilde{v}, \tilde{K})|\tilde{v}]] + \mathrm{E}_{\tilde{v}}\, [\mathrm{Var}[L(\tilde{v}, \tilde{K})|\tilde{v}]]. \end{aligned} \tag{21.6.6}$$

The second and fourth terms of (21.6.6) can be combined to produce

$$\begin{aligned} \mathrm{Var}[S(\tilde{v}, \tilde{K})] = {} & \mathrm{Var}_{\tilde{v}}\, [A(\tilde{v})] + \mathrm{Var}[L(\tilde{v}, \tilde{K})] \\ & - 2\, \mathrm{Cov}_{\tilde{v}}[A(\tilde{v}), \mathrm{E}[L(\tilde{v}, \tilde{K}|\tilde{v}]]. \end{aligned} \tag{21.6.7}$$

We assume that the selection of $\{a(j)\}$, the sequence of asset cash flows, is under

the control of the actuary and that the objective is to minimize $\mathrm{Var}[S(\tilde{v}, \tilde{K})]$. Since the second term of (21.6.5) does not depend on the control variable $\{a(j)\}$, the minimization of $\mathrm{Var}[S(\tilde{v}, \tilde{K})]$ can be achieved by minimizing the term

$$\mathrm{Var}_{\tilde{v}}\, [A(\tilde{v}) - \mathrm{E}[L(\tilde{v}, \tilde{K})|\tilde{v}]].$$

This variance will reach its minimum value of zero if $A(\tilde{v}) - \mathrm{E}[L(\tilde{v}, \tilde{K}|\tilde{v}] = 0$ for each future year. If a sequence of asset cash flows satisfies

$$a(0) + nP_x = 0,$$
$$a(j) - n[{}_{j-1}p_x q_{x+j-1} - P_x\, {}_jp_x] = 0, \quad j = 1, 2, \ldots \tag{21.6.8}$$

could be found, then

$$[a(0) + nP_x] + \sum_{j=1}^{\infty} [a(j) - n({}_{j-1}p_x\, q_{x+j-1} - P_x\, {}_jp_x)]v_j = 0$$

and

$$\mathrm{Var}_{\tilde{v}}[\tilde{A}(\tilde{v}) - \mathrm{E}[L(\tilde{v}, \tilde{K})|\tilde{v}]] = 0.$$

In this very special case, we can substitute into (21.6.7) to obtain

$$\begin{aligned}\mathrm{Var}[S(\tilde{v}, \tilde{K})] &= \mathrm{Var}[A(\tilde{v})] + \mathrm{Var}[L(\tilde{v}, \tilde{K})] - 2\,\mathrm{Cov}[A(\tilde{v}), A(\tilde{v})] \\ &= \mathrm{Var}[L(\tilde{v}, \tilde{K})] - \mathrm{Var}[A(\tilde{v})].\end{aligned}$$

The matching conditions in (21.6.8) resemble an equivalence principle. The matching requirement imposes stringent conditions on asset cash flows. For example, it is possible that the sequence of asset cash flows that will minimize $\mathrm{Var}[S(\tilde{v}, \tilde{K})]$ will require investments $a(j) < 0$ for some j. Other economic and budget restrictions may influence the selection of the asset cash flow sequence.

21.7 Notes and References

The ideas developed in this chapter are built on ideas that are more diverse and of recent origin than are those in earlier chapters. As a consequence, these notes and references are important to an actuary seeking to apply or extend these ideas.

The integration of times series models for stochastic interest rates with random time and cause of decrement has been the subject of intense activity in recent years. Sections 21.3.1 and 21.4.1 follow Frees (1990). The portfolio interest rate risk management rules developed in Section 21.6.2 are also based on this work. More general developments, going beyond the MA(1) model, are by Bellhouse and Panjer (1980) and Giaccotti (1986).

The empirical analysis of interest rate data to determine the adequacy of various models has been a major endeavor in financial economics. Becker (1991) provides a good example of this work. Klein (1993) traces the implications of various distributions of the interest rates on insurance cash flow analysis. In particular, Klein examines the hypothesis that the distribution of random shock terms in interest

rate random models may have heavy tails. The history of formulating and testing hypotheses about rates of return is summarized by Fama (1970).

Jetton (1988) classifies and illustrates methods for generating certain families of interest rate scenarios. Christiansen (1992) starts with Jetton's partial classification but expands the models considered and illustrates their applications. There is an emphasis on interest rate generators that produce rates that regress toward a mean after a perturbation and on shifts from among yield curves according to a matrix of transition probabilities. Tilley (1993) provides more background on the financial economics of interest rate scenario generators. A monograph by Boyle (1992), especially Chapters 2, 3, and 4, provides background for this chapter.

The immunization ideas of Section 21.6.1 have many roots. They were introduced into actuarial science by Redington (1952) and brought to the attention of North American actuaries by Vanderhoof (1972). The Vanderhoof paper contains a model for a life insurance company that is designed to replicate the characteristics of the U.S. life insurance industry in 1971.

In this chapter it is suggested that a stochastic model for generating simultaneously interest rate scenarios for several types of investments and certain other economic variables be constructed. The other economic variables might be an index of consumer prices, or the unemployment rate. The model would be constructed to include observed contemporaneous correlations among the variables, as well as autocorrelations across time. Such a comprehensive and consistent model would obviously be useful in simulating the future operations of a social security system, large pension plan, or insurance company. Since about 1980 a great deal of effort has been spent in constructing, testing, and using such models. A pioneering actuarial effort in this field was by Wilkie (1986). The model has been the subject of intense discussion by Geoghegan et al. (1992).

Exercises

Section 21.1

21.1. If I has a uniform distribution on the interval $(e^{0.03} - 1, e^{0.10} - 1)$ find $\mathrm{E}[(1 + I)^{-1}] - (1 + \mathrm{E}[I])^{-1}$.

21.2. The function $(1 + x)^{-1}$ can be written as the Taylor series expansion

$$(1 + x)^{-1} = 1 - x + x^2 - R(\theta) \qquad |x| < 1$$

where the remainder term

$$R(\theta) = \frac{x^3}{(1 + \theta)^2} \qquad |\theta| < |x| < 1.$$

If I has a uniform distribution on the internal (0.02, 0.12) and the random variable $(1 + I)^{-1}$ is approximated by the first three terms of the Taylor series, evaluate

$$E[1 - I + I^2] - (1 + E[I])^{-1}.$$

21.3. The random variable I has a Pareto distribution with p.d.f.

$$f_I(x) = \frac{11}{(1 + x)^{12}} \quad 0 < x$$
$$= 0 \qquad \text{elsewhere.}$$

a. Find $E[I]$.
b. Find $E[(1 + I)^{-1}] - (1 + E[I])^{-1}$.

21.4. Adopt the model described by (21.3.1) and define the random variable $\tilde{v}_n$ as

$$\tilde{v}_n = \exp\left[-\sum_1^n \log(1 + I_k)\right].$$

a. Using the m.g.f. of $\log(1 + I_k)$ find $E[\tilde{v}_n]$.
b. Using the m.g.f. of $\log(1 + I_k)$ find $E[(\tilde{v}_n)^j]$, $j = 1, 2, 3, \ldots$.
c. Assume that the random variables $\tilde{v}$ and K, completed future life years, are independent and use the results of part b to determine
(i) $E[(\tilde{v}_{K+1})^j] \quad j = 1, 2, 3, \ldots$
(ii) $\mathrm{Var}(\tilde{v}_{K+1})$.
d. Continue with the assumptions made in this exercise and confirm that

$$_*A_x + {}_*d\ {}_*\ddot{a}_x = 1,$$

where $_*d = 1 - e^{-(\delta-\sigma^2/2)}$.

21.5. Adopt the model described by (21.3.1) except that the random variables ϵ_k are independent, identically and uniformly distributed on the interval $(-0.05, 0.05)$, and $\delta = 0.05$. Let $Z = \Sigma_{k=1}^2 \log(1 + I_k)$ be the logarithm of the random two-period interest accumulation function.
a. Evaluate $E[Z]$ and $\mathrm{Var}(Z)$.
b. Display the p.d.f. of Z.
c. Display the p.d.f. of $Y = e^Z$.

21.6. Adopt the assumptions of Exercise 21.5. Let $X_k = \log(1 + I_k)$ and $\bar{X} = \Sigma_1^n X_k / n$.
a. Provide a justification for the statement that

$$\frac{\bar{X} - 0.05}{\sqrt{0.01/12n}}$$

will have an approximate $N(0, 1)$ distribution as n becomes large.
b. Provide a justification for the statement that $\Sigma_1^n X_k$ has an approximate $N(0.5n, 0.01n/12)$ distribution.
c. Use the result in (b) and support the statement that $\Pi_1^n (1 + I_k)$, the random interest accumulation function, has an approximate lognormal distribution with parameters $\mu = 0.5n$ and $\sigma^2 = 0.01n/12$.

Section 21.3

21.7. Adopt the model

$$\log(1 + I_k) = \log(1 + I_{k-1}) + \epsilon_k \quad k = 1, 2, 3, \ldots$$
$$= \delta \quad k = 0$$

for annual interest rates. The assumptions about the random shock terms (ϵ_k) made in connection with (21.3.1) are also adopted. This is called a ***random walk*** model and has frequently been used in studies of rates of return in common stocks. Exhibit (a) E[$\log(1 + I_k)$], (b) Var[$\log(1 + I_k)$], and (c) the distribution of $\log(1 + I_k)$, $k = 1, 2, 3, \ldots$. [Hint: Confirm that

$$\log(1 + I_k) = \delta + \sum_{1}^{k} \epsilon_j.]$$

21.8. Using the model and assumptions of Exercise 21.7, (a) exhibit E[$\log \tilde{v}_n$] and Var($\log \tilde{v}_n$), (b) state the distribution of $\log \tilde{v}_n$, and (c) answer (a) and (b) for $\tilde{v}_n$.

21.9. Another version of the random walk model is

$$\log I_k = \log I_{k-1} + \epsilon_k \quad k = 1, 2, 3$$
$$= r \quad k = 0,$$

where the random shock terms (ϵ_k) are mutually independent with $N(0, \sigma^2)$ distributions. Determine (a) E[$\log I_n$], (b) Var($\log I_n$), and (c) the distribution of $\log I_n$.

21.10. This exercise is a continuation of Exercise 21.9. For the random variable I_n, state its distribution and find E[I_n] and Var(I_n).

Section 21.4

21.11. Confirm that (21.4.5) for E[$(\tilde{v}_n)^2$] reduces to (21.3.4) if $\theta = 0$.

21.12. Confirm that (21.4.6) for E[$\tilde{v}_r\, \tilde{v}_s$] reduces to (21.3.6) if $\theta = 0$.

21.13. Verify the entries under Var($1 + I_k$), $k = 1, 2, 3, \ldots$, in the following table:

Model	**Var(1 + I_k)**	**Name**
Formula (21.3.1)	$(e^{\sigma^2} - 1)e^{(2\delta+\sigma^2)}$	Lognormal
Formula (21.4.1)	$\sigma^2(1 + \theta^2)$	MA(1)
Exercise 21.7	$k\sigma^2$	Random walk

Note the increasing variance with the random walk, Exercise 21.7.

Section 21.5

21.14. Confirm that the model described in (21.5.7) is equivalent to

$$Y(t, n) = Y(t - 1, n)^{1-\phi_n}\, e^{\lambda(t,n)+\sigma_n\epsilon_{t,n}}$$

and

$$\frac{Y(t, n)}{Y(t - 1, n)} = Y(t - 1, n)^{-\phi_n} e^{\lambda(t,n)+\sigma_n\epsilon_{t,n}}.$$

These formulas indicate why (21.5.7) is called a ***multiplicative model***. Note that if the observed value $Y(0, n) = y(0, n) > 0$, $Y(t, n) > 0$.

21.15. If $\phi_n = 0$ and $\lambda(t, n) = \mu$ show that (21.5.7) becomes

$$\log Y(t, n) = \mu + \log Y(t - 1, n) + \sigma_n \epsilon_{t,n},$$

$$\Delta\log Y(t - 1, n) = \mu + \sigma_n \epsilon_{t,n},$$

$$\log Y(t, n) - \log Y(0, n) = t\mu + \sigma_n \sum_{s=1}^{t} \epsilon_{s,n},$$

$$\mathrm{E}[\log Y(t,n)] = \log[y(0, n)] + t\mu,$$

and

$$\mathrm{Var}[\log Y(t, n)] = t\, \sigma_n^2.$$

This is a random walk model with a drift term μ. Show that if $\mu = 0$, the results of Exercise 21.9 are replicated.

21.16. If $[1 + i(s, s + k)] = (1 + i)$, $k = 1, 2, 3, \ldots, n$, and the maturity value of a bond is 1, confirm that $y(s, s + n) = i$.

21.17. The spot rate at times are given in the following:

k	$i(s, s + k)$
1	0.050
2	0.055
3	0.060

Calculate the corresponding par yield rates $y(s, s + t)$ for bond maturities in $k = 1, 2, 3$ periods.

Section 21.6

21.18. Let $\bar{a}(t)$ denote the rate of cash flow from assets and $\bar{l}(t)$ denote the rate of cash flows from insurance operations. For example, $\bar{l}(t)$ would measure claims and expenses less premiums, at time t. These flows are assumed to be deterministic and independent of each other and the interest rate. Let $\bar{A}(\delta)$, $\bar{L}(\delta)$, and $\bar{S}(\delta) = \bar{A}(\delta) - \bar{L}(\delta)$ denote the current value of assets, liabilities, and surplus, respectively, valued at force of interest δ. We have

$$\bar{S}(\delta) = \bar{A}(\delta) - \bar{L}(\delta) = \int_0^\infty e^{-\delta t} [\bar{a}(t) - \bar{l}(t)]\, dt.$$

In addition, let $\bar{R}(\delta) = \bar{S}(\delta) / \bar{A}(\delta)$, which is interpreted as the surplus ratio.

Confirm that if $\bar{R}(\delta_0)$ is a minimum value of $\bar{R}(\delta)$, then

$$\frac{\bar{A}'(\delta_0)}{\bar{A}(\delta_0)} = \frac{\bar{L}'(\delta_0)}{\bar{L}(\delta_0)}$$

and

$$\frac{\bar{A}''(\delta_0)}{\bar{A}(\delta_0)} > \frac{\bar{L}''(\delta_0)}{\bar{L}(\delta_0)}.$$

21.19. The cash flow rate functions in this exercise will be related to the gamma density function; that is, both $\bar{a}(t)$ and $\bar{l}(t)$ of Exercise 21.18 will be of the form

$$f(t) = \frac{k\beta^{\alpha}t^{\alpha-1}\, e^{-\beta t}}{\Gamma(\alpha)} \qquad k > 0.$$

a. Confirm the following result:

$$\int_0^{\infty} t^n e^{-\delta t} f(t)dt = \frac{k\beta^{\alpha}\Gamma(\alpha + n)}{(\beta + \delta)^{n+\alpha}\Gamma(\alpha)}$$

b. If the parameters of the asset cash flow are denoted by α_A, β_A, and k_A and of the liability cash flow by α_L, β_L, and k_L, confirm that the conditions for a minimum of $\bar{R}(\delta)$ as determined in Exercise 21.18 are

$$\frac{\beta_A^{\alpha_A}\, \alpha_A}{(\beta_A + \delta_0)^{\alpha_A+1}} = \frac{\beta_L^{\alpha_L}\, \alpha_L}{(\beta_L + \delta_0)^{\alpha_L+1}}$$

and

$$\frac{\alpha_A + 1}{\beta_A + \delta_0} > \frac{\alpha_L + 1}{\beta_L + \delta_0}.$$

21.20. Accept the results of Exercise 21.18. Verify that

$$\frac{\bar{A}'(\delta_0)}{\bar{A}(\delta_0)} = \frac{\bar{L}'(\delta_0)}{\bar{L}(\delta_0)}$$

is equivalent to

$$\frac{\int_0^{\infty} t\, e^{-\delta_0 t}\, \bar{a}(t)dt}{\int_0^{\infty} e^{-\delta_0 t}\, \bar{a}(t)dt} = \frac{\int_0^{\infty} t\, e^{-\delta_0 t}\, \bar{l}(t)dt}{\int_0^{\infty} e^{-\delta_0 t}\, \bar{l}(t)dt}.$$

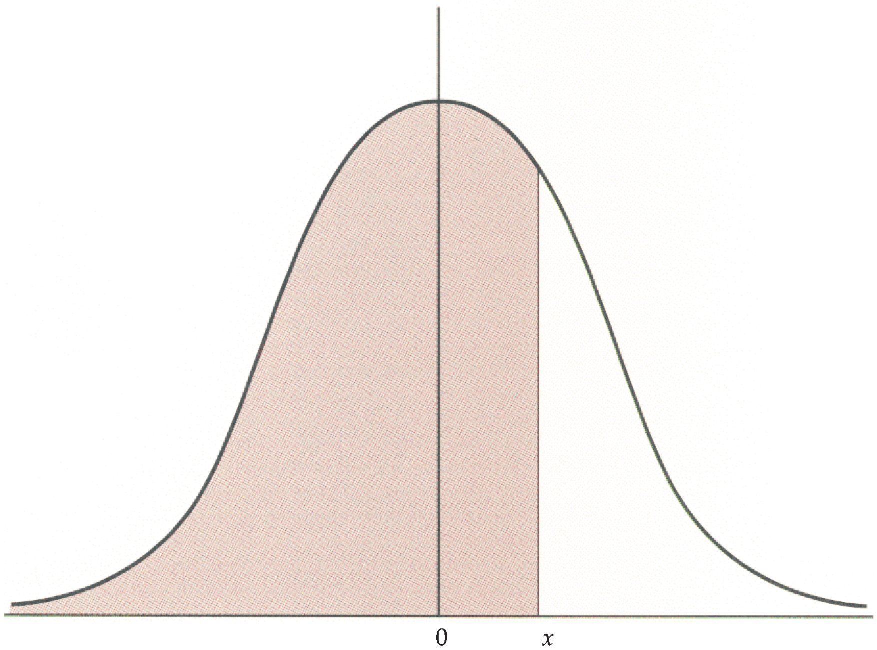
This result leads to the interpretation that the ratio $\bar{A}'(\delta_0)/\bar{A}(\delta_0)$ is a weighted average of t, the time of asset cash flows with the weights provided by $e^{-\delta_0 t}\, \bar{a}(t)/\int_0^{\infty} e^{-\delta_0 t}\, \bar{a}(t)dt$. A similar interpretation can be made of $\bar{L}'(\delta_0)/\bar{L}(\delta_0)$. These weighted times of cash flows motivate calling $\bar{A}'(\delta_0)/\bar{A}(\delta_0)$ and $\bar{L}'(\delta_0)/\bar{L}(\delta_0)$ durations.

21.21. We adopt the second central moment of interia of the times of cash flows as a measure of dispersion of the times of cash flows. Confirm that the conditions developed in Exercise 21.18 are equivalent to the requirement that the dispersion of asset cash flows shall be equal to or greater than dispersion of liability cash flows.
[Hint: The second central moment of inertia for a mass with density at x of $m(x)$ is $\int_0^\infty (x - c)^2 m(x)dx$ where $c = \int_0^\infty xm(x)dx$.]

21.22. Adopt the symbols of Exercise 21.18 and build a model for a set of l_x fully continuous whole life policies. Make the following assumptions:
- The contract premium rate is $\bar{G} = \bar{P}(\bar{A}_x)(1 + \theta)$, $\theta > 0$.
- Expense payments are made at time t for survivors at a rate of ce^{-rt}, $r > 0$.
- Assumptions are realized.
  a. Confirm that $\bar{l}(t) = l_x[\mu_x(t) + ce^{-rt} - \bar{P}(\bar{A}_x)(1 + \theta)]_t p_x$.
  b. Express $\bar{L}(\delta)$ in actuarial present-value symbols.
  c. Express $\bar{L}'(\delta)$ in actuarial symbols.

Appendix 1

NORMAL DISTRIBUTION TABLE



Normal Distribution

The table on page 674 gives the value of

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-w^2/2}\, dw$$

for certain values of x. The integer of x is given in the top row, and the first decimal place of x is given in the left column. Since the density function of x is symmetric, the value of the cumulative distribution function for negative x can be obtained by subtracting from unity the value of the cumulative distribution function for x.

x	0	1	2	3
0.0	0.5000	0.8413	0.9772	0.9987
0.1	0.5398	0.8643	0.9821	0.9990
0.2	0.5793	0.8849	0.9861	0.9993
0.3	0.6179	0.9032	0.9893	0.9995
0.4	0.6554	0.9192	0.9918	0.9997
0.5	0.6915	0.9332	0.9938	0.9998
0.6	0.7257	0.9452	0.9953	0.9998
0.7	0.7580	0.9554	0.9965	0.9999
0.8	0.7881	0.9641	0.9974	0.9999
0.9	0.8159	0.9713	0.9981	1.0000

Selected Points of the Normal Distribution

$\Phi(x)$	x
0.800	0.842
0.850	1.036
0.900	1.282
0.950	1.645
0.975	1.960
0.990	2.326
0.995	2.576

Appendix 2A

ILLUSTRATIVE LIFE TABLE

Illustrative Life Table: Basic Functions

Age	l_x	d_x	1,000 q_x
0	100 000.00	2 042.1700	20.4217
1	97 957.83	131.5672	1.3431
2	97 826.26	119.7100	1.2237
3	97 706.55	109.8124	1.1239
4	97 596.74	101.7056	1.0421
5	97 495.03	95.2526	0.9770
6	97 399.78	90.2799	0.9269
7	97 309.50	86.6444	0.8904
8	97 222.86	84.1950	0.8660
9	97 138.66	82.7816	0.8522
10	97 055.88	82.2549	0.8475
11	96 973.63	82.4664	0.8504
12	96 891.16	83.2842	0.8594
13	96 807.88	84.5180	0.8730
14	96 723.36	86.0611	0.8898
15	96 637.30	87.7559	0.9081
16	96 549.54	89.6167	0.9282
17	96 459.92	91.6592	0.9502
18	96 368.27	93.9005	0.9744
19	96 274.36	96.3596	1.0009
20	96 178.01	99.0569	1.0299
21	96 078.95	102.0149	1.0618
22	95 976.93	105.2582	1.0967
23	95 871.68	108.8135	1.1350
24	95 762.86	112.7102	1.1770
25	95 650.15	116.9802	1.2330
26	95 533.17	121.6585	1.2735
27	95 411.51	126.7830	1.3288
28	95 284.73	132.3953	1.3895
29	95 152.33	138.5406	1.4560

Illustrative Life Table: Basic Functions

Age	l_x	d_x	$1{,}000\ q_x$
30	95 013.79	145.2682	1.5289
31	94 868.53	152.6317	1.6089
32	94 715.89	160.6896	1.6965
33	94 555.20	169.5052	1.7927
34	94 385.70	179.1475	1.8980
35	94 206.55	189.6914	2.0136
36	94 016.86	201.2179	2.1402
37	93 815.64	213.8149	2.2791
38	93 601.83	227.5775	2.4313
39	93 374.25	242.6085	2.5982
40	93 131.64	259.0186	2.7812
41	92 872.62	276.9271	2.9818
42	92 595.70	296.4623	3.2017
43	92 299.23	317.7619	3.4427
44	91 981.47	340.9730	3.7070
45	91 640.50	366.2529	3.9966
46	91 274.25	393.7687	4.3141
47	90 880.48	423.6978	4.6621
48	90 456.78	456.2274	5.0436
49	90 000.55	491.5543	5.4617
50	89 509.00	529.8844	5.9199
51	88 979.11	571.4316	6.4221
52	88 407.68	616.4165	6.9724
53	87 791.26	665.0646	7.5755
54	87 126.20	717.6041	8.2364
55	86 408.60	774.2626	8.9605
56	85 634.33	835.2636	9.7538
57	84 799.07	900.8215	10.6230
58	83 898.25	971.1358	11.5752
59	82 927.11	1 046.3843	12.6181
60	81 880.73	1 126.7146	13.7604
61	80 754.01	1 212.2343	15.0114
62	79 541.78	1 302.9994	16.3813
63	78 238.78	1 399.0010	17.8812
64	76 839.78	1 500.1504	19.5231
65	75 339.63	1 606.2618	21.3203
66	73 733.37	1 717.0334	23.2871
67	72 016.33	1 832.0273	25.4391
68	70 184.31	1 950.6476	27.7932
69	68 233.66	2 072.1177	30.3680

Illustrative Life Table: Basic Functions

Age	l_x	d_x	1,000 q_x
70	66 161.54	2 195.4578	33.1833
71	63 966.08	2 319.4639	36.2608
72	61 646.62	2 442.6884	39.6240
73	59 203.93	2 563.4258	43.2982
74	56 640.51	2 679.7050	47.3108
75	53 960.80	2 789.2905	51.6911
76	51 171.51	2 889.6965	56.4708
77	48 281.81	2 978.2164	61.6840
78	45 303.60	3 051.9717	67.3671
79	42 251.62	3 107.9833	73.5589
80	39 143.64	3 143.2679	80.3009
81	36 000.37	3 154.9603	87.6369
82	32 845.41	3 140.4624	95.6134
83	29 704.95	3 097.6146	104.2794
84	26 607.34	3 024.8830	113.6860
85	23 582.45	2 921.5530	123.8867
86	20 660.90	2 787.9129	134.9367
87	17 872.99	2 625.4088	146.8926
88	15 247.58	2 436.7474	159.8121
89	12 810.83	2 225.9244	173.7533
90	10 584.91	1 998.1533	188.7738
91	8 586.75	1 759.6818	204.9298
92	6 827.07	1 517.4869	222.2749
93	5 309.58	1 278.8606	240.8589
94	4 030.72	1 050.9136	260.7257
95	2 979.81	840.0452	281.9122
96	2 139.77	651.4422	304.4456
97	1 488.32	488.6776	328.3410
98	999.65	353.4741	353.5993
99	646.17	245.6772	380.2041
100	400.49	163.4494	408.1188
101	237.05	103.6560	437.2837
102	133.39	62.3746	467.6133
103	71.01	35.4358	498.9935
104	35.58	18.9023	531.2793
105	16.68	9.4105	564.2937
106	7.27	4.3438	597.8266
107	2.92	1.8458	631.6360
108	1.08	0.7163	665.4495
109	0.36	0.2517	698.9685
110	0.11	0.0793	731.8742

Illustrative Life Table: Single Life Actuarial Functions, $i = 0.06$

Age	$\ddot{a}_x$	$1{,}000\ A_x$	$1{,}000\ ({}^2A_x)$
0	16.80096	49.0025	25.9210
1	17.09819	32.1781	8.8845
2	17.08703	32.8097	8.6512
3	17.07314	33.5957	8.5072
4	17.05670	34.5264	8.4443
5	17.03786	35.5930	8.4547
6	17.01675	36.7875	8.5310
7	16.99351	38.1031	8.6666
8	16.96823	39.5341	8.8553
9	16.94100	41.0757	9.0917
10	16.91187	42.7245	9.3712
11	16.88089	44.4782	9.6902
12	16.84807	46.3359	10.0460
13	16.81340	48.2981	10.4373
14	16.77685	50.3669	10.8638
15	16.73836	52.5459	11.3268
16	16.69782	54.8404	11.8295
17	16.65515	57.2558	12.3749
18	16.61024	59.7977	12.9665
19	16.56299	62.4720	13.6080
20	16.51330	65.2848	14.3034
21	16.46105	68.2423	15.0569
22	16.40614	71.3508	15.8730
23	16.34843	74.6170	16.7566
24	16.28783	78.0476	17.7128
25	16.22419	81.6496	18.7472
26	16.15740	85.4300	19.8657
27	16.08733	89.3962	21.0744
28	16.01385	93.5555	22.3802
29	15.93683	97.9154	23.7900
30	15.85612	102.4835	25.3113
31	15.77161	107.2676	26.9520
32	15.68313	112.2754	28.7206
33	15.59057	117.5148	30.6259
34	15.49378	122.9935	32.6772
35	15.39262	128.7194	34.8843
36	15.28696	134.7002	37.2574
37	15.17666	140.9437	39.8074
38	15.06159	147.4572	42.5455
39	14.94161	154.2484	45.4833

Illustrative Life Table: Single Life Actuarial Functions, $i = 0.06$

Age	$\ddot{a}_x$	1,000 A_x	1,000 $({}^2A_x)$
40	14.81661	161.3242	48.6332
41	14.68645	168.6916	52.0077
42	14.55102	176.3572	55.6199
43	14.41022	184.3271	59.4833
44	14.26394	192.6071	63.6117
45	14.11209	201.2024	68.0193
46	13.95459	210.1176	72.7205
47	13.79136	219.3569	77.7299
48	13.62235	228.9234	83.0624
49	13.44752	238.8198	88.7329
50	13.26683	249.0475	94.7561
51	13.08027	259.6073	101.1469
52	12.88785	270.4988	107.9196
53	12.68960	281.7206	115.0885
54	12.48556	293.2700	122.6672
55	12.27581	305.1431	130.6687
56	12.06042	317.3346	139.1053
57	11.83953	329.8381	147.9883
58	11.61327	342.6452	157.3280
59	11.38181	355.7466	167.1332
60	11.14535	369.1310	177.4113
61	10.90412	382.7858	188.1682
62	10.65836	396.6965	199.4077
63	10.40837	410.8471	211.1318
64	10.15444	425.2202	223.3401
65	9.89693	439.7965	236.0299
66	9.63619	454.5553	249.1958
67	9.37262	469.4742	262.8299
68	9.10664	484.5296	276.9212
69	8.83870	499.6963	291.4559
70	8.56925	514.9481	306.4172
71	8.29879	530.2574	321.7850
72	8.02781	545.5957	337.5361
73	7.75683	560.9339	353.6443
74	7.48639	576.2419	370.0803
75	7.21702	591.4895	386.8119
76	6.94925	606.6460	403.8038
77	6.68364	621.6808	421.0184
78	6.42071	636.5634	438.4155
79	6.16101	651.2639	455.9527

Illustrative Life Table: Single Life Actuarial Functions, $i = 0.06$

Age	$\ddot{a}_x$	1,000 A_x	1,000 $({}^2A_x)$
80	5.90503	665.7528	473.5861
81	5.65330	680.0019	491.2698
82	5.40629	693.9837	508.9574
83	5.16446	707.6723	526.6012
84	4.92824	721.0431	544.1537
85	4.69803	734.0736	561.5675
86	4.47421	746.7428	578.7956
87	4.25710	759.0320	595.7923
88	4.04700	770.9244	612.5133
89	3.84417	782.4056	628.9163
90	3.64881	793.4636	644.9611
91	3.46110	804.0884	660.6105
92	3.28118	814.2726	675.8298
93	3.10914	824.0111	690.5878
94	2.94502	833.3007	704.8565
95	2.78885	842.1408	718.6115
96	2.64059	850.5325	731.8321
97	2.50020	858.4791	744.5010
98	2.36759	865.9853	756.6047
99	2.24265	873.0577	768.1330
100	2.12522	879.7043	779.0793
101	2.01517	885.9341	789.4400
102	1.91229	891.7573	799.2147
103	1.81639	897.1852	808.4054
104	1.72728	902.2295	817.0170
105	1.64472	906.9025	825.0563
106	1.56850	911.2170	832.5324
107	1.49838	915.1860	839.4558
108	1.43414	918.8224	845.8386
109	1.37553	922.1396	851.6944
110	1.32234	925.1507	857.0377

Illustrative Life Table: Joint Life Actuarial Functions, $i = 0.06$

Age	$\ddot{a}_{xx}$	$1{,}000\ A_{xx}$	$1{,}000\ (^2A_{xx})$	$\ddot{a}_{x:x+10}$	$1{,}000\ A_{x:x+10}$	$1{,}000\ (^2A_{x:x+10})$
0	16.13448	86.7274	50.8875	16.28443	78.2400	34.7076
1	16.71842	53.6745	17.4565	16.55328	63.0218	18.1309
2	16.70637	54.3565	16.9753	16.52270	64.7527	18.2195
3	16.68957	55.3072	16.6683	16.48839	66.6947	18.4277
4	16.66839	56.5060	16.5191	16.45053	68.8378	18.7468
5	16.64317	57.9339	16.5121	16.40925	71.1745	19.1700
6	16.61421	59.5733	16.6324	16.36464	73.6996	19.6923
7	16.58178	61.4085	16.8664	16.31677	76.4091	20.3096
8	16.54614	63.4258	17.2017	16.26571	79.2997	21.0188
9	16.50749	65.6137	17.6271	16.21147	82.3696	21.8172
10	16.46599	67.9626	18.1330	16.15408	85.6181	22.7036
11	16.42178	70.4655	18.7116	16.09353	89.0457	23.6776
12	16.37492	73.1176	19.3572	16.02977	92.6543	24.7402
13	16.32547	75.9170	20.0661	15.96277	96.4469	25.8935
14	16.27340	78.8643	20.8373	15.89244	100.4282	27.1413
15	16.21865	81.9632	21.6726	15.81866	104.6042	28.4891
16	16.16111	85.2203	22.5769	15.74131	108.9826	29.9441
17	16.10065	88.6424	23.5556	15.66025	113.5710	31.5141
18	16.03715	92.2366	24.6142	15.57534	118.3771	33.2071
19	15.97049	96.0099	25.7588	15.48645	123.4087	35.0317
20	15.90053	99.9697	26.9958	15.39343	128.6737	36.9970
21	15.82715	104.1234	28.3320	15.29615	134.1800	39.1126
22	15.75021	108.4786	29.7746	15.19448	139.9353	41.3884
23	15.66958	113.0429	31.3311	15.08826	145.9474	43.8349
24	15.58511	117.8241	33.0098	14.97738	152.2240	46.4632
25	15.49667	122.8299	34.8192	14.86169	158.7725	49.2847
26	15.40413	128.0682	36.7681	14.74106	165.6003	52.3114
27	15.30734	133.5468	38.8662	14.61538	172.7144	55.5555
28	15.20617	139.2737	41.1234	14.48452	180.1217	59.0301
29	15.10047	145.2564	43.5502	14.34836	187.8286	62.7483
30	14.99012	151.5028	46.1574	14.20681	195.8411	66.7238
31	14.87498	158.0203	48.9566	14.05976	204.1648	70.9706
32	14.75491	164.8162	51.9595	13.90712	212.8047	75.5028
33	14.62981	171.8977	55.1785	13.74882	221.7652	80.3352
34	14.44953	179.2716	58.6264	13.58478	231.0501	85.4824
35	14.36398	186.9444	62.3164	13.41497	240.6623	90.9593
36	14.22304	194.9221	66.2622	13.23933	250.6040	96.7805
37	14.07662	203.2104	70.4777	13.05785	260.8765	102.9610
38	13.92461	211.8144	74.9770	12.87052	271.4799	109.5154
39	13.76695	220.7386	79.7749	12.67736	282.4136	116.4579

Illustrative Life Table: Joint Life Actuarial Functions, $i = 0.06$

Age	$\ddot{a}_{xx}$	$1{,}000\ A_{xx}$	$1{,}000\ (^2A_{xx})$	$\ddot{a}_{x:x+10}$	$1{,}000\ A_{x:x+10}$	$1{,}000\ (^2A_{x:x+10})$
40	13.60357	229.9867	84.8858	12.47840	293.6755	123.8024
41	13.43441	239.5619	90.3247	12.27370	305.2625	131.5623
42	13.25943	249.4664	96.1064	12.06333	317.1700	139.7502
43	13.07861	259.7015	102.2457	11.84740	329.3924	148.3778
44	12.89194	270.2677	108.7571	11.62604	341.9222	157.4559
45	12.69943	281.1642	115.6552	11.39940	354.7507	166.9939
46	12.50112	292.3892	122.9537	11.16767	367.8678	177.0001
47	12.29706	303.9398	130.6661	10.93105	381.2615	187.4810
48	12.08733	315.8114	138.8051	10.68978	394.9184	198.4414
49	11.87202	327.9986	147.3826	10.44412	408.8233	209.8841
50	11.65127	340.4941	156.4093	10.19438	422.9597	221.8099
51	11.42522	353.2895	165.8951	9.94087	437.3092	234.2171
52	11.19405	366.3746	175.8482	9.68395	451.8518	247.1016
53	10.95797	379.7377	186.2752	9.42400	466.5661	260.4567
54	10.71721	393.3656	197.1814	9.16142	481.4292	274.2728
55	10.47203	407.2435	208.5696	8.89664	496.4168	288.5375
56	10.22273	421.3546	220.4410	8.63011	511.5030	303.2353
57	9.96964	435.6810	232.7940	8.36232	526.6612	318.3475
58	9.71308	450.2029	245.6250	8.09375	541.8633	333.8526
59	9.45345	464.8990	258.9275	7.82491	557.0805	349.7258
60	9.19114	479.7465	272.6922	7.55633	572.2833	365.9390
61	8.92659	494.7213	286.9070	7.28853	587.4417	382.4614
62	8.66024	509.7977	301.5568	7.02206	602.5251	399.2593
63	8.39257	524.9491	316.6234	6.75745	617.5030	416.2961
64	8.12406	540.1477	332.0853	6.49524	632.3449	433.5327
65	7.85522	555.3647	347.9183	6.23597	647.0206	450.9279
66	7.58658	570.5707	364.0947	5.98016	661.5006	468.4383
67	7.31867	585.7356	380.5839	5.72831	675.7560	486.0192
68	7.05202	600.8289	397.3525	5.48092	689.7590	503.6243
69	6.78718	615.8203	414.3642	5.23847	703.4830	521.2065
70	6.52467	630.6790	431.5803	5.00138	716.9030	538.7185
71	6.26504	645.3750	448.9598	4.77008	729.9954	556.1128
72	6.00881	659.8785	466.4595	4.54495	742.7386	573.3422
73	5.75650	674.1606	484.0346	4.32634	755.1127	590.3606
74	5.50858	688.1934	501.6393	4.11456	767.1002	607.1233
75	5.26555	701.9503	519.2266	3.90989	778.6857	623.5869
76	5.02783	715.4057	536.7489	3.71254	789.8559	639.7107
77	4.79586	728.5362	554.1588	3.52273	800.6001	655.4561
78	4.57002	741.3197	571.4091	3.34060	810.9096	670.7874
79	4.35066	753.7364	588.4536	3.16625	820.7782	685.6720

Illustrative Life Table: Joint Life Actuarial Functions, i = 0.06

Age	$\ddot{a}_{xx}$	1,000 A_{xx}	1,000 $(^2A_{xx})$	$\ddot{a}_{x:x+10}$	1,000 $A_{x:x+10}$	1,000 $(^2A_{x:x+10})$
80	4.13809	765.7683	605.2473	2.99977	830.2020	700.0806
81	3.93260	777.3999	621.7467	2.84117	839.1791	713.9874
82	3.73442	788.6175	637.9108	2.69046	847.7098	727.3701
83	3.54375	799.4102	653.7007	2.54760	855.7965	740.2101
84	3.36075	809.7690	669.0804	2.41251	863.4431	752.4921
85	3.18552	819.6876	684.0169	2.28509	870.6554	764.2049
86	3.01814	829.1617	698.4806	2.16521	877.4407	775.3401
87	2.85866	838.1892	712.4451	2.05273	883.8075	785.8931
88	2.70706	846.7701	725.8879	1.94748	889.7655	795.8619
89	2.56332	854.9067	738.7899	1.84925	895.3253	805.2478
90	2.42735	862.6027	751.1355	1.75786	900.4984	814.0543
91	2.29908	869.8636	762.9129	1.67309	905.2969	822.2875
92	2.17836	876.6967	774.1136	1.59471	909.7331	829.9554
93	2.06505	883.1102	784.7323	1.52251	913.8199	837.0680
94	1.95899	889.1137	794.7670	1.45626	917.5703	843.6367
95	1.85998	894.7179	804.2185	1.39571	920.9973	849.6744
96	1.76783	899.9341	813.0901	1.34065	924.1140	855.1951
97	1.68232	904.7742	821.3876	1.29084	926.9335	860.2140
98	1.60324	909.2506	829.1188	1.24605	929.4689	864.7475
99	1.53035	913.3762	836.2934	1.20604	931.7333	868.8126
100	1.46344	917.1638	842.9228	1.17060	933.7399	872.4279
101	1.40226	920.6266	849.0197	1.13946	935.5020	875.6129
102	1.34659	923.7777	854.5980	1.11241	937.0336	878.3888
103	1.29620	926.6301	859.6727	1.08917	938.3489	880.7785
104	1.25086	929.1969	864.2600	1.06949	939.4630	882.8066
105	1.21032	931.4911	868.3771	1.05308	940.3917	884.5002
106	1.17437	933.5261	872.0421	1.03965	941.1518	885.8881
107	1.14277	935.3151	875.2746	1.02889	941.7609	887.0017
108	1.11526	936.8720	878.0956	1.02047	942.2374	887.8735
109	1.09161	938.2110	880.5276	1.01406	942.6001	888.5376
110	1.07154	939.3470	882.5952	1.00934	942.8678	889.0280

Appendix 2B

ILLUSTRATIVE SERVICE TABLE

Age x	$l_x^{(\tau)}$	$d_x^{(d)}$	$d_x^{(w)}$	$d_x^{(i)}$	$d_x^{(r)}$	s_x
30	100 000	100	19 900	—	—	1.00
31	79 910	80	14 376	—	—	1.06
32	65 454	72	9 858	—	—	1.13
33	55 524	61	5 702	—	—	1.20
34	49 761	60	3 971	—	—	1.28
35	45 730	64	2 693	46	—	1.36
36	42 927	64	1 927	43	—	1.44
37	40 893	65	1 431	45	—	1.54
38	39 352	71	1 181	47	—	1.63
39	38 053	72	989	49	—	1.74
40	36 943	78	813	52	—	1.85
41	36 000	83	720	54	—	1.96
42	35 143	91	633	56	—	2.09
43	34 363	96	550	58	—	2.22
44	33 659	104	505	61	—	2.36
45	32 989	112	462	66	—	2.51
46	32 349	123	421	71	—	2.67
47	31 734	133	413	79	—	2.84
48	31 109	143	373	87	—	3.02
49	30 506	156	336	95	—	3.21
50	29 919	168	299	102	—	3.41
51	29 350	182	293	112	—	3.63
52	28 763	198	259	121	—	3.86
53	28 185	209	251	132	—	4.10
54	27 593	226	218	143	—	4.35
55	27 006	240	213	157	—	4.62
56	26 396	259	182	169	—	4.91
57	25 786	276	178	183	—	5.21
58	25 149	297	148	199	—	5.53
59	24 505	316	120	213	—	5.86

Age x	$l_x^{(\tau)}$	$d_x^{(d)}$	$d_x^{(w)}$	$d_x^{(i)}$	$d_x^{(r)}$	S_x
60	23 856	313	—	—	3 552	6.21
61	19 991	298	—	—	1 587	6.56
62	18 106	284	—	—	2 692	6.93
63	15 130	271	—	—	1 350	7.31
64	13 509	257	—	—	2 006	7.70
65	11 246	204	—	—	4 448	8.08
66	6 594	147	—	—	1 302	8.48
67	5 145	119	—	—	1 522	8.91
68	3 504	83	—	—	1 381	9.35
69	2 040	49	—	—	1 004	9.82
70	987	17	—	—	970	10.31

Appendix 3

SYMBOL INDEX

Symbol	Page
$A_{xy:\overline{n\rvert}}$	280
$\bar{A}^{\ \ 1}_{xy:\overline{n\rvert}}$	283
${}^2A_{xy:\overline{n\rvert}}$	281
$\bar{A}_{wxy}^{\ \ \ 2}$	565
$\bar{A}_{\overline{x_1x_2x_3}}$	561
${}_kAS$	486
${}_k\widehat{AS}$	512
$(AS)_{x+h}$	351
(AAI)	526
$b(u)$	587
b_j	230
b_t	94
$b_f(t)$	600
B_t	611
$\hat{B}_{x+k}$	344
$B^{(3)}_{x+t}$	549
$B^{(j)}_{x+t}$	342
${}_hBP$	509
c	410
c_k	486
$\hat{c}_k$	512
$c(t)$	399
C_1	651
C_2	651
C_3	651
C_h	234
${}_kCV$	486
$d_x^{(j)}$	316
${}_nd_x$	59
${}_nd_x^{(j)}$	316
${}_nd_x^{(\tau)}$	316
${}_tD_j$	557
${}_{k+1}D$	513
$(DA)^1_{x:\overline{n\rvert}}$	117
$(D\bar{A})^1_{x:\overline{n\rvert}}$	108
${}_n\mathcal{D}_x$	59
${}_n\mathcal{D}_x^{(j)}$	316
${}_n\mathcal{D}_x^{(\tau)}$	316
e	468
e_{h-1}	483
e_x	69

Symbol	Page
$\mathring{e}_x$	68
$\hat{e}_k$	512
$\mathring{e}_{x:\overline{n\rvert}}$	71
e_{xy}	272
$e_{\overline{xy}}$	272
$\mathring{e}_{xy}$	272
$\mathring{e}_{\overline{xy}}$	272
E	5, A5
E	501
E_0	501
${}_nE_x$	101
$(ES)_{x+h+t}$	351
$ELRA$	525
f	609
$f(u;t)$	490
$f_S(s)$	34
$F_X(x)$	28
F_t	629
$F^{(k)}$	35
$F_S(s)$	34
${}_kF$	513
G	4, 449, 467
$\hat{G}$	531
$G(b)$	407
$G(x{:}\alpha,\beta)$	387
$h(x)$	453, 609
$H(r)$	600
$H(x{:}\alpha,\beta,x_0)$	387
${}_u(hp)^{(\tau)}_{x+t}$	548
$(h\mu)^{(j)}_{x+t}(u)$	548
i'_{k+1}	539
$\hat{\imath}_{k+1}$	512
$i(s,s+t)$	656
I_k	644
I_d	445
$I_d(x)$	17
${}_ji_k$	638
$(IA)_x$	115
$(I\bar{A})_x$	106
$(\bar{I}\bar{A})_x$	106

Appendix 4

GENERAL RULES FOR SYMBOLS OF ACTUARIAL FUNCTIONS

An actuarial function is represented by a principal symbol and a combination of auxiliary symbols such as letters, numerals, double dots, circles, hats, and horizontal and vertical bars. The principal symbol expresses the general definition of the function; choice and placement of the auxiliary symbols at the top and corners give precise meaning. We review the rules for selecting and placing the symbols and show one or more functional forms in common application areas.

This notation is based upon the system of International Actuarial Notation (IAN) that was originally adopted by the Second International Congress of Actuaries in London in 1898 and is modified periodically under the guidance of the Permanent Committee of Actuarial Notations of the International Actuarial Association. IAN is a basic system of principles that does not cover all areas of actuarial applications. In this text these principles have been followed, and sometimes extended, to construct consistent notation where needed.

This Appendix is meant to provide the reader with an overview of basic patterns for expressing the symbols appearing in this book. Although it is a good introduction to IAN, it is not exhaustive. Authoritative sources for further reference are:

- Actuarial Society of America, "International Actuarial Notation," *Transactions,* XLVIII, 1947: 166–176.
- Faculty of Actuaries, *Transactions,* XIX, 1950: 89.
- *Journal of the Institute of Actuaries,* LXXV, 1949: 121.

An actuarial symbol can be viewed as illustrated below. Box I represents the principal symbol, the others subscripts or superscripts. The roman numerals in the boxes correspond to the section designations of this Appendix.

VI	IV	V
	I	
III		II

Section I. Center

Principal Symbol	Description	Topic
i	Effective rate of interest for a time period, usually 1 year, or with a superscript in Position V, a nominal rate.	Interest
v	Present value of 1 due at the end of the effective interest period, usually 1 year.	
δ	Force of interest, usually stated as an annual rate.	
d	Effective rate of interest-in-advance, or discount rate, for a time period of usually 1 year, or with a superscript in Position V, a nominal rate. This symbol never has a subscript in Position II.	
l	Expected number, or number, of survivors at a given age.	Life Tables
d	Expected number, or number, of those dying within a given time period. This symbol always has a subscript in Position II.	
p	Probability of surviving for a given time period.	
q	Probability of dying within a given time period.	
μ	Force of mortality, usually stated on an annual basis.	
m	Central death rate for a given time period.	
L	Expected number, or number, of years lived within a time period by the survivors at the beginning of the period.	
T	Expected total, or total, future lifetime of the survivors at a given age. (The above are survivorship group definitions of the life table functions denoted by l, d, L, and T. For the alternative stationary population definitions, see Chapter 19.)	
A	Actuarial present value (net single premium) of an insurance or pure endowment of 1.	Life Insurance and Pure Endowments

Principal Symbol	*Description*	*Topic*
(IA)	Actuarial present value (net single premium) of an insurance with a benefit amount of 1 at the end of the first year, increasing linearly at a rate of 1 per year.	
(DA)	Actuarial present value (net single premium) of a term insurance with an initial benefit amount equal to the term and decreasing linearly at the rate of 1 per year.	
E	Actuarial present value of a pure endowment of 1.	
a	Actuarial present value of an annuity of 1 per time period, usually 1 year.	Annuities
s	Actuarial accumulated value of an annuity of 1 per time period, usually 1 year.	
(Ia)	Actuarial present value of an annuity payable at the rate of 1 per year at the end of the first year and increasing linearly at a rate of 1 per year.	
(Da)	Actuarial present value of a temporary annuity with an initial payment rate equal to the term and decreasing linearly at a rate of 1 per period.	
P	Level annual premium rate to cover only benefits, usually determined by the equivalence principle.	Premiums
V	Reserve to cover future benefits in excess of future benefit premiums.	Reserves
W	Face amount of a paid-up policy purchased with a cash value equal to the reserve. (Principle symbols for benefit premiums, reserves, and amounts of reduced paid-up insurance, P, V, and W, are combined with benefit symbols unless the benefit is a level unit insurance payable at the end of the year of death.)	[Examples: $\bar{P}_x$; $P(\bar{A}_x)$; ${}_{10}V^{(4)}\ (\bar{A}_{x:\overline{n}\rvert})$; $P^{(12)}({}_{30\rvert}\ddot{a}_{35}^{(12)})$]
S	Salary scale function used to project salaries.	Pensions
Z	Average of a given number of salary scale function values, usually at unit intervals in the independent variable.	

Section II. Lower Space to the Right

Auxiliary Symbol	*Description*	*Examples*
x; 10	A single letter or numeral is the individual's age at the commencement of the overall time period implied by the principal symbol.	a_x; $\ddot{a}_{10}$ q_x; ${}_5q_{10}$ $\bar{A}_x$; A_{10}
$\overline{n}\rvert$; $\overline{10}\rvert$	A term certain is indicated by a single letter or numeral under an angle.	$A_{x:\overline{n}\rvert}$; $\ddot{a}_{\overline{10}\rvert}$
$[x]$; [35] $[x] + t$; $[35 - n]$ $+ n$	Alphanumeric expressions enclosed by brackets indicate the age at which the life was selected. A term, representing duration since selection, may be added to the bracketed expression to express the attained age of the life.	$l_{[x]}$; $l_{[x]+10}$ $A_{[35]}$; $\ddot{a}^{i}_{[35-n]+n}$
xyz or $x{:}y{:}z$ $25{:}\overline{10}\rvert$	Two or more alphanumeric characters indicate a joint status that survives until the first death or expiration of the indicated lives and terms certain.	l_{xyz}; $A_{x:y:z}$ $\ddot{a}_{25:\overline{10}\rvert}$; $P_{25:\overline{10}\rvert}$
$\diagup\overline{\quad}\diagdown$	This symbol emphasizes the joint status when ambiguity is possible.	$A^{\;1}_{\diagup\overline{xy}\diagdown:z}$
$\overset{1}{x}{:}\overline{10}\rvert$; $\overset{2}{x}\underset{1}{y}z$	Numerals can be placed above or below the individual statuses of a collection of alphanumeric characters to show the order in which the units are to fail for an (insurable) event to occur. Benefits are payable upon the failure of the status with a numeral above it.	$\bar{A}^{1}_{x:\overline{n}\rvert}$; ${}_{\infty}q_{\underset{1}{x}\underset{2}{y}\overset{3}{z}}$
$\overline{xyz}$; $\overline{65{:}60{:}\overline{10}\rvert}$	A horizontal bar over a collection of alphanumeric characters defines a status that survives until the last survivor of the individual statuses fails.	$a_{\overline{xyz}}$; $\bar{A}_{\overline{xy:\overline{n}\rvert}}$
$\overline{xyz}^{\,r}$; $\overline{x{:}y{:}10}^{\,[r]}$	A single alphanumeric character, say, r, above the right end of the bar over the set of alphanumeric characters defines a status that survives as long as at least r of the individual statuses survive. If the r is enclosed in brackets, the status exists only while exactly r of the individual statuses survive.	$\bar{a}_{\overline{xyz}}^{\,[2]}$; $\bar{A}_{\overline{xyz}}^{\,2}$

Auxiliary Symbol	Description	Examples
$y\|x$; $60\|55$ $\overset{1}{y}z\|x$	A vertical bar separating the alphanumeric characters indicates that the income or coverage of the principal symbol commences upon the failure, as specified, of the status before the bar and continues until the failure of the status following the bar, providing the statuses fail in that order.	$a_{y\|x}$ $a_{\underset{1}{w}y\overset{3}{z}\|x}$

Section III. Lower Space to the Left

Auxiliary Symbol	Description	Examples
n; 15	A single alphanumeric character shows the time for which the principal symbol is evaluated. For an annual premium, P, this position shows the maximum number of years for which the premiums are paid if this is less than the period of coverage of an insurance or the period of deferral for a deferred annuity.	${}_np_x$; ${}_{15}E_{30}$ ${}_{20}P_{25}$; ${}_{20}V_{40:\overline{30\|}}$
$n\|m$; $n\|$	An alphanumeric pair separated by a vertical bar indicates a period of deferment (left of the bar) and a period following deferment (right of the bar). In some cases, when either is equal to 1 or infinity, it can be omitted.	${}_{n\|m}q_x$; ${}_{n\|}\bar{a}_x$

Section IV. Top Center

Auxiliary Symbol	Description	Examples
$\ddot{}$	The double dot (dieresis) on an annuity symbol indicates that the payments are at the beginning of the periods, that is, an annuity-due. Without the dieresis, the annuity is an annuity-immediate with payments at the ends of the periods.	$\ddot{a}_x$; $\ddot{s}_{\overline{40}\rvert}$
$\bar{}$	A horizontal bar indicates that the frequency of events is infinite. For annuities the payments are considered to be made continuously, and for insurances the benefit is paid at the moment of failure.	$\bar{a}_x$; $\bar{A}_x$ ${}_3\bar{V}_x$; $\bar{P}(\bar{A}_x)$
$\circ$	A circle (degree sign) means that the benefit or lifetime is complete, that is, credited up to the time of death.	$\mathring{a}_x$; $\mathring{e}_x$

Section V. Upper Space to the Right

Auxiliary Symbol	Description	Examples
(m); (12)	An alphanumeric character in parentheses shows the number of annuity payments in an interest period, usually 1 year. For an insurance it is the number of periods in a year at the end of which the death benefit can be paid. On multiple decrement symbols it indicates the cause of decrement to be used or that the total of all decrements is to be used.	$s_{\overline{10}\rvert}^{(12)}$; $A_x^{(m)}$ $q_x^{(2)}$; ${}_tp_x^{(\tau)}$
$\{m\}$; $\{12\}$	An alphanumeric character in braces shows the number of apportionable annuity-due payments in a time period, usually 1 year. On a principal symbol of a premium or a reserve, it shows that premiums are paid on this basis.	$\ddot{a}_{30:\overline{20}\rvert}^{\{12\}}$; $P_{30}^{\{1\}}$ ${}_tV^{\{2\}}(\bar{A}_x)$
r; i	An alphabetic character indicates the special basis used for the actuarial present value.	$\ddot{a}_{65}^{r}$; $\ddot{a}_{\{x\}}^{i}$

Section VI. Upper Space to the Left

Auxiliary Symbol	*Description*	*Examples*
h; 2	Alphanumeric character indicating the number of years during which premiums are paid if this is less than the coverage period of the insurance or the deferral period of the deferred annuity. This is used only on the principal symbols V or W where Position III is used for the time for which the function is evaluated.	${}^{h}_{5}V_{30}$
	In this text a new use for this position is to show that the actuarial present value of an annuity or an insurance is calculated at a multiple of the assumed force of interest.	${}^{2}\bar{A}_{x}$; ${}^{2}\ddot{a}_{20:\overline{10}\rvert}$

Appendix 5

SOME MATHEMATICAL FORMULAS USEFUL IN ACTUARIAL MATHEMATICS

The purpose here is not to recall familiar standard formulas and techniques, but to indicate some that may be less familiar to actuarial students.

Calculus

If

$$F(t) = \int_{\alpha(t)}^{\beta(t)} f(x, t)\, dx,$$

then

$$\frac{dF(t)}{dt} = \int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial t} f(x, t)\, dx + f(\beta(t), t) \frac{d}{dt}\beta(t)$$
$$- f(\alpha(t), t) \frac{d}{dt}\alpha(t).$$

Calculus of Finite Differences

Operators

a. Shift:

$$\mathrm{E}[f(x)] = f(x+1)$$

b. Difference:

$$\Delta f(x) = f(x+1) - f(x) = (\mathrm{E} - 1) f(x)$$

c. Repeated differences:

$$\begin{aligned}\Delta^n f(x) &= \Delta[\Delta^{n-1} f(x)] \\ &= (\mathrm{E} - 1)^n f(x) \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k)\end{aligned}$$

d. Difference of a product:

$$\Delta[f(x)\,g(x)] = f(x+1)\,\Delta g(x) + g(x)\,\Delta f(x)$$

e. Antidifference:

If

$$\Delta f(x) = g(x),$$

then

$$\Delta^{-1} g(x) = f(x) + w(x)$$

where

$$w(x) = w(x+1).$$

Applications

a. Representation of a polynomial (Newton's formula): Let $p_n(x)$ be a polynomial of degree n; then

$$p_n(x) = \sum_{k=0}^{n} \binom{x-a}{k} \Delta^k p_n(a).$$

b. Summation of series:

If

$$\Delta F(x) = f(x),$$

then

$$\begin{aligned} f(1) &= F(2) - F(1) \\ f(2) &= F(3) - F(2) \\ &\ \vdots \\ f(n) &= F(n+1) - F(n) \end{aligned}$$

$$\sum_{x=1}^{n} f(x) = F(n+1) - F(1) = \Delta^{-1} f(x)\Big|_{1}^{n+1}.$$

c. Summation by parts:

$$\sum_{x=1}^{n} g(x)\,\Delta f(x) = f(x)\,g(x)\Big|_{1}^{n+1} - \Delta^{-1}[f(x+1)\,\Delta g(x)]\Big|_{1}^{n+1}$$

[Proof: Sum each side of the equation for $\Delta[f(x)g(x)]$ from $x = 1$ to $x = n$.]

Probability Distributions

Discrete Distributions	p.f.	Restrictions on Parameters	Moment Generating Function, $M(s)$	Moments	
				Mean	Variance
Binomial	$\binom{n}{x}p^x q^{n-x}$, $x = 0, 1, \ldots n$	$0 < p < 1$ $q = 1 - p$	$(pe^s + q)^n$	np	npq
Bernoulli	Special case $n = 1$				
Negative Binomial	$\binom{r+x-1}{x}p^r q^x$, $x = 0, 1, 2, \ldots$	$0 < p < 1$ $q = 1 - p$ $r > 0$	$\left(\frac{p}{1 - qe^s}\right)^r$, $qe^s < 1$	$\frac{rq}{p}$	$\frac{rq}{p^2}$
Geometric	Special case $r = 1$				
Poisson	$\frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$	$\lambda > 0$	$e^{\lambda(e^s-1)}$	λ	λ
Uniform	$\frac{1}{n}$, $x = 1, \ldots, n$	n, a positive integer	$\frac{e^s(1 - e^{sn})}{n(1 - e^s)}$, $s \neq 0$ 1, $s = 0$	$\frac{n+1}{2}$	$\frac{n^2 - 1}{12}$

Continuous Distributions	p.d.f.	Restrictions on Parameters	Moment Generating Function, $M(s)$	Moments: Mean	Moments: Variance
Uniform	$\frac{1}{b-a}, \quad a < x < b$	—	$\frac{e^{bs} - e^{as}}{(b-a)s}, s \neq 0$ $1, s = 0$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$
Normal	$\frac{1}{\sigma\sqrt{2\pi}} \exp[-(x-\mu)^2/2\sigma^2]$, $-\infty < x < \infty$	$\sigma > 0$	$\exp(\mu s + \sigma^2 s^2/2)$	μ	σ^2
Gamma	$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0$	$\alpha > 0, \beta > 0$	$\left(\frac{\beta}{\beta - s}\right)^\alpha, s < \beta$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$
Exponential	Special case $\alpha = 1$				
Chi-square	Special case $\alpha = \frac{k}{2}, \beta = \frac{1}{2}$	k, a positive integer			
Inverse Gaussian	$\frac{\alpha}{\sqrt{2\pi\beta}} x^{-3/2} \exp\left[-\frac{(\beta x - \alpha)^2}{2\beta x}\right]$, $x > 0$	$\alpha > 0, \beta > 0$	$\exp\left[\alpha\left(1 - \sqrt{1 - \frac{2s}{\beta}}\right)\right]$, $s < \frac{\beta}{2}$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$
Pareto	$\alpha x_0^\alpha / x^{\alpha+1}, x > x_0$	$x_0 > 0, \alpha > 0$		$\frac{\alpha x_0}{\alpha - 1}$ $\alpha > 1$	$\frac{\alpha x_0^2}{(\alpha-2)(\alpha-1)^2}$ $\alpha > 2$
Lognormal	$\frac{1}{x\sigma\sqrt{2\pi}} \exp[-(\log x - m)^2/2\sigma^2]$, $x > 0$	$-\infty < m < \infty$ $\sigma > 0$		$e^{m+\sigma^2/2}$	$(e^{\sigma^2} - 1)e^{2m+\sigma^2}$

Appendix 6

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Appendix 7

ANSWERS TO EXERCISES

Chapter 1

1.1. a. and b.

w	$u(w)$	$u(w_1,w_2)$	$u(w_1,w_2,w_3)$
0	-1.00	125×10^{-6}	-48×10^{-10}
4 000	-0.500	93×10^{-6}	-34×10^{-10}
6 700	-0.250	78×10^{-6}	-14×10^{-10}
8 300	-0.125	74×10^{-6}	—
10 000	0.000	—	—

1.2. b. 2, 2 d. 2 log 2

1.3. c. Var(X)

1.7. a. Yes, for all w b. $(90 < w < 100) \cup (w > 110)$

1.11. $-\dfrac{n}{2\alpha}\log(1 - 2\alpha)$

1.12. a. $G = 400 \log \dfrac{13}{12} = 32.02$ b. $G = 150 \log \dfrac{3}{2} = 60.82$

1.13. a. 30 b. 26

1.14. Complete insurance

1.17. a. $-[1 - F(d)]$

1.18. a. 10, 100

1.19. a. $50, \dfrac{2{,}500}{3}$ b. $k = 0.25$, $d = 50$

c. $\text{Var}[X - I_1(X)] = 468.75$, $\text{Var}[X - I_2(X)] = 260.42$

Chapter 2

2.1. $\dfrac{1}{2}, \dfrac{19}{4}$

2.2. $\dfrac{1}{2}, \dfrac{77}{12}$

2.3. $\frac{35}{4}, \frac{1{,}085}{48}$

2.4. $\frac{7}{4}, \frac{77}{48}$

2.5. $\frac{49}{4}, \frac{735}{16}$

2.6. $\frac{a}{100}$ and $a^2\left(\frac{197}{30{,}000}\right)$

2.7.

x	$F_S(x)$
0	0.2268
1	0.2916
2	0.4374
3	0.6210
4	0.7434
5	0.8586
6	0.9018
7	0.9582
8	0.9762
9	0.9918
10	0.9948
11	0.9988
12	0.9996
13	1.0000

2.8. c. $\frac{1}{48}, \frac{1}{6}, \frac{1}{2}$

2.9. $\mathrm{E}[X] = \frac{\alpha}{\beta}$, $\mathrm{Var}(X) = \frac{\alpha}{\beta^2}$

2.10. $\mathrm{E}[X] = 1$, $\mathrm{Var}(X) = \frac{1}{3}$; $\mathrm{E}[Y] = \frac{3}{2}$, $\mathrm{Var}(Y) = \frac{3}{4}$; $\Pr(X + Y > 4) \cong 0.0748$;
$\Pr(X + Y > 4) = 0.0833$

2.11. a. $b = -1, c = 1, d = a$ or $b = 1, c = 0, d = -a$
b. 0.0228, 0.1587, 0.5000

2.12. a. 18, 36 b. 27.8713, 31.9607

2.13. a. 0.0041 b. 0.0045

2.14. 3.56; that is, 35,600

2.15. a. 6.4, 6.144 b. $7(10^4)$, $17.072(10^8)$ c. 1.37341

2.16. 0.0062

Chapter 3

3.1.

$s(x)$	$F(x)$	$f(x)$	$\mu(x)$
$\cos x$	$1 - \cos x$	$\sin x$	—
—	$1 - e^{-x}$	e^{-x}	1
$\dfrac{1}{1+x}$	—	$\dfrac{1}{(1+x)^2}$	$\dfrac{1}{1+x}$

3.2. a. $\exp\left[\dfrac{-B}{\log c}(c^x - 1)\right]$ b. $\exp(-ux^{n+1})$ $u = \dfrac{k}{n+1}$

c. $\left(1 + \dfrac{x}{b}\right)^{-a}$

3.3. $\mu(x) = \dfrac{x^2}{4}$, $f(x) = \dfrac{x^2}{4} e^{-x^3/12}$, $F(x) = 1 - e^{-x^3/12}$

3.4. a. $\int_0^\infty \mu(x)\,dx < \infty$ b. $s'(x) > 0$ for some x including $x = 1, 2$

c. $\int_0^\infty f_X(x)\,dx = 2^n\Gamma(n) > 1$ for $n \geq 1$

3.5. a. $\dfrac{1}{100 - x}$ b. $\dfrac{x}{100}$ c. $\dfrac{1}{100}$ d. $\dfrac{3}{10}$

3.6. a. $1 - \dfrac{t}{60}$ b. $\dfrac{1}{60 - t}$ c. $\dfrac{1}{60}$

3.7. a. $\dfrac{8}{9}$ b. $\dfrac{1}{8}$ c. $\dfrac{1}{8}$ d. $\dfrac{1}{128}$ e. $\dfrac{128}{3}$

3.9. 0.001994

3.10. $f_X(x) = \binom{10}{x}(0.77107)^x(0.22893)^{10-x}$ $x = 0, 1, 2, \ldots, 10$

$E[\mathcal{L}(x)] = 7.7107$, $\text{Var}[\mathcal{L}(x)] = 1.765211$

3.11. a. $\dfrac{9}{4}$ for each b. $\dfrac{27}{16}$ for each c. $-\dfrac{1}{3}$

3.12. a. ${}_5q_0 = 0.01505$ is more than 10 times ${}_5q_5 = 0.001503$
b. ${}_{55|5}q_{25} = 0.156729$

3.15. 1,436.19

3.18. a. $\dfrac{1}{c}$ b. $\dfrac{1}{c^2}$ c. $\dfrac{1}{c}\log 2$ d. 0

3.19. a. $te^{-t^2/2}$ b. $\sqrt{\dfrac{\pi}{2}}$

3.20. a. $\dfrac{(100 - x)}{2}$ b. $\dfrac{(100 - x)^2}{12}$ c. $\dfrac{(100 - x)}{2}$

3.23. a. $\mathring{e}_x = \dfrac{10 - x}{2} \quad x = 0, 1, 2, \ldots, 9$

$e_x = \dfrac{9 - x}{2} \quad x = 0, 1, 2, \ldots, 9$

3.24. a. $u(0) = e^{-\lambda}, \quad -\dfrac{c(x)}{d(x)} = 0, \quad \dfrac{1}{d(x)} = \dfrac{\lambda}{x + 1}$

b. $u(0) = (1 - p)^n, \quad -\dfrac{c(x)}{d(x)} = 0, \quad \dfrac{1}{d(x)} = \dfrac{(n - x)p}{(x + 1)(1 - p)}$

3.25. a. $u(0) = 0 \quad -\dfrac{c(x)}{d(x)} = 1 \quad \dfrac{1}{d(x)} = v$

b. $u(0) = 0 \quad -\dfrac{c(x)}{d(x)} = 1 + i \quad \dfrac{1}{d(x)} = 1 + i$

3.28. Uniform distribution: 0.989709
Constant force: 0.989656
Balducci: 0.989602

3.29. a. 77.59 b. 29.11

3.30. a. 0.044 b. 0.04421

3.31.

	Uniform Distribution	Constant Force	Balducci
a.	0.012696	0.012616	0.012537
b.	0.013676	0.013770	0.013865
c.	0.013770	0.013770	0.013770

3.35. a. $\dfrac{\alpha}{\omega - x}$ b. $\dfrac{\omega - x}{\alpha + 1}$

3.36. a. 0.000877 b. 0.999189

3.37. a. 0.4076 b. 0.1786

3.39. 0.97920

3.40. $\log\left(1 - \dfrac{q_{[x]}}{2}\right) - \log(1 - q_{[x]})$

3.41. $q'_x < 2q_x$

3.43. a. $\left(\dfrac{1 + Bc^x}{1 + B}\right)^{-A/(B \log c)}$

3.44. a. $\dfrac{5^7}{4^{10}}$ b. 77.2105

3.45. b. $-\log(1 - q_x)$ c. $\dfrac{-q_x^2}{(1 - q_x)\log(1 - q_x)}$ d. $\dfrac{1}{45}$

3.49. $q^1_{40} = 0.0055547, \quad q_{40} = 0.0027812$

3.50. Check value $e_{40} = 35.367$

3.51. $e_{20} = 46.038, \quad e_{40} = 28.366, \quad e_{60} = 13.264, \quad e_{80} = 3.889, \quad e_{100} = 0.503$

3.52. Check value $\mathring{e}_{40} = 35.867$

3.53. Starting value $e_{y:\overline{0}|} = 0$, Check value $e_{25:\overline{20}|} = 19.369$

3.54. Starting value $e_{\omega:\overline{10}|} = 0$, Check value $e_{40:\overline{10}|} = 9.809$

3.55. $e_{15:\overline{25}|} = 24.610$

Chapter 4

4.5. b. $\bar{A}^{1}_{x:\overline{n}|} = \dfrac{\mu_{x+n}}{\delta + \mu_{x+n}} A_{x:\overline{n}|}^{\ \ \ 1}$ c. $-\dfrac{\mu_{x+n}}{\delta + \mu_{x+n}} (A_{x:\overline{n}|}^{\ \ \ 1})^2$, where n satisfies (b)

d. $n = \dfrac{\log 2}{\mu + \delta}$, $\min \operatorname{Cov}(Z_1, Z_2) = -\dfrac{\mu}{4(\mu + \delta)}$

4.6. a. 0.237832 b. 0.416667

4.7. a. 0.092099 b. 0.055321

4.8. a. $\dfrac{20}{3(100 - x)}\left[1 - \left(\dfrac{20}{120 - x}\right)^3\right]$,

$$\frac{20}{7(100 - x)}\left[1 - \left(\frac{20}{120 - x}\right)^7\right] - \left\{\frac{20}{3(100 - x)}\left[1 - \left(\frac{20}{120 - x}\right)^3\right]\right\}^2$$

b. $\dfrac{20}{3(100 - x)}\left[10 - 10\left(\dfrac{20}{120 - x}\right)^2 - (100 - x)\left(\dfrac{20}{120 - x}\right)^3\right]$

4.10. a. $\dfrac{\mu}{(\mu + \delta)^2}$ b. $\mu\left[\dfrac{2}{(\mu + 2\delta)^3} - \dfrac{\mu}{(\mu + \delta)^4}\right]$

4.11. a. $f_Z(z) = \begin{cases} 0.2\, z^{(-0.8)} & 0 < z < 1 \\ 0.0 & \text{elsewhere} \end{cases}$

c. 1/6, 25/396.

4.12. $F_Z(z) = \begin{cases} 0.0 & z < v^n \\ 1.0 - F_T(\log z/\log v) & v^n \le z < 1 \\ 1.0 & 1 \le z \end{cases}$

4.13. a. $F_Z(z) = \begin{cases} 0.0 & z < e^{(-1.0)} \\ z^{0.2} & e^{(-1.0)} \le z < 1 \\ 1.0 & 1 \le z \end{cases}$

c. $1/6 + (5/6)e^{(-1.2)}$

4.14. a. 0.407159 b. 5.554541

4.16. a. 0.5 b. 0.05

4.17. b. $(IA)_{x:\overline{m}|} = (IA)^{1}_{x:\overline{m}|} + mA_{x:\overline{m}|}^{\ \ \ 1}$

4.19. a. $v^{[k+(j+1)/m]}$ b. $A_x^{(m)} = \sum_{k=0}^{\infty} v^{k+1}\, {}_kp_x \sum_{j=0}^{m-1} {}_{j/m|1/m}q_{x+k}\, (1 + i)^{[1-(j+1)/m]}$

4.23. $A_x + A^{1}_{x:\overline{65-x}|}$

4.24. 4,007.85

4.26. a. $\dfrac{9{,}100}{14 - k}$

b. $1{,}000{,}000[{}^2A_{x:\overline{n}|}^{\ \ \ 1} - (A_{x:\overline{n}|}^{\ \ \ 1})^2] + (k\pi)^2[{}^2\bar{A}^{1}_{x:\overline{n}|} - (\bar{A}^{1}_{x:\overline{n}|})^2] - 2{,}000\, k\pi\, \bar{A}^{1}_{x:\overline{n}|}\, A_{x:\overline{n}|}^{\ \ \ 1}$

where π is the net single premium in (a)

4.27. a. 0.307215

4.34. a. $A^{\,1}_{20:\overline{20|}} = 0.01827$
$^2A^{\,1}_{20:\overline{20|}} = 0.01143$
b. 110,933,839

4.35. b. $A_{\omega:\overline{0|}} = 0.0$

4.36. a. 3.06569 b. $(I\bar{A})_x = (\bar{A}^{\,1}_{x:\overline{1|}} + vp_x\,\bar{A}_{x+1}) + vp_x\,(I\bar{A})_{x+1}, \quad (I\bar{A})_\omega = 0$
c. $(\bar{I}\bar{A})_x = [(\bar{I}\bar{A})^{1}_{x:\overline{1|}} + vp_x\,\bar{A}_{x+1}] + vp_x\,(\bar{I}\bar{A})_{x+1}, \quad (\bar{I}\bar{A})_\omega = 0$
d. $(I\bar{A})_x = (d/\delta)[q_x + p_x\,\bar{A}_{x+1}] + vp_x\,(I\bar{A})_{x+1},$

$$(\bar{I}\bar{A})_x = \frac{d}{\delta}\left[\left(\frac{i-\delta}{i\delta}\right) q_x + p_x\,\bar{A}_{x+1}\right] + vp_x\,(\bar{I}\bar{A})_{x+1}$$

4.38. b. $\bar{A}_{y:\overline{0|}} = 1$ c. $\bar{A}_{45:\overline{65-45|}} = 0.34743$

4.39. mean = 38,056.82, variance = 42,337,224.63

Chapter 5

5.1. a. 16.008, 12.761, 5.397 b. 3.137, 10.230, 9.523

5.2. a. 0.111, 0.251, 0.572 b. 0.0251

5.4. $-\mathrm{Var}(v^T) = -({}^2\bar{A}_x - \bar{A}_x^2)$

5.6. a. $$F_y(y) = \begin{cases} 1-(1-\delta y)^{\mu/\delta} & 0 \le y < \dfrac{1}{\delta} \\ 1 & \dfrac{1}{\delta} \le y \end{cases}$$

b. $$F_y(y) = \begin{cases} 1-(1-\delta y)^{\mu/\delta} & 0 \le y < \bar{a}_{\overline{n|}} \\ 1 & \bar{a}_{\overline{n|}} \le y \end{cases}$$

c. $$F_y(y) = \begin{cases} 1-(v^n-\delta y)^{\mu/\delta} & 0 \le y < \dfrac{v^n}{\delta} \\ 1 & \dfrac{v^n}{\delta} \le y \end{cases}$$

d. $$F_y(y) = \begin{cases} 0 & 0 < y \le \bar{a}_{\overline{n|}} \\ 1-(1-\delta y)^{\mu/\delta} & \bar{a}_{\overline{n|}} < y < \dfrac{1}{\delta} \\ 1 & \dfrac{1}{\delta} \le y \end{cases}$$

5.7. $\ddot{a}_{x:\overline{n|}} = \ddot{a}_{x:\overline{1|}} - v^n\,{}_np_x\,\ddot{a}_{x+n:\overline{1|}} + vp_x\,\ddot{a}_{x+1:\overline{n|}} \qquad \begin{array}{l} n = 1, 2, \ldots \\ x = 0, 1, \ldots \end{array}$
$\ddot{a}_{\omega:\overline{n|}} = \ddot{a}_{\omega:\overline{1|}}$ which equals 1/2 by the trapezoidal rule

5.8. ${}_{n|}\ddot{a}_x = v^n\,{}_np_x\,\ddot{a}_{x+n:\overline{1|}} + vp_x\,{}_{n|}\ddot{a}_{x+1} \qquad \begin{array}{l} x = 0, 1, \ldots \\ n = 0, 1, 2, \ldots \end{array}$
${}_{n|}\ddot{a}_\omega = 0$

5.9. $\ddot{a}_{\overline{x:\overline{n|}}} = v^n\,{}_np_x\,\ddot{a}_{x+n:\overline{1|}} + \ddot{a}_{\overline{n|}}(1 - vp_x) + vp_x\,\ddot{a}_{\overline{x+1:\overline{n|}}}$
$\ddot{a}_{\overline{\omega:\overline{n|}}} = \ddot{a}_{\overline{n|}}$

5.14. $\dfrac{2}{i}\,(a_{x:\overline{n|}} - {}^2a_{x:\overline{n|}}) - {}^2a_{x:\overline{n|}} - (a_{x:\overline{n|}})^2$

5.15. $\dfrac{s_{\overline{1}|}^{(m)} - 1}{d^{(m)}} A_x$

5.16. a. $\dfrac{1}{m} \sum_{h=0}^{m-1} v^{h/m} \ {}_{h/m}p_x + \dfrac{1}{m} \sum_{h=m}^{(y-x)m-1} v^{h/m} \ {}_{h/m}p_x$

b. $\alpha(m) - \beta(m)(1 - vp_x)$

c. $c(x) = \alpha(m) - \beta(m)(1 - vp_x), \quad d(x) = vp_x, \quad \ddot{a}_{y:\overline{0}|}^{(m)} = 0$

5.17. $\ddot{a}_{x:\overline{n}|} - \dfrac{m-1}{2m} (1 - {}_nE_x), \quad {}_{n|}\ddot{a}_x - \dfrac{m-1}{2m} \ {}_nE_x$

5.22. a. $\alpha(m) \ \ddot{s}_{25:\overline{40}|} - \beta(m) \left(\dfrac{1}{{}_{40}E_{25}} - 1 \right)$ b. (i) 15.038 (ii) 196.380

5.23. a. $Y = \begin{cases} (I\ddot{a})_{\overline{K+J+1/m}|}^{(m)} & K = 0, 1, \ldots, n-1, \quad J = 0, 1, \ldots, m-1 \\ (I\ddot{a})_{\overline{n}|}^{(m)} & K = n, n+1, \ldots \end{cases}$

5.24. a. $Y = \begin{cases} (D\ddot{a})_{\overline{K+J+1/m}|}^{(m)} & K = 0, 1, \ldots, n-1, \quad J = 0, 1, \ldots, m-1 \\ (D\ddot{a})_{\overline{n}|}^{(m)} & K = n, n+1, \ldots \end{cases}$

5.25. a. $Y = \begin{cases} (I\ddot{a})_{\overline{K+J+1/m}|}^{(m)} & K = 0, 1, \ldots, n-1, \quad J = 0, 1, \ldots, m-1 \\ n \ \ddot{a}_{\overline{K+J+1/m}|}^{(m)} & K = n, n+1, \ldots, \quad J = 0, 1, \ldots, m-1 \end{cases}$

5.31. a. $\ddot{a}_x + 0.03(Ia)_x$

b. $\sum_{k=0}^{\infty} (1.03)^k \ v^k \ {}_kp_x = \ddot{a}'_x$ evaluated at interest rate $i' = \dfrac{i - 0.03}{1.03}$

5.32. $\int_0^n (n - t) \ v^t \ {}_tp_x \ dt$

5.33. $1{,}200 \left(\dfrac{a_{30}^{(12)} + {}_{10|}a_{30}^{(12)} + 3 \ {}_{20|}a_{30}^{(12)} + 5 \ {}_{30|}a_{30}^{(12)} - 10 \ {}_{40|}a_{30}^{(12)}}{{}_{40}E_{30}} \right)$

5.34. $\bar{a}_{35:\overline{25}|} - {}_{25}p_{35} \ \bar{a}_{\overline{25}|}$

5.35. $\ddot{a}_{x:\overline{n}|} - {}_np_x \ \ddot{a}_{\overline{n}|}$

5.36. $\dfrac{1}{12} \ddot{a}_{x:\overline{25}|} - \dfrac{25}{12} \ {}_{25}E_x$

5.38. $v^{2n} \ {}_np_x \ (1 - {}_np_x) \ \ddot{a}_{x+n}^2 + v^{2n} \ {}_np_x \ \dfrac{{}^2A_{x+n} - A_{x+n}^2}{d^2}$

5.41. a. $\alpha(m) = 1 + \dfrac{m^2 - 1}{12m^2} \delta^2 + \dfrac{2m^4 - 5m^2 + 3}{720m^4} \delta^4 + \cdots$

$\beta(m) = \dfrac{m-1}{2m} \left[1 + \dfrac{m+1}{3m} \delta + \dfrac{m(m+1)}{12m^2} \delta^2 + \dfrac{(m+1)(6m^2 - 4)}{360m^3} \delta^3 + \cdots \right]$

b. $\alpha(\infty) = 1 + \dfrac{1}{12} \delta^2 + \dfrac{1}{360} \delta^4 + \cdots$

$\beta(\infty) = \dfrac{1}{2} \left[1 + \dfrac{1}{3} \delta + \dfrac{1}{12} \delta^2 + \dfrac{1}{60} \delta^3 + \cdots \right]$

5.44. $\frac{I}{\delta} + \left(J - \frac{I}{\delta}\right) v^T, \frac{I}{\delta} + \left(J - \frac{I}{\delta}\right) \bar{A}_x, \left(J - \frac{I}{\delta}\right)^2 ({}^2\bar{A}_x - \bar{A}_x^2)$

5.45. a. 14.353 b. 13.350 c. 1.002

5.51. a. 488.23 b. 700.48 c. 531.77

5.53. $\ddot{a}_{55:\overline{10}|} = 7.45735$

5.54. $\bar{a}_{55:\overline{10}|} = 7.19783$

5.56. $\bar{a}_{55:\overline{10}|} = 7.19783$

5.57. $\bar{a}_{60:\overline{10}|} = 6.46348, \quad \mathrm{Var}(X) = 1.82621$

5.58. $\ddot{a}_{65}^{(12)} = 10.13343, \quad \mathrm{Var}(Y) = 16.87662$

5.59. 10.41532

Chapter 6

6.1. −0.43202, 0.39760

6.3. 0.303598

6.4. a. 0.02 b. 0.00857 c. 0.02885

6.6. $\dfrac{\mu}{\mu + 2\delta} = {}^2\bar{A}_x$

6.10.

	Annual Premiums for (35)		
Insurance	**Fully Continuous**	**Semicontinuous**	**Fully Discrete**
10-Year endowment	0.075128	0.072885	0.072810
30-Year endowment	0.015371	0.014894	0.014751
60-Year endowment	0.008913	0.008621	0.008374
Whole life	0.008903	0.008611	0.008362
30-Year term	0.005117	0.004958	0.004815
10-Year term	0.002669	0.002589	0.002514

6.12. $A_x = \dfrac{1 - r}{1 + i - r}, \; P_x = \dfrac{1 - r}{1 + i},$

$\ddot{a}_x = \dfrac{1 + i}{1 + i - r},$

$\dfrac{{}^2A_x - A_x^2}{(d\ddot{a}_x)^2} = \dfrac{(1 - r)r}{1 + 2i + i^2 - r}$

6.13. 0.019139

6.15. 0.032868

6.16. 0.0413

6.17. With the common $(\bar{A}_{40:\overline{25}|})$ omitted from the premium symbols,
$P \le P^{(2)} \le P^{\{4\}} \le P^{\{12\}} \le \bar{P}$

6.18. $\dfrac{100}{99}$

6.19. 740.93

6.21. $P(A'^{\,1}_{45:\overline{20}|})$ where $A'^{\,1}_{45:\overline{20}|}$ is the actuarial present value of a 20-year term insurance on (45) under which $b_{k+1} = \ddot{s}_{\overline{k+1}|}$

6.22. a. 11.5451, 20.4106 b. 6.3099, 25.6458

6.24. ${}_{25}P_{40}$

6.25. b. $P^{(12)}(A^{\;(12)}_{\overset{1}{65}:\overline{10}|}) + d^{(12)}$

6.26. $\dfrac{100{,}000}{(1.1\ \ddot{s}_{\overline{30}|} - 0.1\ \ddot{s}_{35:\overline{30}|})}$

6.27. 0.008

6.28. $\dfrac{11{,}000\ A_x + 25\ \ddot{a}_{x:\overline{20}|}}{\ddot{a}_{x:\overline{20}|} - 1.1(I_{\overline{20}|}A)_x}$

6.29. $\dfrac{2\ A_{25} - A^{\,1}_{25:\overline{10}|}}{2\ \ddot{a}_{25:\overline{40}|} - \ddot{a}_{25:\overline{10}|}}$

6.30. $L_1 = v^T - \bar{P}(\bar{A}_x)\bar{a}_{\overline{T}|} \equiv 1 - \left(\dfrac{1}{\bar{a}_x}\right)\bar{a}_{\overline{T}|} = L_2$

6.31. a. -0.08 b. 0.1296 c. 0.1587

6.32. $\dfrac{\bar{A}_x}{2\ddot{a}_x - \ddot{a}_{x:\overline{5}|}}$

6.33. ${}_{20}P^{\{m\}}(\bar{A}_x) - {}_{20}P^{(m)}(\bar{A}_x) = {}_{20}\bar{P}(\bar{A}_x)\left(\dfrac{\bar{A}^{\,1}_{x:\overline{20}|} - A^{\,1\,(m)}_{x:\overline{20}|}}{\delta\ddot{a}^{(m)}_{x:\overline{20}|}}\right)$

6.35. a. $F_L(u) = \begin{cases} \dfrac{\mu}{\delta u + \bar{P}}\left(\dfrac{\delta u + \bar{P}}{\delta + \bar{P}}\right)^{\mu/\delta} & -\dfrac{\bar{P}}{\delta} < u < 1 \\ 0 & \text{elsewhere} \end{cases}$

6.36. a. $\dfrac{1}{\sqrt{3}}$

b. 0.02

Chapter 7

7.1. ${}_1V = 0.15111$ ${}_2V = 0.30809$ ${}_3V = 0.47118$ ${}_4V = 0.64067$

7.2. ${}_1V = 0.14925$ ${}_2V = 0.30492$ ${}_3V = 0.46741$ ${}_4V = 0.63712$

7.3. ${}_1V = 1.2871$ ${}_2V = 2.6996$ ${}_3V = 4.2553$ ${}_4V = 5.9748$

7.4. ${}_1V = 0.15064$ ${}_2V = 0.30730$ ${}_3V = 0.47025$ ${}_4V = 0.63980$

7.5. a. $1 = \dfrac{1}{5}\displaystyle\int_0^5 e^{0.1[1.06^{-t}(1+\bar{P}/\delta)-\bar{P}/\delta]}\,dt$ where $\delta = \log(1.06)$

${}_1\bar{V} = 10\log\left(\dfrac{1}{4}\displaystyle\int_0^4 e^{0.1[1.06^{-t}(1+\bar{P}/\delta)-\bar{P}/\delta]}\,dt\right)$ where $\bar{P}$ and δ are as in (a)

b. $\bar{P} = 0.388380, \quad {}_1\bar{V} = 0.182825$

7.6. $${}_tL = \begin{cases} v^U - \bar{P}(\bar{A}_{x:\overline{n}|})\, \bar{a}_{\overline{U}|} & U < n - t \\ v^{n-t} - \bar{P}(\bar{A}_{x:\overline{n}|})\, \bar{a}_{\overline{n-t}|} & U \ge n - t \end{cases}$$

7.7. $\mathrm{E}[{}_tL] = \bar{a}_{x+t:\overline{n-t}|}$, $\mathrm{Var}({}_tL) = \dfrac{{}^2\bar{A}_{x+t:\overline{n-t}|} - \bar{A}^2_{x+t:\overline{n-t}|}}{\delta^2}$

7.8. a. $\bar{A}_{45:\overline{20}|} - {}_{20}\bar{P}(\bar{A}_{35:\overline{30}|})\, \bar{a}_{45:\overline{10}|}$ b. $\bar{A}_{50:\overline{5}|}^{\;\;\,1}$

7.9. a. $u_0 = \dfrac{-\log(\bar{A}_x)}{\delta}$ b. 23.2476

7.10. 41.7524

7.11. $F_{{}_tL}(y) = 0 \qquad y < v^{n-t} - \bar{P}(\bar{A}_{x:\overline{n}|})\, \bar{a}_{\overline{n-t}|}$

$$F_{{}_tL}(y) = \frac{1 - F_{T(x)}\left(t - \dfrac{1}{\delta} \log \dfrac{\delta y + \bar{P}(\bar{A}_{x:\overline{n}|})}{\delta + \bar{P}(\bar{A}_{x:\overline{n}|})}\right)}{1 - F_{T(x)}(t)} \qquad v^{n-t} - \bar{P}(\bar{A}_{x:\overline{n}|})\, \bar{a}_{\overline{n-t}|} \le y < 1$$

$F_{{}_tL}(y) = 1 \qquad y \ge 1$

7.12. $F_{{}_tL}(y) = 0 \qquad y < \bar{P}(\bar{A}^{\,1}_{x:\overline{n}|})\, \bar{a}_{\overline{n-t}|}$

$$F_{{}_tL}(y) = \frac{1 - F_{T(x)}(n)}{1 - F_{T(x)}(t)} = \frac{{}_np_x}{{}_tp_x} \qquad -\bar{P}(\bar{A}_{x:\overline{n}|})\, \bar{a}_{\overline{n-t}|} \le y < v^{n-t} - \bar{P}(\bar{A}^{\,1}_{x:\overline{n}|})\, \bar{a}_{\overline{n-t}|}$$

$$F_{{}_tL}(y) = \frac{1 - F_{T(x)}\left(t - \dfrac{1}{\delta} \log \dfrac{\delta y + \bar{P}(\bar{A}^{\,1}_{x:\overline{n}|})}{\delta + \bar{P}(\bar{A}^{\,1}_{x:\overline{n}|})}\right)}{1 - F_{T(x)}(t)} \qquad v^{n-t} - \bar{P}(\bar{A}^{\,1}_{x:\overline{n}|}) \le y < 1$$

7.14. $\bar{A}_{50} - {}_{20}\bar{P}(\bar{A}_{40})\, \bar{a}_{50:\overline{10}|}$, $[{}_{10}P(A_{50}) - {}_{20}P(A_{40})]\, \bar{a}_{50:\overline{10}|}$,

$\left[1 - \dfrac{{}_{20}\bar{P}(\bar{A}_{40})}{{}_{10}\bar{P}(\bar{A}_{50})}\right] \bar{A}_{50}$, ${}_{20}\bar{P}(\bar{A}_{40})\, \bar{s}_{40:\overline{10}|} - {}_{10}\bar{k}_{40}$

7.15. $\bar{A}_{50:\overline{10}|} - \bar{P}(\bar{A}_{40:\overline{20}|})\, \bar{a}_{50:\overline{10}|}$, $[\bar{P}(\bar{A}_{50:\overline{10}|}) - \bar{P}(\bar{A}_{40:\overline{20}|})]\, \bar{a}_{50:\overline{10}|}$,

$\left[1 - \dfrac{\bar{P}(\bar{A}_{40:\overline{20}|})}{\bar{P}(\bar{A}_{50:\overline{10}|})}\right] \bar{A}_{50:\overline{10}|}$, $\bar{P}(\bar{A}_{40:\overline{20}|})\, \bar{s}_{40:\overline{10}|} - {}_{10}\bar{k}_{40}$,

$1 - \dfrac{\bar{a}_{50:\overline{10}|}}{\bar{a}_{40:\overline{20}|}}$, $\dfrac{\bar{P}(\bar{A}_{50:\overline{10}|}) - \bar{P}(\bar{A}_{40:\overline{20}|})}{\bar{P}(\bar{A}_{50:\overline{10}|}) + \delta}$, $\dfrac{\bar{A}_{50:\overline{10}|} - \bar{A}_{40:\overline{20}|}}{1 - \bar{A}_{40:\overline{20}|}}$

7.16. $\bar{P}({}_{30|}\bar{a}_{35})\, \bar{s}_{35:\overline{20}|}$

7.18. (7.3.3)

7.19. $A_{50} - {}_{20}P_{40}\, \ddot{a}_{50:\overline{10}|}$, $({}_{10}P_{50} - {}_{20}P_{40})\, \ddot{a}_{50:\overline{10}|}$, $\left(1 - \dfrac{{}_{20}P_{40}}{{}_{10}P_{50}}\right) A_{50}$,

${}_{20}P_{40}\, \ddot{s}_{40:\overline{10}|} - {}_{10}k_{40}$

7.20. $A_{50:\overline{10}|} - P_{40:\overline{20}|}\, \ddot{a}_{50:\overline{10}|}$, $(P_{50:\overline{10}|} - P_{40:\overline{20}|})\, \ddot{a}_{50:\overline{10}|}$,

$\left(1 - \dfrac{P_{40:\overline{20}|}}{P_{50:\overline{10}|}}\right) A_{50:\overline{10}|}$, $P_{40:\overline{20}|}\, \ddot{s}_{40:\overline{10}|} - {}_{10}k_{40}$,

$1 - \dfrac{\ddot{a}_{50:\overline{10}|}}{\ddot{a}_{40:\overline{20}|}}$, $\dfrac{P_{50:\overline{10}|} - P_{40:\overline{20}|}}{P_{50:\overline{10}|} + d}$, $\dfrac{A_{50:\overline{10}|} - A_{40:\overline{20}|}}{1 - A_{40:\overline{20}|}}$

7.22. $\dfrac{1}{5}$

7.23.

Insurance	Fully Continuous	Semicontinuous	Fully Discrete
30-Year endowment	0.17530	0.17504	0.17407
Whole life	0.08604	0.08566	0.08319
30-Year term	0.03379	0.03370	0.03273

7.24. (b) and (c)

7.26. All but (d)

7.27. All

7.29. 0.008

7.30. 0.240

7.31. a. 0.005527
b. 0.051255
c. 0.946122
d. 0.132109

7.32. a. 0.0241821 b. 0.0189660

Chapter 8

8.1. a. $\dfrac{1 - r}{1 + i}$

b. $\dfrac{(1 - r)(1 + i + r)}{(1 + i)(1 + i - r)}$

8.2. $\dfrac{\int_0^\infty b_t\, v^t\, {}_tp_x\, \mu_x(t)\, dt}{\int_0^\infty w(t)\, v^t\, {}_tp_x\, dt}$

8.3. a. $\dfrac{\mu}{\delta + \mu}$

b. $\dfrac{\mu t}{\delta + \mu}$

8.5. a. $(P_{x+1} - vq_{x+h})\, {}_hp_x\, vq_x$
b. $(P_{x+1} - vq_{x+h})\, {}_hp_x(v\, {}_jp_x\, q_{x+j} + {}_jq_x\, P_{x+1})$

c. If $P_{x+1} - vq_{x+n} < 0$, then $\mathrm{Cov}(C_j, C_h) < 0$ for all $j < h$

8.6. If $1 - vq_{x+h}\,\ddot{s}_{\overline{h+1}|} < 0$, then $\mathrm{Cov}(C_j, C_h) < 0$ for all $j < h$

8.13. $(\bar{A}_{x:\overline{40}|})$ is omitted from the reserve and premium symbols

a. $\frac{1}{2}\,{}_{20}V + \frac{1}{2}\,{}_{21}V + \frac{1}{2}\,P$ b. $\frac{1}{2}\,{}_{20}\bar{V} + \frac{1}{2}\,{}_{21}\bar{V}$

c. $\frac{1}{2}\,{}_{20}V^{(2)} + \frac{1}{2}\,{}_{21}V^{(2)}$ d. $\frac{1}{3}\,{}_{20}V^{(2)} + \frac{2}{3}\,{}_{21}V^{(2)} + \frac{1}{3}\,P^{(2)}$

e. Same as (b) f. $\frac{1}{3}\,{}_{20}\bar{V} + \frac{2}{3}\,{}_{21}\bar{V} + \frac{1}{3}\,P^{\{2\}}$

8.14. 0.05448

8.17. b. $\mathrm{Var}(L) = 0.076090$

8.18. a. 0.0067994 b. 0.1858077 c. 0.2012024
d. 0.0275369 e. 0.0255406

8.21. $-{}_tp_x[\delta\ {}_t\bar{V}(\bar{A}_x) + \bar{P}(\bar{A}_x)]$

8.22. a. ${}_tp_x\ [\pi_t + \delta_t {}_t\bar{V} - b_t\ \mu_x(t)]$
b. $v^t[\pi_t + \mu_x(t)\ {}_t\bar{V} - b_t\ \mu_x(t)]$
c. $v^t\ {}_tp_x\ [\pi_t - b_t\ \mu_x(t)]$

8.26. a. and b. 1,491.03 c. 343.84 d. 0

8.27. a. 1,490,915
b. 6,450,962; 1,495,093, which is 1.00280 times the reserve
c. 5,311,375; supplement is 3,791, which is 0.00254 times the reserve
d. For b.: 645,096,250; 149,133,281, which is 1.00028 times the reserve
For c.: 531,137,500; supplement is 37,911, which is 0.00025 times the reserve

8.28. a. 1,104,260 is the reserve for these policies
b. 6,450,962; 1,108,438, which is 1.00378 times the reserve
c. 5,311,375; supplement is 3,791, which is 0.00343 times the reserve
d. For b.: 645,096,250; 110,467,781, which is 1.00038 times the reserve
For c.: 531,137,500; supplement is 37,911, which is 0.00034 times the reserve

8.29. $5{,}000[{}_{10}\bar{V}(\bar{A}_{30}) + P^{\{1\}}(\bar{A}_{30}) + {}_{11}\bar{V}(\bar{A}_{30})]$

8.30. a. 0.2 b. 0.25 c. 0.7584 d. 0.27

8.32. 0.081467

8.34. a. 355.6563
b. 2,614.2511

8.35. a. $100{,}000\ \dfrac{A^{\ 1}_{35:\overline{30}|}}{1 - A^{\ 1}_{35:\overline{30}|}}$

b. $100{,}000\ A^{\ 1}_{35+k:\overline{30-k}|} + SA^{\ 1}_{35+k:\overline{30-k}|}$

c. $\dfrac{S - SA^{\;1}_{35:\overline{k}|}}{A^{\;1}_{35:\overline{k}|}}$

d. $S = 18{,}575.08, \quad {}_{20}V = 53{,}962.62$

Chapter 9

9.1. a. $1 - \dfrac{1}{(1+s)^{n-2}} - \dfrac{1}{(1+t)^{n-2}} + \dfrac{1}{(1+s+t)^{n-2}} \qquad s > 0, t > 0$

b. $f_{T(x)}(s) = \dfrac{n-2}{(1+s)^{n-1}} \qquad s > 0$

$F_{T(x)}(s) = 1 - \dfrac{1}{(1+s)^{n-2}} \qquad s > 0$

$\mu(x+s) = \dfrac{n-2}{1+s} \qquad s > 0$

c. $\text{Cov}[T(x), T(y)] = \dfrac{1}{(n-4)(n-3)^2} \qquad \rho_{T(x)T(y)} = \dfrac{1}{n-2}$

9.2. $\dfrac{1}{(1+s+t)^{n-2}} \qquad s \geq 0, t \geq 0$

9.3. $F_{T(x)T(y)}(s, t) = \left[1 - \dfrac{1}{(1+s)^{n-2}}\right]\left[1 - \dfrac{1}{(1+t)^{n-2}}\right] \qquad s > 0, t > 0$

$s_{T(x)T(y)}(s, t) = \dfrac{1}{(1+s)^{n-2}} \dfrac{1}{(1+t)^{n-2}} \qquad s \geq 0, t \geq 0$

9.4. a. ${}_np_x \, {}_np_y$

b. ${}_np_x + {}_np_y - 2 \, {}_np_x \, {}_np_y$

c. ${}_np_x + {}_np_y - {}_np_x \, {}_np_y$

d. $1 - {}_np_x \, {}_np_y$

e. Same as for (d)

f. $(1 - {}_np_x)(1 - {}_np_y) = 1 - {}_np_x - {}_np_y + {}_np_x \, {}_np_y$

9.6. ${}_nq_{xx}$

9.7. $F_{T(xy)}(t) = 1 - \dfrac{1}{(1+2t)^{n-2}} \qquad t > 0$

$s_{T(xy)}(t) = \dfrac{1}{(1+2t)^{n-2}} \qquad t \geq 0$

$\text{E}[T(xy)] = \dfrac{1}{2(n-3)}$

9.8. $f_{T(xy)}(t) = \begin{cases} \dfrac{(10-t)^3}{2{,}500} & 0 < t < 10 \\ 0 & \text{elsewhere} \end{cases}$

9.10. ${}_{n|}q_x + {}_{n|}q_y - {}_{n|}q_x \, {}_{n|}q_y$

No, since for ${}_{n|}q_{\overline{xy}}$ the second death must occur in year $n + 1$, and this is not the case for the requested probability.

9.11. a. $F_{T(\overline{xy})}(t) = 1 - \frac{2}{(1+t)^{n-2}} + \frac{1}{(1+2t)^{n-2}} \qquad t > 0$

$$f_{T(\overline{xy})}(t) = 2(n-2)\left[\frac{1}{(1+t)^{n-1}} - \frac{1}{(1+2t)^{n-1}}\right] \qquad t > 0$$

b. $\mathrm{E}[T(\overline{xy})] = \frac{3}{2(n-3)}$

c. $\mu_{\overline{xy}}(t) = \frac{2(n-2)\,[1/(1+t)^{n-1} - 1/(1+2t)^{n-1}]}{2/(1+t)^{n-2} - 1/(1+2t)^{n-2}} \qquad t > 0$

9.12. $\frac{2}{9}$

9.13. a. $\frac{2}{3}$ b. $\frac{29}{30}$ c. 18.06 d. 36.94 e. 160.11 f. 182.33
g. 82.95 h. 0.49

9.14. $\mu_{xx}(0)\, \mathring{e}_{xx} - 1$

9.17. $\frac{531}{2{,}000}$

9.18. a. $\frac{1}{\alpha}\log\left[1 + \frac{(e^{\alpha t}-1)^2}{e^{\alpha}-1}\right]$ b. $\frac{(e^{\alpha t}-1)(e^{\alpha}-1)}{(e^{\alpha}-1) + (e^{\alpha t}-1)^2}$

9.19. $\frac{1}{\alpha}\log\left[1 + \frac{(e^{0.05\alpha}-1)(e^{0.03\alpha}-1)}{e^{\alpha}-1}\right]$

9.20. a. 0.001500 b. 0.000266 c. 0.004232

9.21. An annuity of 1 payable at the end of each year for n years and for as long thereafter as (xy) exists.

9.22. An insurance of 1 payable on the death of (x), or at the end of n years, whichever is later.

9.24. $\bar{a}_{25:\overline{25}|} + \bar{a}_{30:\overline{20}|} - \bar{a}_{25:30:\overline{20}|}$

9.25. ${}_{20|}a_{30} + {}_{25|}a_{25} - {}_{25|}a_{25:30}$

9.26. $\frac{1}{6}\,\ddot{a}_{xy:\overline{n}|} + \frac{1}{2}\,\ddot{a}_{y:\overline{n}|} + \frac{1}{3}\,\ddot{a}_{x:\overline{n}|}$

9.27. $a_{x:\overline{n}|} + v^n\,{}_np_x\,a_{x+n:y:\overline{m-n}|}$

9.28. ${}_{5|}\bar{a}_{55} + {}_{20|}\bar{a}_{40} - {}_{5|10}\bar{a}_{40:55} - {}_{20|}\bar{a}_{40:55}$

9.29. a. $[\ddot{a}_x^{(m)} + p(\ddot{a}_y^{(m)} - \ddot{a}_{xy}^{(m)})]$ b. $\frac{\ddot{a}_x^{(m)}}{\ddot{a}_x^{(m)} + p(\ddot{a}_y^{(m)} - \ddot{a}_{xy}^{(m)})}$

9.32. a. 7.0753 b. 7.0756

9.35. $w = \frac{3}{5}x + \frac{2}{5}y$

9.37. $\frac{1}{3}$

9.41. ${}_\infty q_{xy}^{1} = {}_\infty q_{xy}^{\ 2}$

9.44. $\bar{A}_{50} - \bar{A}^{\;\;1}_{50:20:\overline{20}|}$

9.45. $\bar{A}^{1}_{x:\overline{n}|} - \bar{A}^{1}_{xy} + {}_{n}E_x\, \bar{A}^{\;\;1}_{\overline{x+n}:y}$

9.46. $\dfrac{1}{12}$

9.47. a. 0.2755 b. $\dfrac{1}{4}\,\bar{A}_{40:50} + 0.0015\bar{a}_{40:50}$

9.48. $\dfrac{1}{3}$, 52.68

9.49. a. $\bar{a}_x + \bar{a}_{\overline{n}|} - \bar{a}_{x:\overline{n}|}$ b. $v^n\ {}_{n}q_x$

9.51. $\mu(x)\ \mathring{e}_{xy} - {}_{\infty}q^{1}_{xy}$

9.52. a. $(\mu_2 + \lambda)e^{-(\mu_2+\lambda)t}$ $\quad t > 0$

b. $e^{-(\mu_2+\lambda)t}$ $\quad t \geq 0$

c. $\dfrac{\lambda}{\mu_1 + \mu_2 + \lambda}$

Chapter 10

10.1. a. $e^{-t\mu_x^{(\tau)}}\ \mu_x^{(j)}$ b. $\dfrac{\mu_x^{(j)}}{\mu_x^{(\tau)}}$ c. $e^{-t\mu_x^{(\tau)}}\ \mu_x^{(\tau)}$

10.2. a. $\dfrac{j(50 - t)^2}{50^3}$ b. $\dfrac{3(50 - t)^2}{50^3}$ c. $\dfrac{j}{3}$ d. $\dfrac{j}{3}$

10.3. a. $f_T(t) = p(u_1 + v_1)e^{-(u_1+v_1)t} + (1 - p)(u_2 + v_2)e^{-(u_2+v_2)t}$,

$f_J(1) = \dfrac{pu_1}{u_1 + v_1} + (1 - p)\,\dfrac{u_2}{u_2 + v_2},\quad f_J(2) = p\,\dfrac{v_1}{u_1 + v_1} + (1 - p)\,\dfrac{v_2}{u_2 + v_2}$

b. $s_T(t) = pe^{-(u_1+v_1)t} + (1 - p)e^{-(u_2+v_2)t}$

10.4. ${}_{3}p_{65}^{(\tau)} = 0.75321$, ${}_{3|}q_{65}^{(1)} = 0.03766$, ${}_{3}q_{65}^{(2)} = 0.16504$

10.5. a. 302.4 and 210.95 b. 231.0 and 177.64

10.6. a. $h(1) = 0.231$, $h(2) = 0.4666$, $h(3) = 0.3024$

b. $h(1|k = 2) = 0.25$, $h(2|k = 2) = 0.75$, $h(3|k = 2) = 0$

10.7. $l_x^{(\tau)} = (a - x)e^{-x}$, $d_x^{(1)} = e^{-x}(1 - e^{-1})$,

$d_x^{(2)} = (a - x - 1)e^{-x} - (a - x - 2)e^{-x-1}$

10.8. $1{,}000\left(\dfrac{a - x^2}{a}\right)e^{-cx}$

10.9. a. ${}_{t}p_x^{(\tau)}\,(\mu_{(x+t)}^{(\tau)} - \mu_{(x)}^{(\tau)})$ b. ${}_{t}p_x^{(\tau)}\,\mu_{(x+t)}^{(j)} + {}_{t}q_x^{(j)}\,\mu_{(x)}^{(\tau)} - \mu_{(x+t)}^{(j)}$ c. ${}_{t}p_x^{(\tau)}\,\mu_{(x+t)}^{(j)}$

10.10.

k	$q_k'^{(1)} = 1 - (p_k^{(\tau)})^{q_k^{(1)}/q_k^{(\tau)}}$	$q_k'^{(2)} = 1 - (p_k^{(\tau)})^{q_k^{(2)}/q_k^{(\tau)}}$
0	0.17433	0.27332
1	0.11210	0.21163
2	0.05426	0.15410
3	0.00000	0.10000

10.11. a. $1 - e^{-c}$ b. c c. $c \int_0^1 {}_tp_x^{(\tau)}\, dt$

10.13. $m_x'^{(j)} \geq q_x'^{(j)} \geq q_x^{(j)}$

10.14. 0.0592

10.15. a. 0.0909 b. 0.0906

10.16.

k	$m_k^{(1)}$	$m_k^{(2)}$
0	0.18750	0.31250
1	0.11765	0.23529
2	0.05556	0.16667
3	0.00000	0.10526

10.17.

x	$p_x^{(\tau)}$	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$
62	0.76048	0.01767	0.02665	0.19520
63	0.85027	0.02054	0.03193	0.09726
64	0.82115	0.02578	0.03705	0.11603

10.18.

x	$m_x^{(1)} = m_x'^{(1)}$	$m_x^{(2)} = m_x'^{(2)}$	$m_x^{(3)} = m_x'^{(3)}$	$m_x^{(\tau)}$	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$
62	0.02020	0.03046	0.22222	0.27288	0.01777	0.02680	0.19554
63	0.02224	0.03459	0.10526	0.16209	0.02057	0.03200	0.09737
64	0.02840	0.04082	0.12766	0.19688	0.02585	0.03716	0.11622

10.20.

x	$m_x^{(1)}$	$m_x^{(2)}$	$m_x'^{(1)}$	$m_x'^{(2)}$
65	0.02073	0.05181	0.02073	0.05183
66	0.03141	0.06283	0.03144	0.06286
67	0.04233	0.07407	0.04237	0.07412
68	0.05348	0.08556	0.05355	0.08565
69	0.06486	0.09730	0.06499	0.09744

10.21. Revise (a) to $m_x^{(j)} \Big/ \left(1 + \frac{1}{2} m_x^{(\tau)}\right)$

10.22. c. i. ${}_tq_x^{(j)} = K_j\,(1 - e^{-[k/(n+1)]t^{n+1}}) = K_j\,{}_tq_x^{(\tau)}$

${}_tq_x'^{(j)} = 1 - e^{-[k/(n+1)]t^{n+1}} = 1 - ({}_tp_x^{(\tau)})^{K_j}$

ii. ${}_tq_x^{(j)} = K_j\,(1 - e^{(B/\log\, c)(c^t-1)}) = K_j\,{}_tq_x^{(\tau)}$

${}_tq_x'^{(j)} = 1 - e^{K_t(B/\log\, c)(c^t-1)} = 1 - ({}_tp_x^{(\tau)})^{K_j}$

10.25.

k	$q_k'^{(1)}$	$q_x'^{(2)}$
0	0.17143	0.27027
1	0.11111	0.21053
2	0.05405	0.15385
3	0.00000	0.10000

10.27. a. From $q_x^{(3)} = q_x'^{(3)}\left[1 - \frac{1}{2}(q_x'^{(1)} + q_x'^{(2)}) + \frac{1}{3}q_x'^{(1)}\, q_x'^{(2)}\right]$, obtain $q_x'^{(3)}$, then use (9.6.3).

b. Obtain $q_x^{(1)}$ from a relation derived in Exercise 9.18,

$$q_x^{(1)} \cong q_x'^{(1)}\left[1 - \frac{1}{2}(q_x^{(2)} + q_x^{(3)})\right]$$

10.28. $q_{69}^{(3)} = 0.94434$

10.29. $q_{50}'^{(1)} = 0.015$

10.30. $1 - \sum_{k=0}^{44} \frac{d_{20+k}^{(2)}}{l_{20}^{(\tau)}} = 1 - \frac{l_{20}^{(2)} - l_{65}^{(2)}}{l_{20}^{(\tau)}}$

10.31. a. Approximate $m_x^{(1)}$, $m_x^{(2)}$ from $q_x'^{(1)}$, $q_x'^{(2)}$, or approximate $q_x'^{(3)}$, $q_x'^{(4)}$ from $m_x^{(3)}$, $m_x^{(4)}$

b. $1 - \sum_{k=0}^{\infty} \frac{d_{y+k}^{(4)}}{l_y^{(\tau)}} = 1 - \frac{l_y^{(4)}}{l_y^{(\tau)}}$

10.32. (The probability of decrement due to cause j when all causes are operating) = (the absolute rate of decrement due to cause j)

− (the probability that decrement will occur due to causes k, $k \neq j$, and thereafter the event associated with j will occur prior to (x) attaining age $x + 1$)

10.34. a.
$$\begin{aligned} f_{T,J}(t, j) &= \frac{\theta\beta^\alpha}{\Gamma(\alpha)}\, t^{\alpha-1}\, e^{-\beta t} \qquad j = 1,\ t \ge 0 \\ &= \frac{(1-\theta)\beta^\alpha\, t^{\alpha-1}\, e^{-\beta t}}{\Gamma(\alpha)} \qquad j = 2,\ t \ge 0, \end{aligned}$$

$$f_J(j) = \begin{cases} \theta & j = 1 \\ 1 - \theta & j = 2, \end{cases}$$

$$f_T(t) = \frac{\beta^\alpha\, t^{\alpha-1}\, e^{-\beta t}}{\Gamma(\alpha)}$$

b. $E[T] = \frac{\alpha}{\beta}$, $\mathrm{Var}(T) = \frac{\alpha}{\beta^2}$

Chapter 11

11.1. If 1 denotes death and 2 withdrawal for any other reason, the actuarial present value is

$$20{,}000 \int_0^{40} v^t \, {}_tp_{30}^{(\tau)} \, \mu_{30(t)}^{(1)} \, dt + 300 \int_0^{40} v^t \, {}_tp_{30}^{(\tau)} \, \mu_{30(t)}^{(2)} \, t \; {}_{40-t|}\bar{a}_{30+t} \, dt$$
$$+ 12{,}000 \, v^{40} \, {}_{40}p_{30}^{(\tau)} \, \bar{a}_{70}.$$

11.2. 0.31075, 0.19717

11.4. $S_x = \dfrac{x}{15} - \dfrac{1}{3} \qquad 20 \le x$

11.5. 2,250

11.6. a. Take $S_{30} = 1$. Then

$$S_{30+k} = \begin{cases} (1.05)^k & 0 \le k < 10 \\ (1.1)(1.05)^k & 10 \le k < 20 \\ (1.1)^2(1.05)^k & 20 \le k < 30 \\ (1.1)^3(1.05)^k & k \ge 30. \end{cases}$$

b. $1{,}200 \sum_{k=0}^{\omega-31} v^{k+1/2} \; {}_{k+1/2}p_{30}^{(\tau)} \; S_{30+k}$

11.7. $0.1 \sum_{k=0}^{\omega-36} v^{k+1/2} \; {}_{k+1/2}p_{35}^{(\tau)} \left[25{,}000\left(\dfrac{S_{35+k}}{S_{35}}\right) - 10{,}000(1.05)^k \right]$

11.8. a. $\sum_{k=5}^{\infty} v^{k+1/2} \; {}_kp_{50}^{(\tau)} \; q_{50+k}^{(r)} \left(20 + k + \dfrac{1}{2}\right) \left(\dfrac{{}_3Z_{50+k}}{S_{50}}\right) 640 \; \bar{a}_{50+k+1/2}^{r}$

$$+ \sum_{k=5}^{14} v^{k+1/2} \; {}_kp_{50}^{(\tau)} \; q_{50+k}^{(r)} \left(20 + k + \frac{1}{2}\right) \left(\frac{{}_3Z_{50+k}}{S_{50}}\right) 320 \; \bar{a}_{50+k+1/2:\overline{15-k-1/2|}}^{r}.$$

Since $q_{50+k}^{(r)} = 0$ for $k < 5$, these sums could be extended down to $k = 0$.

b. In the first sum, the terms for $k > 14$ are changed to

$$\sum_{k=15}^{\infty} v^{k+1/2} \; {}_kp_{50}^{(\tau)} \; q_{50+k}^{(r)} \left(\frac{{}_3Z_{50+k}}{S_{50}}\right)(22{,}400) \; \bar{a}_{50+k+1/2}^{r}.$$

c. In the answer to part (a), replace $(20 + k + 1/2)$ by 20.

11.9. a. $R(30, 20, 15) = 8{,}000 + 720 \sum_{j=0}^{14} \dfrac{S_{50+j}}{S_{50}}$

b. $R(30, 20, 15\text{-}1/2) = 8{,}000 + 720 \, \dfrac{\sum_{j=0}^{14} S_{50+j} + (1/2)\, S_{65}}{S_{50}}$

c. $8{,}000 \sum_{k=8}^{17} v^{k+1/2} \; {}_kp_{50}^{(\tau)} \; q_{50+k}^{(r)} \; \bar{a}_{50+k+1/2}^{r}$

d. $\sum_{k=0}^{17} v^{k+1/2} \; {}_kp_{50}^{(\tau)} \; q_{50+k}^{(r)} \; 720 \left[\dfrac{\sum_{j=0}^{k-1} S_{50+j} + (1/2) S_{50+k}}{S_{50}} \right] \bar{a}_{50+k+1/2}^{r},$

which equals the sum with $k = 8$ to 17, or

$$\frac{720}{S_{50}} \left[\sum_{j=0}^{17} S_{50+j} \left(\frac{1}{2} \, v^{j+1/2} \; {}_jp_{50}^{(\tau)} \; q_{50+j}^{(r)} \; \bar{a}_{50+j+1/2}^{r} + \sum_{k=j+1}^{17} v^{k+1/2} \; {}_kp_{50}^{(\tau)} \; q_{50+k}^{(r)} \; \bar{a}_{50+k+1/2}^{r} \right) \right]$$

11.10. 1.0756

11.11. a. $20{,}000 \int_{15}^{\infty} v^t\, {}_tp_{40}^{(\tau)}\, \mu_{25}^{(r)}(15+t)\, \dfrac{S_{40+t}}{S_{40}}\, \bar{a}_{40+t}^{r}\, dt$

b. $600 \int_{15}^{\infty} v^t\, {}_tp_{40}^{(\tau)}\, \mu_{25}^{(r)}(15+t)\, \dfrac{S_{40+t}}{S_{40}}\, (15+t)\, \bar{a}_{40+t}^{r}\, dt$

c. $600 \int_{0}^{15} v^t\, {}_tp_{40}^{(\tau)}\, \mu_{25}^{(w)}(15+t)\, \dfrac{S_{40+t}}{S_{40}}\, (15+t)\, {}_{15-t|}\bar{a}_{40+t}^{w}\, dt$

where the w superscript on the annuity symbol indicates that it is to be calculated using mortality rates appropriate for a life after withdrawal from employee status.

d. $1{,}000 \int_{15}^{\infty} v^t\, {}_tp_{40}^{(\tau)}\, \mu_{25}^{(r)}(15+t) \left(8 + \int_0^t \dfrac{S_{40+u}\, du}{S_{40}}\right) \bar{a}_{40+t}^{r}\, dt$

Assuming S_x is a step function between integral ages,

$$1{,}000 \int_{15}^{\infty} v^t\, {}_tp_{40}^{(\tau)}\, \mu_{25}^{(r)}(15+t) \left[8 + \frac{\left(\sum_{k=0}^{\lfloor t\rfloor-1} S_{40+k}\right) + S_{40+\lfloor t\rfloor}(t - \lfloor t\rfloor)}{S_{40}}\right] \bar{a}_{40+t}^{r}\, dt$$

e. $1{,}000 \int_{0}^{15} v^t\, {}_tp_{40}^{(\tau)}\, \mu_{25}^{(w)}(15+t) \left(8 + \dfrac{\int_0^t S_{40+u}\, du}{S_{40}}\right) {}_{15-t|}\bar{a}_{40+t}^{w}\, dt$

Assuming S_x is a step function between integral ages,

$$1{,}000 \int_{0}^{15} v^t\, {}_tp_{40}^{(\tau)}\, \mu_{25}^{(w)}(15+t) \left[8 + \frac{\sum_{k=0}^{\lfloor t\rfloor-1} S_{40+k} + S_{40+\lfloor t\rfloor}(t - \lfloor t\rfloor)}{S_{40}}\right] {}_{15-t|}\bar{a}_{40+t}^{w}\, dt$$

11.12. $\displaystyle\sum_{k=30}^{39} v^{k+1/2}\, {}_kp_{30}^{(\tau)}\, q_{30+k}^{(r)}\, (0.60)35{,}000\, \frac{S_{30+k+1/2}}{S_{30}}\, \bar{a}_{30+k+1/2}^{r}$

$\displaystyle + \sum_{k=40}^{\infty} v^{k+1/2}\, {}_kp_{30}^{(\tau)}\, q_{30+k}^{(r)}\, (0.60)35{,}000\, \frac{S_{69+1/2}}{S_{30}}\, \bar{a}_{30+k+1/2}^{r}$

11.13. a. $\displaystyle 24{,}000\, \frac{\sum_{k=0}^{24} v^{k+1}\, {}_kp_{35}^{(\tau)}\, q_{35+k}^{(i)}\, \ddot{a}_{[36+k]:\overline{28-k}|}^{(12)i}}{\ddot{a}_{35:\overline{25}|}^{(\tau)}}$

b. $24{,}000\, ({}^{20}_{15}\Pi^{i}_{45} - {}^{30}_{25}\Pi^{i}_{35})\ddot{a}^{(\tau)}_{45:\overline{15}|}$ where $24{,}000\, {}^{30}_{25}\Pi^{i}_{35}$ and $24{,}000\, {}^{20}_{15}\Pi^{i}_{45}$ are the annual benefit premiums for the benefit for (35) and (45), respectively.

Chapter 12

12.1. $S = X_1 + X_2 + \cdots + X_N$ where N is the number of cars and X_i is the number of passengers in an i.

12.2. Let N denote the number of rainfalls and X_i the number of inches of rain in rainfall i.

12.3. a. npp_1 b. $npp_2 - np^2p_1^2$ c. $[p\,M_x(t) + 1 - p]^n$

12.4. a. 1.7 b. 0.81 c. 1.6 d. 0.44 e. 2.72 f. 2.8216

12.5. a. 2.72 b. 5.1

12.6. e^{-2}, $0.2e^{-2}$, $0.42e^{-2}$, $0.681333e^{-2}$, $1.008067e^{-2}$

12.8. $r = \dfrac{\lambda}{-\log(1-c)}$, $p = 1 - c$

12.9. $\dfrac{3^{24}\, x^{23}\, e^{-3x}}{23!}$

12.10. Poisson with parameter λp

12.12. Compound Poisson with $\lambda = 8$, $p(1) = 0.05$, $p(2) = 0.15$, $p(3) = 0.425$, $p(4) = 0.375$

12.13. Compound Poisson with $\lambda = 14$, $p(-2) = 1/14$, $p(1) = 4/14$, $p(3) = 9/14$

12.14. $\Pr(N = n+1) = \lambda \dfrac{\Pr(N = n)}{n+1}$

12.16. a. 0.425 b. 0.3984 c. 0.184; no: $\Pr(N_1 = 1, N_2 = 1) \neq \Pr(N_1 = 1)\Pr(N_2 = 1)$

12.17.

$x/f(x)$	Compound Poisson	Compound Negative Binomial	Compound Binomial
0	0.011109	0.044194	0.001953
1	0.034993	0.069606	0.012305
2	0.070112	0.096827	0.039727
3	0.105111	0.108230	0.085805
4	0.130100	0.110967	0.137767
5	0.138723	0.104988	0.173661
	$E(N) = 4.5$	$E(N) = 4.5$	$E(N) = 4.5$
	$Var(N) = 4.5$	$Var(N) = 9$	$Var(N) = 2.25$

12.25. a. $\Phi(2) = 0.9772$ b. $G\left(\frac{44}{3}; \frac{256}{9}, \frac{8}{3}\right)$

12.26. b. i. $\dfrac{p_2}{\lambda[p_1(1+\theta)]^2}$ ii. $\dfrac{p_2 + (q/p)p_1^2}{(rq/p)(p_1(1+\theta))^2}$

12.28. a. $\alpha, \dfrac{\beta}{\pi_i}$

Chapter 13

13.1. a. -1 b. $e^{-\tilde{R}(u+1)}$ c. $\log \dfrac{p}{q}$ d. $\left(\dfrac{q}{p}\right)^{u+1}$

13.3. $\dfrac{f(t-s)}{1 - F(t-s)}\, dt$

13.5. $0; \gamma$

13.7. a. 3 b. 1

13.8. $\dfrac{10}{7 \log 2} - 1$

13.12. a. $\left(\dfrac{1}{2}\right)\left(\dfrac{p_2}{p_1}\right)$ b. $\left(\dfrac{1}{3}\right)\left(\dfrac{p_3}{p_1}\right)$ c. $\left(\dfrac{1}{3}\right)\left(\dfrac{p_3}{p_1}\right) - \left(\dfrac{1}{4}\right)\left(\dfrac{p_2}{p_1}\right)^2$

13.13. b. $\mathrm{E}[L] = \left(\dfrac{1}{2}\right)\left(\dfrac{p_2}{\theta p_1}\right)$, $\mathrm{Var}(L) = \left(\dfrac{1}{3}\right)\left(\dfrac{p_3}{\theta p_1}\right) + \left(\dfrac{1}{4}\right)\left(\dfrac{p_2}{\theta p_1}\right)^2$

13.14. $\dfrac{2\theta r}{1 + (1 + \theta)2r - e^{2r}}$

13.15. a. $\dfrac{2}{3}$ b. 2

13.16. $p_1 = \displaystyle\sum_{i=1}^{n} \frac{A_i}{\beta_i}$

13.17. a. $\dfrac{5}{27}$ b. $\dfrac{4}{5}$ c. $\dfrac{-17r + 54}{3(3 - r)(6 - r)}$

d. $\left(\dfrac{4}{9}\right)\left(\dfrac{10 - 3r}{8 - 6r + r^2}\right) = \left(\dfrac{4}{9}\right)\left(\dfrac{2}{2 - r}\right) + \left(\dfrac{1}{9}\right)\left(\dfrac{4}{4 - r}\right)$

e. $\psi(u) = \dfrac{4}{9} e^{-2u} + \dfrac{1}{9} e^{-4u}$ Check: $\psi(0) = \dfrac{5}{9} = \dfrac{1}{1 + \theta}$

13.18. a. $\dfrac{10}{3}$ b. 2 c. $\dfrac{9}{(3 - 5r)^2}$

d. $\left(\dfrac{2}{3}\right)\left(\dfrac{0.12 - 0.1r}{0.24 - 1.1r + r^2}\right) = \dfrac{0.4(0.3)}{0.3 - r} - \dfrac{0.067(0.8)}{0.8 - r}$

e. $\psi(u) = 0.4e^{-0.3u} - 0.067e^{-0.8u}$

13.19. a. $\xi = \dfrac{\theta}{1 + \theta}$, $\mathrm{E}[L] = \left(\dfrac{1}{1 + \theta}\right)\left(\dfrac{\alpha}{\beta}\right)$, $\mathrm{E}[L^2] = \dfrac{1}{1 + \theta}\left(\dfrac{\alpha^2}{\beta^2} + \dfrac{\alpha}{\beta^2}\right)$

b. $\psi(u) = \dfrac{1}{1 + \theta}[1 - G(u{:}\alpha,\beta)]$

13.20. a. $\displaystyle\sum_{i=1}^{n} \left[\frac{A_i/\beta_i}{\displaystyle\sum_{j=1}^{n} (A_j/\beta_j)}\right] \beta_i e^{-\beta_i x}$

b. $\dfrac{A_i/\beta_i}{\left[\displaystyle\sum_{i=1}^{n} (A_i/\beta_i)\right]}$, $i = 1, 2, \ldots, n$

c. $\sum_{i=1}^{n} \left[\frac{A_i/\beta_i}{\sum_{j=1}^{n} (A_j/\beta_j)} \right] \frac{1}{\beta_i}$

13.21. a. $\frac{21}{10}(e^{-3x} + e^{-7x})$

b. $\frac{29}{105}$

c. $\frac{1{,}009}{11{,}025}$

13.23. a. $fc, f\lambda, \psi(u, ft)$ b. $\frac{1}{\lambda}$

13.24. $\frac{10}{3}$

13.25. a. $\psi'(u) = \begin{cases} \frac{\lambda}{c}\psi(u) - \frac{\lambda}{c} & 0 \le u \le 1 \\ \frac{\lambda}{c}\psi(u) - \frac{\lambda}{c}\psi(u-1) & u > 1 \end{cases}$

b. $\psi(u) = 1 - \left(1 - \frac{\lambda}{c}\right) e^{(\lambda/c)u} \quad 0 \le u \le 1$

Chapter 14

14.1. b. $b^2 pq$
c. $s = 0.233$

14.3. a. 31.35 days b. 2.99848 c

14.4. a. 0.13022 b. 0.09823 c. 0.05683

14.5. a. $E[S] = 4.7$, $Var(S) = 16.40$
b. $\lambda = 1.7$, $p(1) = \frac{7}{17}$, $p(4) = \frac{10}{17}$, 16.7

14.6. a. $\sum_{j=1}^{n} b_j \tilde{\lambda}_j$, $\sum_{j=1}^{n} b_j^2 \tilde{\lambda}_j$ c. 167, 293

14.7. $\frac{q_i q_j}{\sum\sum_{k<l} q_k q_l}$

14.9. $\sigma\phi\left(\frac{d-\mu}{\sigma}\right) - (d-\mu)\left[1 - \Phi\left(\frac{d-\mu}{\sigma}\right)\right]$

14.10. a. $-[1 - F_S(x)]$ b. $f_S(x+1)$

14.11. $f_S(x) = 2x \quad 0 < x < 1$

14.12.

x	$f_S(x)$	$F_S(x)$	$E[I_x]$
0	0.050	0.050	3.500
1	0.124	0.174	2.550
2	0.180	0.354	1.724

14.13. a. 0.3885 b. 0.985

14.14. $0.8\ E[I_d] - 0.8\ E[I_l]$ where $l = d + \dfrac{m}{0.8}$

14.15. 3.758

14.16. a. $\alpha = \dfrac{\theta}{\xi} - \dfrac{|\xi - \theta|}{\xi\sqrt{1 + \xi}}$ b. $\theta < \xi$ and $(1 + \theta)^2 < 1 + \xi$

14.17. a. $\alpha = \dfrac{2\theta}{\xi} - 1$ b. $\xi > 2\theta$

14.18. $\dfrac{\theta - \xi e^{-\beta}}{1 - e^{-\beta}}$, $1 + [(1 + \theta) - (1 + \xi)e^{-\beta}]r = \dfrac{1 - re^{-\beta(1-r)}}{1 - r}$

14.19. $\tilde{R} = \dfrac{1.25 - 2\alpha}{(1 - \alpha)^2}$, $\alpha = 0.25$

14.20. $$H_d = \frac{1}{\alpha} \log \left\{\Phi\left(\frac{d - \mu}{\sigma}\right) + \left[1 - \Phi\left(\frac{d - \mu}{\sigma} - \alpha\sigma\right)\right] \exp\left[\alpha(\mu - d) + \frac{\alpha^2\sigma^2}{2}\right]\right\}$$

Note: The answer of Exercise 14.9 is the limit of H_d when $\alpha \to 0$.

14.21. Inverse Gaussian distribution with parameters α and β.

14.23. a. $e^{-\delta t}\left\{e^{tm + t\sigma^2/2}\left[1 - \Phi\left(\dfrac{\log d - (tm + t\sigma^2)}{\sqrt{t}\,\sigma}\right)\right] - d\left[1 - \Phi\left(\dfrac{\log d - tm}{\sqrt{t}\,\sigma}\right)\right]\right\}$

b. $m = \delta - \sigma^2/2$

c. $\Phi\left(\dfrac{-\log d + t\delta + t\sigma^2/2}{\sigma\sqrt{t}}\right) - de^{-\delta t}\,\Phi\left(\dfrac{-\log d + t\delta - t\sigma^2/2}{\sigma\sqrt{t}}\right)$

Chapter 15

15.1. a.

Savings Account	566.50	Reserve	485.44
		Surplus	81.06
	566.50		566.50

Premium Income	550.00
Interest Income	16.50
	566.50
Increase in Reserves	485.44
Net Income	81.06

b. $1 - \Psi(5.28) = 0.00000$

15.3. b. i. 7,425.56 ii. 183,833
c. i. 7,500 ii. 187,500
d. 7,425.56 e. 7,500 f. 0.171914

15.4. a. 288.41 b. 332.35

15.5. $$\frac{1{,}000\ \bar{A}_{[40]:\overline{25}|} + 4\ \ddot{a}_{[40]:\overline{25}|} + 8.5}{0.93\ \ddot{a}_{[40]:\overline{25}|} + 0.05\,{}_{10}\mathrm{E}_{[40]}\ \ddot{a}_{[40]+10:\overline{15}|} - 0.35}$$

15.6. $$\frac{1{,}000\ \bar{A}_{x:\overline{n}|} + 2.5\ \ddot{a}_{x:\overline{n}|} + 2.5}{0.935}$$

15.7. $a = 1 + e_0 + e_2 + e_3, \qquad c = e_1 + de_0$

15.8. a. $\dfrac{\bar{P}(\bar{A}_x) + \alpha}{1-p} + \dfrac{\theta}{1-p}\dfrac{1}{b} = \pi + \dfrac{f}{b}$

b. $\dfrac{\bar{P}(\bar{A}_x) + \alpha}{1-p} + \dfrac{\theta}{1-p}\dfrac{1}{\mathrm{E}[B]} = \pi + \dfrac{f}{\mathrm{E}[B]}$

15.9. a. 200 b. 20 c. $\sqrt{200}$

15.10. a. 1st Year $a'b + 14.29$ where $a' = \dfrac{1{,}000\bar{A}_x + 0.5\ddot{a}_x + 2.5}{0.95\bar{a}_x - 0.25}$

Renewal Years $a'b + 2.63$

b. All Years $a'b + \dfrac{2.5\ddot{a}_x - 7.5}{0.95\ddot{a}_x - 0.25}$

15.12. $(G_2 - G_1)\displaystyle\sum_{k=0}^{9}(1 - c_K)l_{x+k}^{(\tau)}\frac{(1+i)^{10-k}}{l_{x+10}^{(\tau)}}$

15.14. b. $\displaystyle\int_0^\infty (e^{-\delta' u} - e^{-\delta u})[b_{t+u}\ \mu_x(t+u) - \pi_{t+u}]\ {}_u p_{x+t}\ du$

Chapter 16

16.1. ${}_tW'_x \gtreqless {}_tW_x$ according as $\dfrac{P'_{x+t}}{P_{x+t}} \gtreqless \dfrac{P'_x}{P_x}$

16.2. $\dfrac{b}{2} + \left(\dfrac{b}{2} - 1\right)\dfrac{A^{\ 1}_{x+t:\overline{n-t}|}}{{}_{n-t}E_{x+t}}$

16.3. a. $\dfrac{{}_{10}CV - L - (1-L)\ \bar{A}^{\ 1}_{40:\overline{10}|}}{{}_{10}E_{40}}$ b. $(1-L)\ \bar{A}^{\ 1}_{45:\overline{5}|} + E\ {}_5E_{45}$

16.4. ${}^{20}_{10}W_{40} = 0.5829$, ${}_{10}W_{40:\overline{20}|} = 0.6232$, proportional amount $= 0.5$

16.7. Whole life: $1 - \dfrac{P^a_x}{P_{x+k}}$, n-Payment life: $1 - \dfrac{{}_nP^a_x}{{}_{n-k}P_{x+k}}$,

n-Year endowment: $\dfrac{1 - P^a_{x:\overline{n}|}}{P_{x+k:\overline{n-k}|}}$

16.8. Whole life: $1 - \dfrac{P_{x+1}}{P_{x+k}}$,

n-Payment life: $1 - \dfrac{\beta^{CRVM}}{{}_{n-k}P_{x+k}}$

$$\text{where } \beta^{CRVM} = {}_nP_x + \frac{{}_{19}P_{x+1} - A^{1}_{x:\overline{1}|}}{\ddot{a}_{x:\overline{n}|}},$$

$$n\text{-Year endowment: } 1 - \frac{\beta^{CRVM}}{P_{x+k:\overline{n-k}|}}$$

$$\text{where } \beta^{CRVM} = P_{x:\overline{n}|} + \frac{{}_{19}P_{x+1} - A^{1}_{x:\overline{1}|}}{\ddot{a}_{x:\overline{n}|}}$$

16.9. c. $\bar{a}_x\ G\ e^{\delta t} + \bar{a}_{x+k+t}\ (\mu_{x+k+t} + \delta) - 1$

16.10. $(G_2 - G_1) \sum_{k=0}^{9} (1 - c_k)\ l^{(\tau)}_{x+k}\ \frac{(1+i)^{10-k}}{l^{(\tau)}_{x+10}}$

16.13. b. $\left(\frac{1 + \hat{\imath}_{h+1}}{1 + i}\right)\left(\frac{p_{x+h}}{p'_{x+h}}\right)$

16.15. $\bar{\beta} = \frac{\bar{A}_x}{(\bar{I}\bar{a})_{x:\overline{m}|}/m + {}_{m|}\bar{A}_x}$

16.16. $\alpha^{Mod}_x = A^{1}_{x:\overline{1}|} + K_1 E_x,\ \beta^{Mod}_x = P_{x+1} - \frac{K}{\ddot{a}_{x+1}}$

16.23. a. $\beta = 0.03$, $\alpha = 0.01$, $\beta - \alpha < 0.05$ b. 0.28 d. 0.0867 e. 0.0278

16.24. $\alpha = \beta^{CRVM}_{x:\overline{20}|} - ({}_{19}P_{x+1} - A^{1}_{x:\overline{1}|}),\ \beta = \frac{P_{x:\overline{20}|}\ \ddot{a}_{x:\overline{15}|} - \alpha}{a_{x:\overline{14}|}}$

16.25. $T = \beta^{CRVM} - {}_{19}P_{x+1}$

Chapter 17

17.1. $\bar{a}_{\overline{n}|} + \int_n^\infty v^s\ {}_{s-n}p_x\ ds$

17.7. $\frac{\bar{A}^{1}_{x:\overline{15}|} - v^{20}\ {}_{15}q_x}{\delta} + \bar{a}_{\overline{5}|}(\bar{A}^{1}_{x:\overline{20}|} - \bar{A}^{1}_{x:\overline{15}|})$, or

$\bar{a}_{\overline{20}|} - \bar{a}_{x:\overline{20}|} + v^{20}\ {}_{15}p_x\ \bar{a}_{x+15:\overline{5}|} - v^{20}\ {}_{20}p_x\ \bar{a}_{\overline{5}|}$

17.8. $\frac{[1{,}000\ A_{x:\overline{20}|}^{\ \ 1} + 120(a^{(12)}_{\overline{20}|} - a^{(12)}_{x:\overline{20}|})]}{\ddot{a}_{x:\overline{20}|}}$

17.10. a.

$$Z = \begin{cases} v^T\ \bar{a}_{\overline{25-T}|} & T \le 15 \\ v^T\ \bar{a}_{\overline{10}|} & 15 < T \le 25 \\ v^{25}\ \bar{a}_{\overline{10}|} & 25 < T \le 35 \\ v^{25}\ \bar{a}_{\overline{T-25}|} & T > 35 \end{cases}$$

b. $\int_0^{15} v^t\ \bar{a}_{\overline{25-t}|}\ {}_tp_{40}\ \mu_{40}(t)\ dt + \bar{a}_{\overline{10}|} \int_{15}^{25} v^t\ {}_tp_{40}\ \mu_{40}(t)\ dt$

$+ v^{25}\ \bar{a}_{\overline{10}|} \int_{25}^{35} {}_tp_{40}\ \mu_{40}(t)\ dt + v^{25} \int_{35}^{\infty} \bar{a}_{\overline{t-25}|}\ {}_tp_{40}\ \mu_{40}(t)\ dt$

17.15. 203,421

17.16. a. 55 b. 53,759.04

17.18. c. as $\delta \to 0$ $\mu^{(3)}$
as $\delta \to \infty$ $\mu^{(3)}/4$

Chapter 18

18.1. a. Either 2 or 4 of w, x, y, z survive to time t
b. Either 1 or 3 of w, x, y, z survive to time t

18.3. a. 6.5 b. 3,236.71

18.4. ${}_tD_3 - 3\ {}_tD_4$ where ${}_tD_3 = {}_tp_{wxy} + {}_tp_{wxz} + {}_tp_{wyz} + {}_tp_{xyz}$ and ${}_tD_4 = {}_tp_{wxyz}$

18.5. $a_w - (a_{wxy} + a_{wxz} + a_{wyz}) + 2a_{wxyz}$

18.6. 0.624010

18.7. a. $15(\bar{a}_x + \bar{a}_y + \bar{a}_z) - 10(\bar{a}_{xy} + \bar{a}_{xz} + \bar{a}_{yz}) + 9\bar{a}_{xyz}$
b. $15\bar{a}_x - 5(\bar{a}_{xy} + \bar{a}_{xz}) + 3\bar{a}_{xyz}$

18.8. 4.6

18.9. a. $12{,}000\ (a^{(12)}_{40:\overline{25|}} + a^{(12)}_{35:\overline{25|}} - 2a^{(12)}_{40:35:\overline{25|}})$
b. $12{,}000\ (a^{(12)}_{40:25} + a^{(12)}_{35:\overline{30|}} - a^{(12)}_{40:35:\overline{25|}})$

18.10. a. $\bar{a}_x + \bar{a}_y + \bar{a}_{\overline{n|}} - \bar{a}_{y:\overline{n|}} - \bar{a}_{xy} - \bar{a}_{x:\overline{n|}} + \bar{a}_{xy:\overline{n|}}$
b. $\bar{a}_{\overline{30|}} + \bar{a}_{25:\overline{40|}} - \bar{a}_{25:\overline{30|}}$

18.11. a. $\frac{1}{7}$ b. $\frac{26}{105}$ c. $\frac{64}{105}$

18.12. a. I is false. Change right-hand side to $\bar{A}^4_{wxyz} + \bar{A}_{w\overset{4}{x}yz} + \bar{A}_{wx\overset{4}{y}z} + \bar{A}_{wxy\overset{4}{z}}$.
b. II is true.
c. III is false. Change right-hand side to $\bar{A}_{w\overset{1}{z}} + \bar{A}_{x\overset{1}{z}} + \bar{A}_{y\overset{1}{z}} - 2(\bar{A}_{wx\overset{1}{z}} + \bar{A}_{wy\overset{1}{z}} + \bar{A}_{xy\overset{1}{z}}) + 3\bar{A}_{wxy\overset{1}{z}}$.

18.13. $\int_0^\infty v^t\ {}_tp_{xy}\ \mu_y(t)\ t\ \bar{A}_{x+t}\ dt$

18.14. a. $\frac{5}{7}$ b. $\frac{3}{7}$

18.15. $A_{z:\overset{1}{\overline{10|}}}\ (A_{y:\overset{1}{\overline{z+10}|}} - A_{xy:\overset{1}{\overline{z+10}|}}) - A_{yz:\overset{1}{\overline{10|}}}\ (A_{y+10:\overset{1}{\overline{z+10}|}} - A_{xy+10:\overset{1}{\overline{z+10}|}})$

18.16. $v^{10}(\bar{A}^1_{xy} + \bar{A}^1_{xz} - \bar{A}^1_{xyz}) - A_{y:\overset{1}{\overline{10|}}}\ \bar{A}^1_{x:y+10} - A_{z:\overset{1}{\overline{10|}}}\ \bar{A}^1_{x:z+10}$
$+ A_{yz:\overset{1}{\overline{10|}}}\ \bar{A}^1_{x:y+10:z+10}$

18.17. $(\bar{A}^1_{30:\overline{5|}} + A_{30:\overset{1}{\overline{5|}}}\ \bar{A}^1_{35:60}) \Big/ \left(\frac{1}{1.075} - A_{30:\overset{1}{\overline{5|}}}\ \bar{A}^1_{35:60}\right)$

18.20. a. $\int_0^\infty (1 - {}_tp_w)\ {}_tp_x\ \mu_x(t)\ {}_tp_{yz}\ dt$ b. ${}_tq^1_{xyz} - {}_\infty q^{\ 1}_{wxyz}$

18.22. a. $\frac{25}{72}$ b. $\frac{19}{36}$ c. $\frac{5}{8}$

18.23. 0.07

18.24. a. 0.3 b. 0.2 c. $\frac{1}{30}$

18.25. $\int_{10}^{15} (1 - {}_{t-10}p_x)\ {}_tp_y\ \mu_y(t)\ ({}_{t+10}p_z - {}_{25}p_z)\ dt$

18.26. $\int_0^{30} (1 - {}_tp_{10})\ {}_tp_{20}\ \mu_{20}(t)({}_tp_{30} - {}_{30}p_{30})\ dt$

$$+ \int_0^{30} (1 - {}_tp_{30})\ {}_tp_{20}\ \mu_{20}(t)\ ({}_tp_{10} - {}_{50}p_{10})\ dt$$

$$+ \int_{30}^{40} (1 - {}_{30}p_{30})\ {}_tp_{20}\ \mu_{20}(t)({}_tp_{10} - {}_{50}p_{10})\ dt$$

18.27. 0.2145

18.28. 0.2704

18.29. I. False. Insurance factor should be $\bar{A}_{\overline{y+t}:z+t}^{\ \ 1}$.
II True
III True

18.30. a. $\bar{a}_{\overline{10}|} - \bar{a}_{x:\overline{10}|} + v^{10}\ \bar{a}_y - v^{10}\ {}_{10}p_x\ \bar{a}_{x+10:y}$
b. $\bar{a}_{\overline{10}|}\ \bar{A}_{xy}^{1} + v^{10}\ \bar{a}_y - v^{10}\ \bar{a}_{xy}$

18.31. $G = \left[\frac{2}{3}\ (\ddot{a}_{x|y} + {}_{n|}\ddot{a}_x) + \frac{1}{3}\ {}_{n|}\ddot{a}_{xy}\right] \Big/ (0.92 - {}_nA_{xy}^{2})$

18.33. $\dfrac{\bar{A}_{x+n:y}^{\ \ \ \ 1}}{\ddot{s}_{x:\overline{n}|} + \ddot{a}_{x:y+n}}$

18.34. $\ddot{a}_{xyz}$

18.35. b. $\ddot{a}_{\overline{xy}:\overline{m}|} + v^m\ {}_np_x\ (1 - {}_mp_y)\ \ddot{a}_{x+m:\overline{n-m}|}$

18.37. Insurance payable at the moment of z's death, if the deaths have occurred in the order x, y, z and only if the death of z is less than 10 years after the death of y.

Chapter 19

19.1. a. 45 b. 2 c. 2

19.2. $2{,}500\ \sqrt{2} + \dfrac{10{,}000}{\pi}$

19.3. $10{,}000 \left[e^{-1/4} - e^{-1/2} - \left(\frac{e^{-1}}{4}\right)\right]$

19.4. $100^2\ [e^{-51/100} - e^{-50/100} + e^{-28/100} - e^{-27/100}] + 100\ [e^{-1/4} + e^{-53/100}]$

19.5. $T_{20} - T_{40} - 20l_{70}$

19.6. $\int_{20}^{50} l(x, -x)\ dx - \int_{70}^{80} l(x, 50 - x)\ dx - \int_{30}^{50} l(80, t - 80)\ dt$

19.8. $b\ \dfrac{\sqrt{2\pi}}{\sqrt{a}} \left(1 - \Phi\ \frac{R}{\sqrt{a}}\right) \exp \left(Rt + \frac{R^2}{2a}\right)$

19.9. $a + \dfrac{(\bar{I}\bar{a})'_{a:\overline{r-a}|}}{\bar{a}'_{a:\overline{r-a}|}}, \displaystyle\int_a^r x l_x \frac{dx}{T_a - T_r}$

19.11. a. $e^{-Rx}\, s(x)(R + \mu_x),\ 1 - e^{-Rx}\, s(x)$

19.15. 0.02

19.17. a. $[\Gamma(\alpha)]^{1/a} - \beta$ b. stationary

19.18. b. a c. $\dfrac{\log b}{c}$

Chapter 20

20.1. $(T_a - T_r)w$

20.2. $ne^{R(t+a)+\tau t} \displaystyle\int_a^r e^{-Ry}\, s(y)\, w(y)\, dy$

20.3. b. $n(t - r + a)s(r)\, \bar{a}_r^h \left(\dfrac{f}{b}\right) \displaystyle\int_0^b w(r - y)e^{\tau(t-y)}\, dy$

c. $^TP_{t+u} = ne^{R(t+u-r+a)}\, s(r)\, \bar{a}'_r \left(\dfrac{f}{b}\right) \displaystyle\int_0^b w(r - y)\, e^{\tau(t+u-y)}\, dy$
$= e^{\rho u}\, {}^TP_t$

20.4. a. $c(r - a)w(r)\, e^{\tau t}\, n(t - r + a)s(r)\, \bar{a}_h^r$

b. $^TP_{t+u} = c(r - a)\, w(r)\, e^{\tau(t+u)}\, n\, e^{R(t-r+a)}\, s(r)\, \bar{a}'_r = e^{\rho u}\, {}^TP_t$

20.5. $e^{\rho t}\, e^{-(R+\mu)(r-a)}\, fw(r)\, \bar{a}'_r$

20.6. a. $\dfrac{1}{0.06} - \dfrac{1 - e^{-6+0.06x}}{(100 - x)(0.06)^2}$ $25 < x < 100$
$\bar{a}_{65}^h = 9.7020$

b. $3{,}056.14\, e^{0.02t}$ $0 \le t$
0 $t < 0$

20.8. a. $34{,}175.71\, e^{0.02t}$ $0 \le t$
b. $4{,}423.17\, e^{0.02t}$ $0 \le t$

20.9. $M(x) = \begin{cases} 0 & x < r \\ 1 & x \ge r \end{cases}$

20.10. $fw(r)\, \bar{a}'_r\, e^{-(R+\mu)(r-a)}\, e^{\rho t}\, \dfrac{\bar{a}_{\overline{r-a}|\theta}}{r - a}$ where $\theta = \delta - \rho$

20.14. a. $P_t = \exp(-\delta[r - X(\delta)])\, {}^TP_t,$
$(aV)_t = {}^TP_t\, \bar{a}_{\overline{r-X(\delta)}|\delta} = P_t\, \bar{s}_{\overline{r-X(\delta)}|\delta}$

b. $P_t = {}^TP_t$ $(aV)_t = {}^TP_t(r - \mu)$ where $\mu = \displaystyle\int_a^r xm(x)\, dx$

20.15. a. $\dfrac{fw(r)\, e^{(\tau-\delta)r}\, s(r)\, \bar{a}_r^h}{\displaystyle\int_a^r w(y)\, e^{(\tau-\delta)y}\, s(y)\, dy}$

b. $\dfrac{e^{-(\delta-\tau)x}\, s(x)\, w(x)}{\displaystyle\int_a^r e^{-(\delta-\tau)y}\, s(y)\, w(y)\, dy}$

20.20. a. $60{,}977.92\ e^{0.02t}$

b. $M(x) = \dfrac{x - 25}{40}$, $m(x) = \dfrac{1}{40}$

i. $38{,}292.33\ e^{0.02t}$

ii. $1{,}524.45\ e^{0.02t}$

c. $M(x) = \dfrac{e^{-1} - e^{-0.04x}}{e^{-1} - e^{-2.6}}, \quad m(x) = \dfrac{0.04e^{-0.04x}}{e^{-1} - e^{-2.6}}$

i. $45{,}479.00\ e^{0.02t}$

ii. $1{,}236.88\ e^{0.02t}$

20.21. $\dfrac{d}{dt}(aF)_t = \left[\dfrac{(aV)_0}{\bar{a}_{\overline{15}|}} + P_t\right] + \delta(aF)_t - {}^{T}P_t \qquad 0 \le t < 15$

Initial condition $(aF)_0 = 0$

$$\frac{d}{dt}(aF)_t = P_t + \delta(aF)_t - {}^{T}P_t \qquad 15 \le t$$

Initial condition $(aF)_{15} = (aV)_{15}$

a. $26{,}234.75\ e^{0.06t} + 38{,}292.33\ e^{0.02t} - 64{,}527.08 \qquad 0 \le t < 15,$
$38{,}292.33\ e^{0.02t} \qquad 15 \le t$

b. $31{,}158.47\ e^{0.06t} + 45{,}479.00\ e^{0.02t} - 76{,}637.48 \qquad 0 \le t < 15,$
$45{,}479.00\ e^{0.02t} \qquad 15 \le t$

20.23. a. $\bar{a}_{\overline{X(\theta)-a}|\theta}$ b. $\mu - a$ where $\mu = \displaystyle\int_a^r x\, m(x)\, dx$

20.25. a. $21{,}000.00\ e^{0.02t}$

b. $263{,}122.29\ e^{0.02t}$

Chapter 21

21.1. 0.000382

21.2. 0.001154

21.3. a. $\dfrac{1}{10}$ b. $\dfrac{1}{132}$

21.4. a. $e^{-n(\delta-\sigma^2/2)}$ b. $e^{-jn(\delta-j\sigma^2/2)}$

c. i. $\displaystyle\sum_{k=0}^{\infty} e^{-j(k+1)(\delta-j\sigma^2/2)}\ {}_kp_x\, q_{x+k} = {}^{j}_{*}A_x$

ii. ${}^{2}_{*}A_x - ({}_{*}A_x)^2$

21.5. a. $0.10, \dfrac{2}{1{,}200}$ b. $f_z(z) = \begin{cases} 100\ z & 0 < z < 0.1 \\ 20 - 100\ z & 0.1 < z < 0.2 \\ 0 & \text{elsewhere} \end{cases}$

c. $f_y(y) = \begin{cases} 100 \log y/y & 1 < y < e^{0.1} \\ (20 - 100 \log y)/y & e^{0.1} < y < e^{0.2} \\ 0 & \text{elsewhere} \end{cases}$

21.7. a. δ b. $k\sigma^2$ c. $\log(1 + I_k) \sim N(\delta, k\sigma^2)$

21.8. a. $-n\delta, \dfrac{n(n+1)(2n+1)}{6}\sigma^2$ b. $\log(\tilde{v}_n) \sim N\left[-n\delta, \dfrac{n(n+1)(2n+1)}{6}\sigma^2\right]$

c. $e^{-n(\delta-[(n+1)(2n+1)/12]\sigma^2)}, e^{-2n(\delta-[(n+1)(2n+1)/12]\sigma^2)}(e^{[n(n+1)(2n+1)/6]\sigma^2} - 1)$

d. $\tilde{v}_n \sim \text{lognormal}\left(-n\delta, \dfrac{n(n+1)(2n+1)}{6}\sigma^2\right)$

21.9. a. r b. $n\sigma^2$ c. $\log I_n \sim N(r, n\delta^2)$

21.10. $e^{r+(n\sigma^2/2)}$, $e^{2r+n\sigma^2}(e^{(n\sigma^2/2)} - 1)$

21.17. 0.050000, 0.054866, 0.059611

21.22. b. $l_x(c^*\bar{a}_x - \theta\bar{A}_x)$

c. $-l_x[(\bar{I}\bar{A})_x + c(I^*\bar{a})_x - \bar{P}(\bar{A}_x)(1+\theta)(\bar{I}\bar{a})_x]$

where ${}^*\bar{a}_x$ is valued at ${}^*\delta = \delta + r$

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