# Area in Polar Coordinates

Objective: To find areas of regions that are bounded by polar curves.

# Area of Polar Coordinates

- *We will begin our investigation of area in polar coordinates with a simple case.*

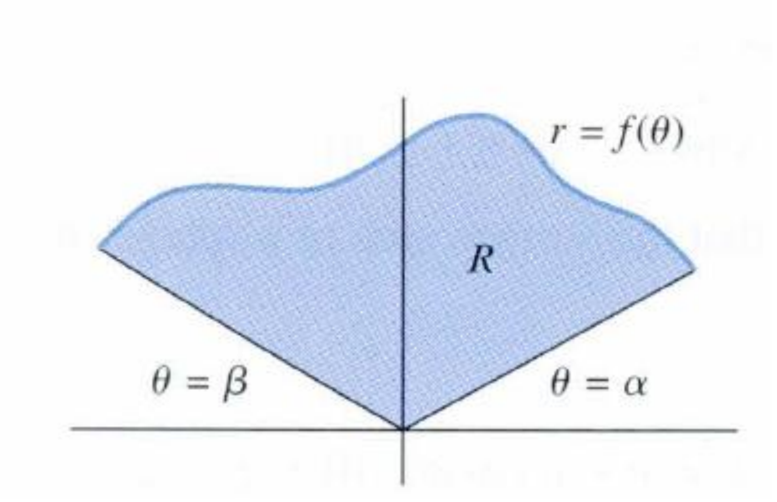
**11.3.1** AREA PROBLEM IN POLAR COORDINATES. Suppose that $\alpha$ and $\beta$ are angles that satisfy the condition

$$\alpha < \beta \leq \alpha + 2\pi$$

and suppose that $f(\theta)$ is continuous and either nonnegative or nonpositive for $\alpha \leq \theta \leq \beta$. Find the area of the region $R$ enclosed by the polar curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ (Figure 11.3.1).

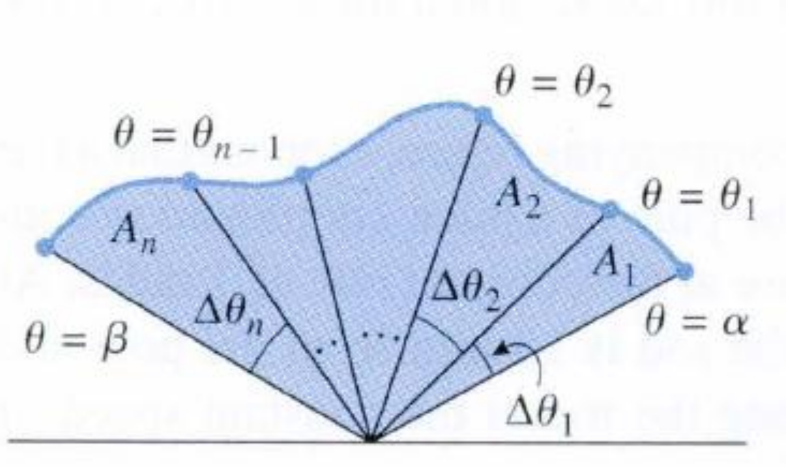
# Area of Polar Coordinates

- *In rectangular coordinates we obtained areas under curves by dividing the region into an increasing number of vertical strips, approximating the strips by rectangles, and taking a limit.  In polar coordinates rectangles are clumsy to work with, and it is better to divide the region into wedges by using rays.*



# Area of Polar Coordinates

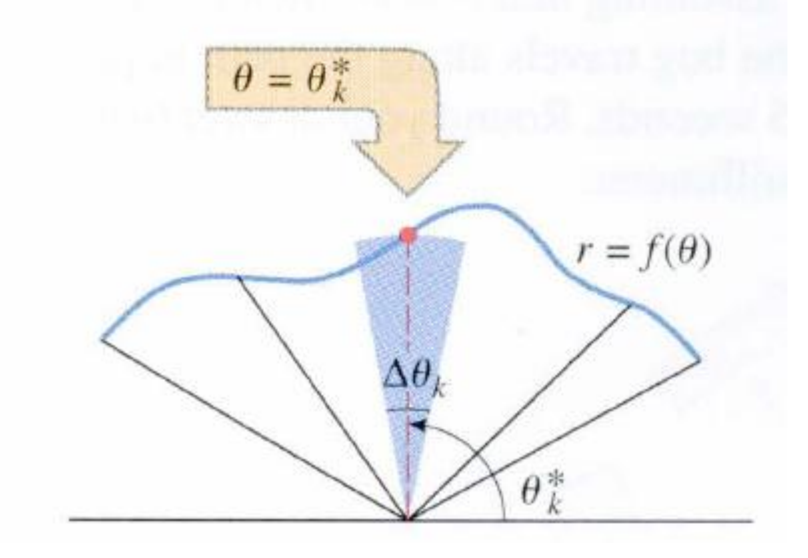
- *As shown in the figure, the rays divide the region R into n wedges with areas $A_1, A_2, \ldots A_n$ and central angles* $\Delta\theta_1, \Delta\theta_2, \ldots \Delta\theta_n$ *. The area of the entire region can be written as* $A = A_1 + A_2 + \ldots A_n = \sum_{k=1}^{n} A_k$



# Area of Polar Coordinates

- *If $\Delta\theta_k$ is small, then we can approximate the area $A_k$ of the kth wedge by the area of a sector with central angle $\Delta\theta_k$ and radius $f(\theta_k^*)$ where $\theta = \theta_k^*$ is any ray that lies in the kth wedge. Thus, the area of the sector is*

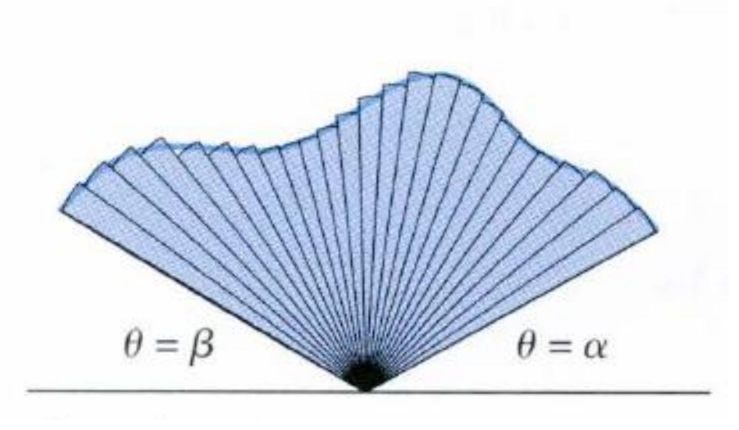
$$A = \sum_{k=1}^{n} A_k \approx \sum_{k=1}^{n} \tfrac{1}{2}[f(\theta_k^*)]^2 \Delta\theta_k$$



# Area of Polar Coordinates

- *If we now increase n in such a way that max* $\Delta\theta_k \to 0$*, then the sectors will become better and better approximations of the wedges and it is reasonable to expect that the approximation will approach the exact value.*

$$A = \lim_{\max \Delta\theta_k \to 0} \sum_{k=1}^{n} \tfrac{1}{2}[f(\theta_k^*)]^2 \Delta\theta_k = \int_{\alpha}^{\beta} \tfrac{1}{2}[f(\theta)]^2 d\theta$$



# Area of Polar Coordinates

- *This all leads to the following.*

**11.3.2** AREA IN POLAR COORDINATES. If $\alpha$ and $\beta$ are angles that satisfy the condition

$$\alpha < \beta \leq \alpha + 2\pi$$

and if $f(\theta)$ is continuous and either nonnegative or nonpositive for $\alpha \leq \theta \leq \beta$, then the area $A$ of the region $R$ enclosed by the polar curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ is

$$A = \int_{\alpha}^{\beta} \frac{1}{2}[f(\theta)]^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta \tag{3}$$

# Area of Polar Coordinates

- *The hardest part of this is determining the limits of integration. This is done as follows:*

**Step 1.** Sketch the region $R$ whose area is to be determined.

**Step 2.** Draw an arbitrary "radial line" from the pole to the boundary curve $r = f(\theta)$.

**Step 3.** Ask, "Over what interval of values must $\theta$ vary in order for the radial line to sweep out the region $R$?"

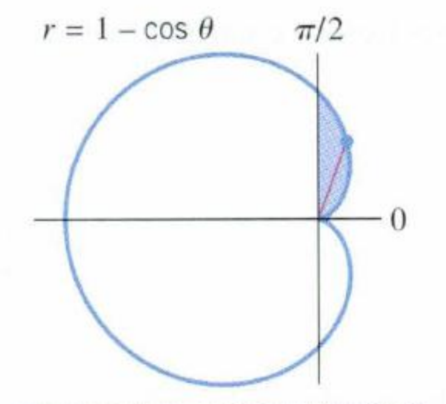
**Step 4.** Your answer in Step 3 will determine the lower and upper limits of integration.

# Example 1

- *Find the area of the region in the first quadrant that is within the cardioid* $r = 1 - \cos\theta$.

# Example 1

- *Find the area of the region in the first quadrant that is within the cardioid r = 1 – cos$\theta$.*
- *The region and a typical radial line are shown. For the radial line to sweep out the region, $\theta$ must vary from 0 to $\pi$/2. So we have*




The shaded region is swept out by the radial line as $\theta$ varies from 0 to $\pi/2$.

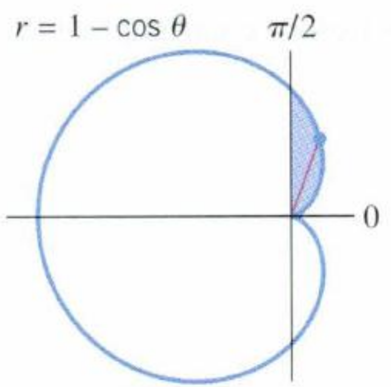
$$A = \int_0^{\pi/2} \tfrac{1}{2} r^2 d\theta = \tfrac{1}{2}\int_0^{\pi/2} (1-\cos\theta)^2 d\theta = \tfrac{3}{8}\pi - 1$$

# Example 2

- *Find the entire area within the cardioid* $r = 1 - \cos\theta$.

# Example 2

- *Find the entire area within the cardioid $r = 1 - \cos\theta$.*
- *For the radial line to sweep out the entire cardioid, $\theta$ must vary from 0 to $2\pi$. So we have*




The shaded region is swept out by the radial line as $\theta$ varies from 0 to $\pi/2$.

$$A = \int_0^{2\pi} \tfrac{1}{2} r^2 d\theta = \tfrac{1}{2} \int_0^{2\pi} (1 - \cos\theta)^2 d\theta = \frac{3\pi}{2}$$

# Example 2

- *Find the entire area within the cardioid r = 1 – cos$\theta$.*
- *For the radial line to sweep out the entire cardioid, $\theta$ must vary from 0 to 2$\pi$. So we have*

$$A = \int_0^{2\pi} \tfrac{1}{2} r^2 d\theta = \tfrac{1}{2} \int_0^{2\pi} (1 - \cos\theta)^2 d\theta = \frac{3\pi}{2}$$

- *We can also look at it this way.*

$$A = 2\int_0^{\pi} \tfrac{1}{2} r^2 d\theta = \int_0^{\pi} (1 - \cos\theta)^2 d\theta = \frac{3\pi}{2}$$

# Example 3

- *Find the area of the region enclosed by the rose curve $r = \cos 2\theta$.*

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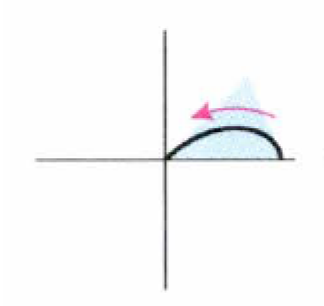
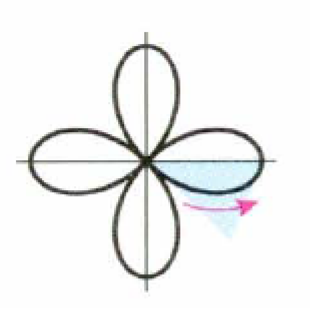
- *Find the area of the region enclosed by the rose curve $r = \cos 2\theta$.*
- *Using symmetry, the area in the first quadrant that is swept out for $0 \leq \theta \leq \pi/4$ is 1/8 of the total area.*



$$A = 8\int_0^{\pi/4} \tfrac{1}{2} r^2 d\theta = 4\int_0^{\pi/4} \cos^2 2\theta d\theta = \frac{\pi}{2}$$



# Example 4

- *Find the area of the region that is inside of the cardioid* $r = 4 + 4\cos\theta$ *and outside of the circle* $r = 6$.

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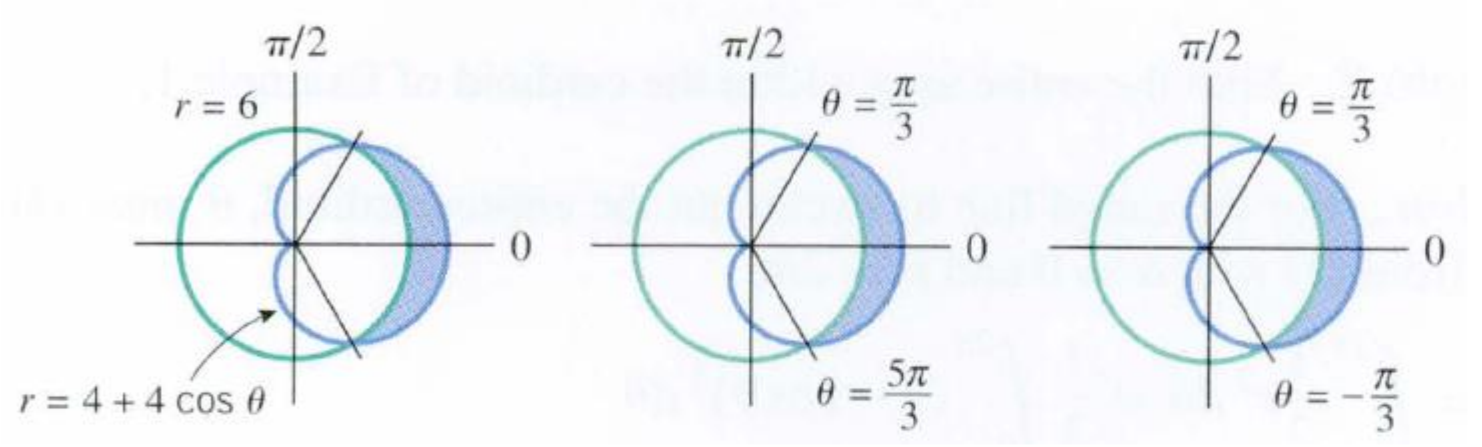
- *Find the area of the region that is inside of the cardioid r = 4 + 4 cos$\theta$ and outside of the circle r = 6.*
- *First, we need to find the bounds.*

$$6 = 4 + 4\cos\theta$$

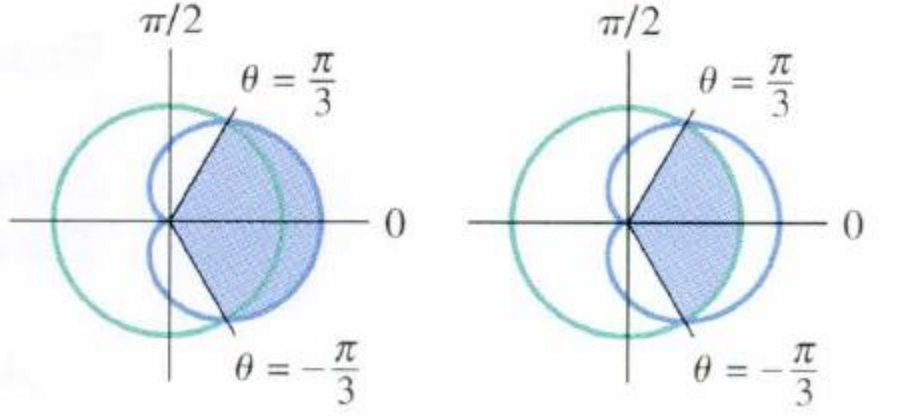
$$\cos\theta = \frac{1}{2}$$



# Example 4

- *Find the area of the region that is inside of the cardioid r = 4 + 4 cos$\theta$ and outside of the circle r = 6.*
- *The area of the region can be obtained by subtracting the areas in the figures below.*

$$A = \int_{-\pi/3}^{\pi/3} \tfrac{1}{2}(4+4\cos\theta)^2\, d\theta - \int_{-\pi/3}^{\pi/3} \tfrac{1}{2}(6)^2\, d\theta = 18\sqrt{3} - 4\pi$$



# Homework

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- 1-15 odd