## An Introduction to Linear Programming

- Linear Programming Problem
- Problem Formulation
- A Maximization Problem
- Graphical Solution Procedure
- Extreme Points and the Optimal Solution
- Computer Solutions
- A Minimization Problem
- Special Cases
- **Linear Programming Applications**

## Linear Programming (LP) Problem

- The maximization or minimization of some quantity is the objective in all linear programming problems.
- All LP problems have constraints that limit the degree to which the objective can be pursued.
- A feasible solution satisfies all the problem's constraints.
- An optimal solution is a feasible solution that results in the largest possible objective function value when maximizing (or smallest when minimizing).
- A graphical solution method can be used to solve a linear program with two variables.

## Linear Programming (LP) Problem

- If both the objective function and the constraints are linear, the problem is referred to as a linear programming problem.
- Linear functions are functions in which each variable appears in a separate term raised to the first power and is multiplied by a constant (which could be 0).
- Linear constraints are linear functions that are restricted to be "less than or equal to", "equal to", or "greater than or equal to" a constant.

## Problem Formulation

- Problem formulation or modeling is the process of translating a verbal statement of a problem into a mathematical statement.

## Guidelines for Model Formulation

- Understand the problem thoroughly.
- Describe the objective.
- Describe each constraint.
- Define the decision variables.
- Write the objective in terms of the decision variables.
- Write the constraints in terms of the decision variables.

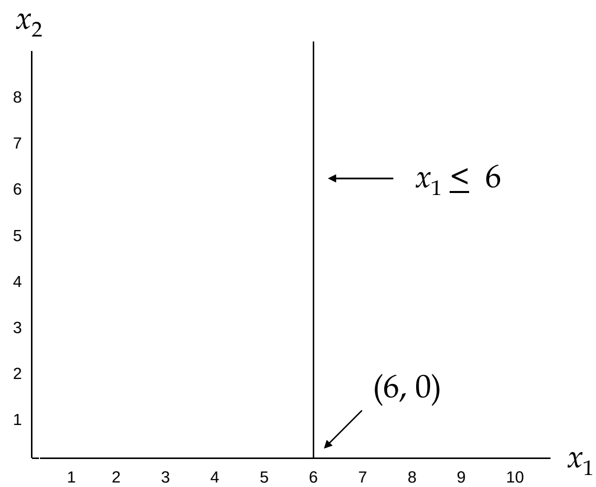
## Example 1: A Maximization Problem

- LP Formulation

$$
\begin{array}{lrcl}
\text{Max} & 5x_1 + 7x_2 & & \\
\text{s.t.} & x_1 \phantom{{}+3x_2} & \leq & 6 \\
 & 2x_1 + 3x_2 & \leq & 19 \\
 & x_1 + \phantom{3}x_2 & \leq & 8 \\
 & x_1, x_2 \geq 0 & &
\end{array}
$$

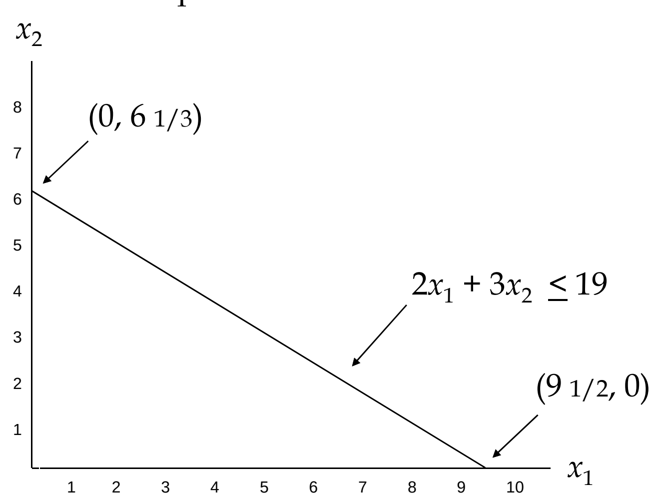
## Example 1: Graphical Solution

- Constraint #1 Graphed



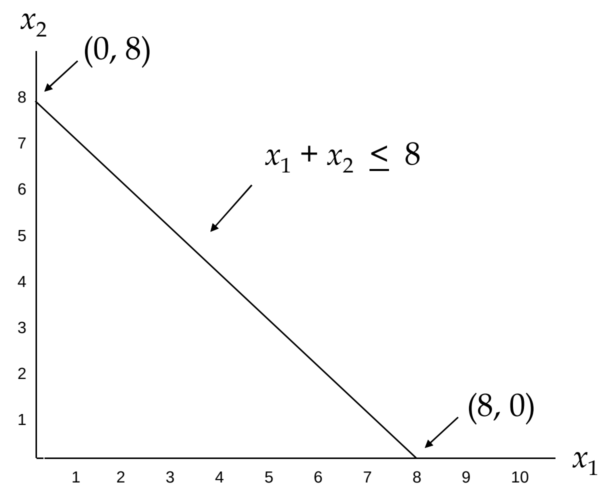

## Example 1: Graphical Solution

- Constraint #2 Graphed



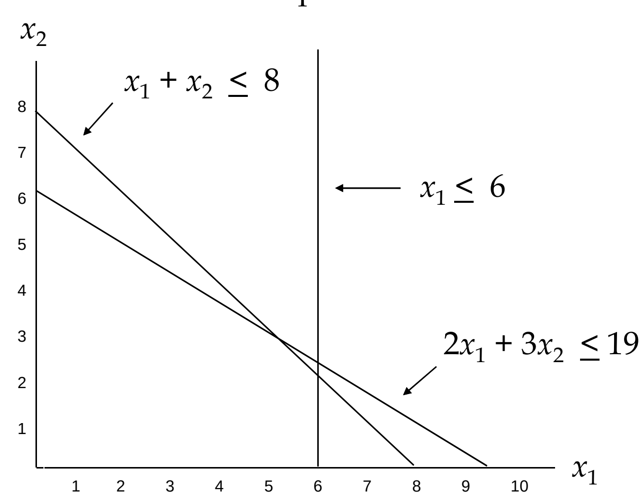
# Example 1: Graphical Solution

- Constraint #3 Graphed



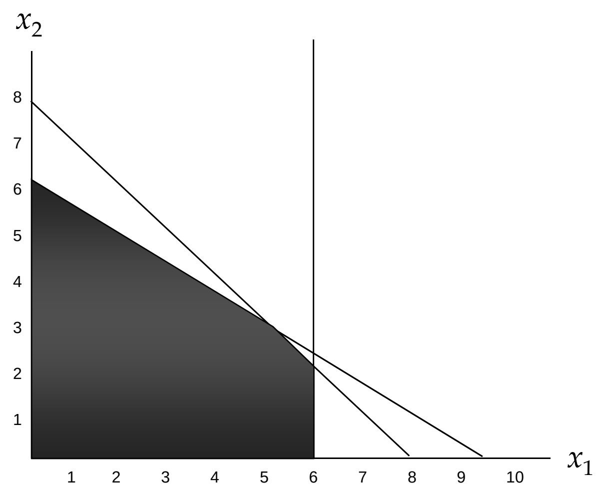

# Example 1: Graphical Solution

- Combined-Constraint Graph



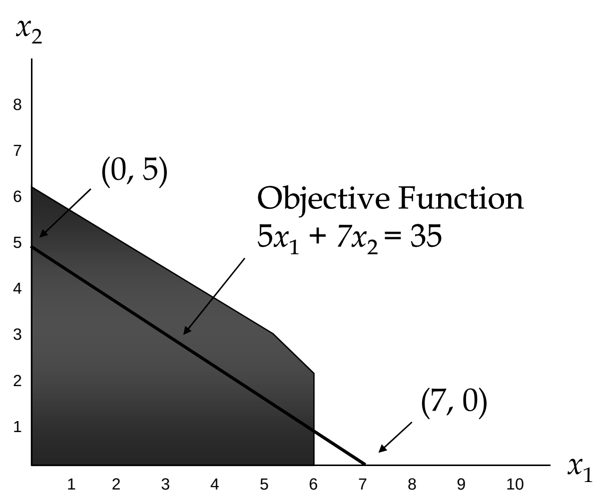
## Example 1:  Graphical Solution

- Feasible Solution Region



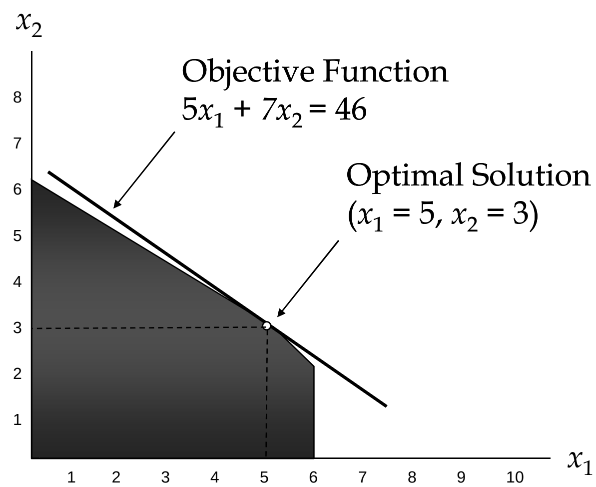

## Example 1:  Graphical Solution

- Objective Function Line



## Example 1: Graphical Solution

- Optimal Solution




## Summary of the Graphical Solution Procedure for Maximization Problems

- Prepare a graph of the feasible solutions for each of the constraints.
- Determine the feasible region that satisfies all the constraints simultaneously..
- Draw an objective function line.
- Move parallel objective function lines toward larger objective function values without entirely leaving the feasible region.
- Any feasible solution on the objective function line with the largest value is an optimal solution.

## Slack and Surplus Variables

- A linear program in which all the variables are non-negative and all the constraints are equalities is said to be in standard form.
- Standard form is attained by adding slack variables to "less than or equal to" constraints, and by subtracting surplus variables from "greater than or equal to" constraints.
- Slack and surplus variables represent the difference between the left and right sides of the constraints.
- Slack and surplus variables have objective function coefficients equal to 0.

## Example 1

- Standard Form

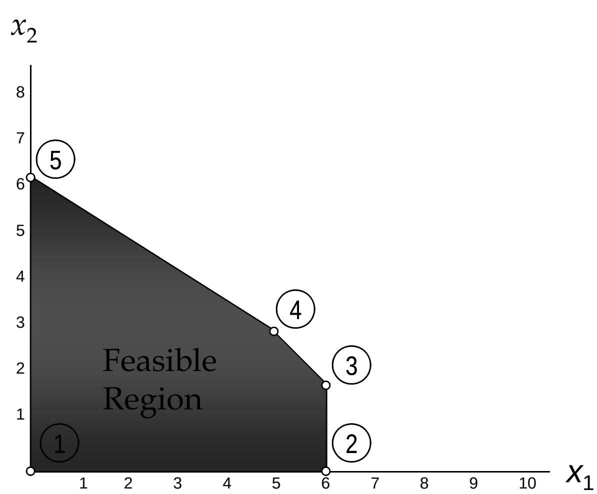
$$\begin{array}{llcl}
\text{Max} & 5x_1 + 7x_2 + 0s_1 + 0s_2 + 0s_3 & & \\
\text{s.t.} & x_1 + s_1 & = & 6 \\
 & 2x_1 + 3x_2 + s_2 & = & 19 \\
 & x_1 + x_2 + s_3 & = & 8 \\
 & x_1, x_2, s_1, s_2, s_3 \geq 0 & &
\end{array}$$

## Extreme Points and the Optimal Solution

- The corners or vertices of the feasible region are referred to as the extreme points.
- An optimal solution to an LP problem can be found at an extreme point of the feasible region.
- When looking for the optimal solution, you do not have to evaluate all feasible solution points.
- You have to consider only the extreme points of the feasible region.

## Example 1: Graphical Solution

- The Five Extreme Points



## Computer Solutions

- Computer programs designed to solve LP problems are now widely available.
- Most large LP problems can be solved with just a few minutes of computer time.
- Small LP problems usually require only a few seconds.
- Linear programming solvers are now part of many spreadsheet packages, such as *Microsoft Excel*.

## Interpretation of Computer Output

- In this chapter we will discuss the following output:
  - objective function value
  - values of the decision variables
  - reduced costs
  - slack/surplus
- In Chapter 3 we will discuss how an optimal solution is affected by a change in:
  - a coefficient of the objective function
  - the right-hand side value of a constraint

## Example 1:  Spreadsheet Solution

- Partial Spreadsheet Showing Problem Data

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | LHS Coefficients | | |
| 2 | Constraints | X1 | X2 | RHS Values |
| 3 | #1 | 1 | 0 | 6 |
| 4 | #2 | 2 | 3 | 19 |
| 5 | #3 | 1 | 1 | 8 |
| 6 | Obj.Func.Coeff. | 5 | 7 | |

## Example 1:  Spreadsheet Solution

- Partial Spreadsheet Showing Solution

| | A | B | C | D |
|---|---|---|---|---|
| 8 | | Optimal Decision Variable Values | | |
| 9 | | X1 | X2 | |
| 10 | | 5.0 | 3.0 | |
| 11 | | | | |
| 12 | Maximized Objective Function | | 46.0 | |
| 13 | | | | |
| 14 | Constraints | Amount Used | | RHS Limits |
| 15 | #1 | 5 | <= | 6 |
| 16 | #2 | 19 | <= | 19 |
| 17 | #3 | 8 | <= | 8 |

## Example 1: Spreadsheet Solution

- Interpretation of Computer Output

  We see from the previous slide that:

| | | |
|---|---|---|
| Objective Function Value | = | 46 |
| Decision Variable #1 ($x_1$) | = | 5 |
| Decision Variable #2 ($x_2$) | = | 3 |
| Slack in Constraint #1 | = | 1 (= 6 - 5) |
| Slack in Constraint #2 | = | 0 (= 19 - 19) |
| Slack in Constraint #3 | = | 0 (= 8 - 8) |

## Reduced Cost

- The <u>reduced cost</u> for a decision variable whose value is 0 in the optimal solution is the amount the variable's objective function coefficient would have to improve (increase for maximization problems, decrease for minimization problems) before this variable could assume a positive value.
- The reduced cost for a decision variable with a positive value is 0.

## Example 1: Spreadsheet Solution

- Reduced Costs

Adjustable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $B$8 | X1 | 5.0 | 0.0 | 5 | 2 | 0.333333333 |
| $C$8 | X2 | 3.0 | 0.0 | 7 | 0.5 | 2 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $B$13 | #1 | 5 | 0 | 6 | 1E+30 | 1 |
| $B$14 | #2 | 19 | 2 | 19 | 5 | 1 |
| $B$15 | #3 | 8 | 1 | 8 | 0.333333333 | 1.666666667 |

## Example 2: A Minimization Problem

- LP Formulation

$$\begin{array}{ll} \text{Min} & 5x_1 + 2x_2 \\ \text{s.t.} & 2x_1 + 5x_2 \geq 10 \\ & 4x_1 - x_2 \geq 12 \\ & x_1 + x_2 \geq 4 \\ & x_1, x_2 \geq 0 \end{array}$$

## Example 2: Graphical Solution

- Graph the Constraints

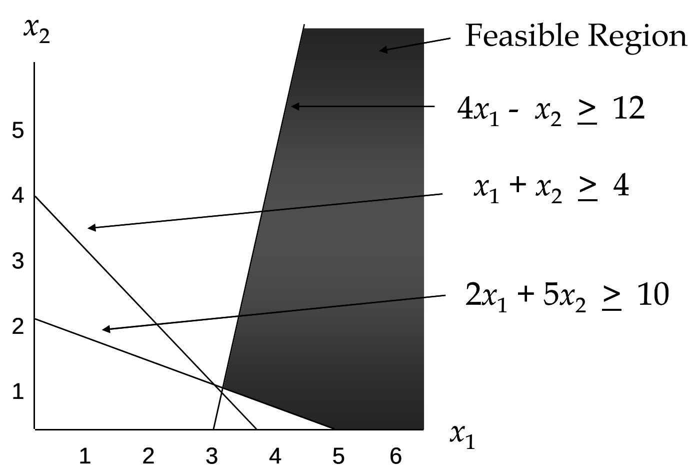
  Constraint 1: When $x_1 = 0$, then $x_2 = 2$; when $x_2 = 0$, then $x_1 = 5$. Connect (5,0) and (0,2). The ">" side is above this line.

  Constraint 2: When $x_2 = 0$, then $x_1 = 3$. But setting $x_1$ to 0 will yield $x_2 = -12$, which is not on the graph. Thus, to get a second point on this line, set $x_1$ to any number larger than 3 and solve for $x_2$: when $x_1 = 5$, then $x_2 = 8$. Connect (3,0) and (5,8). The ">" side is to the right.

  Constraint 3: When $x_1 = 0$, then $x_2 = 4$; when $x_2 = 0$, then $x_1 = 4$. Connect (4,0) and (0,4). The ">" side is above this line.

## Example 2: Graphical Solution

- Constraints Graphed



## Example 2: Graphical Solution

- Graph the Objective Function

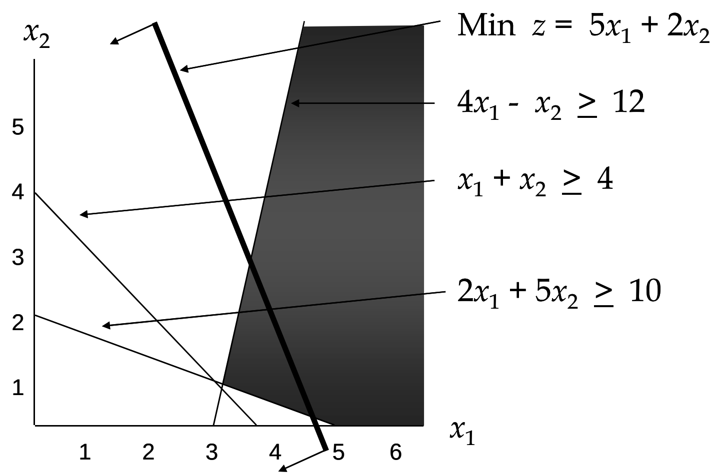
  Set the objective function equal to an arbitrary constant (say 20) and graph it. For $5x_1 + 2x_2 = 20$, when $x_1 = 0$, then $x_2 = 10$; when $x_2 = 0$, then $x_1 = 4$. Connect (4,0) and (0,10).

- Move the Objective Function Line Toward Optimality

  Move it in the direction which lowers its value (down), since we are minimizing, until it touches the last point of the feasible region, determined by the last two constraints.

## Example 2: Graphical Solution

- Objective Function Graphed



## Example 2: Graphical Solution

- Solve for the Extreme Point at the Intersection of the Two Binding Constraints

$$4x_1 - x_2 = 12$$

$$x_1 + x_2 = 4$$

Adding these two equations gives:

$$5x_1 = 16 \quad \text{or} \quad x_1 = 16/5.$$

Substituting this into $x_1 + x_2 = 4$ gives: $x_2 = 4/5$

## Example 2: Graphical Solution

- Solve for the Optimal Value of the Objective Function

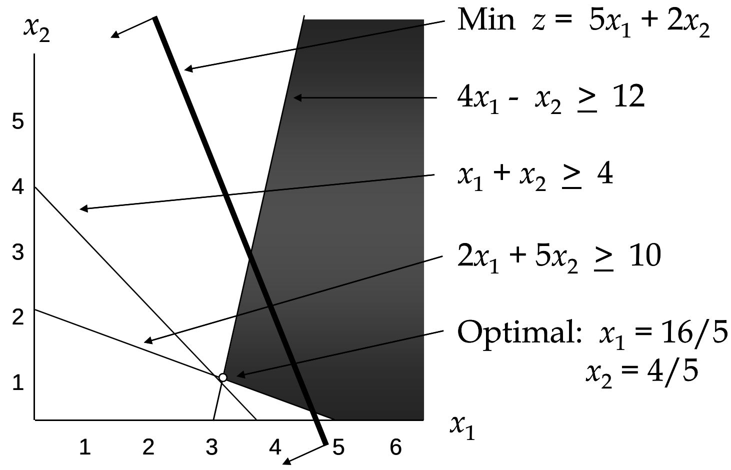
Solve for $z = 5x_1 + 2x_2 = 5(16/5) + 2(4/5) = 88/5$.

Thus the optimal solution is

$$x_1 = 16/5; \quad x_2 = 4/5; \quad z = 88/5$$

## Example 2: Graphical Solution

- Optimal Solution




## Example 2: Spreadsheet Solution

- Partial Spreadsheet Showing Problem Data

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | LHS Coefficients | | |
| 2 | Constraints | X1 | X2 | RHS |
| 3 | #1 | 2 | 5 | 10 |
| 4 | #2 | 4 | -1 | 12 |
| 5 | #3 | 1 | 1 | 4 |
| 6 | Obj.Func.Coeff. | 5 | 2 | |

## Example 2: Spreadsheet Solution

- Partial Spreadsheet Showing Formulas

| | A | B | C | D |
|---|---|---|---|---|
| 9 | | Decision Variables | | |
| 10 | | X1 | X2 | |
| 11 | Dec.Var.Values | | | |
| 12 | | | | |
| 13 | Minimized Objective Function | | =B6*B11+C6*C11 | |
| 14 | | | | |
| 15 | Constraints | Amount Used | | Amount Avail. |
| 16 | #1 | =B3*$B$11+C3*$C$11 | >= | =D3 |
| 17 | #2 | =B4*$B$11+C4*$C$11 | >= | =D4 |
| 18 | #3 | =B5*$B$11+C5*$C$11 | >= | =D5 |

## Example 2: Spreadsheet Solution

- Partial Spreadsheet Showing Solution

| | A | B | C | D |
|---|---|---|---|---|
| 9 | | Decision Variables | | |
| 10 | | X1 | X2 | |
| 11 | Dec.Var.Values | 3.20 | 0.800 | |
| 12 | | | | |
| 13 | Minimized Objective Function | | 17.600 | |
| 14 | | | | |
| 15 | Constraints | Amount Used | | Amount Avail. |
| 16 | #1 | 10.4 | >= | 10 |
| 17 | #2 | 12 | >= | 12 |
| 18 | #3 | 4 | >= | 4 |

## Feasible Region

- The feasible region for a two-variable linear programming problem can be nonexistent, a single point, a line, a polygon, or an unbounded area.
- Any linear program falls in one of three categories:
  - is infeasible
  - has a unique optimal solution or alternate optimal solutions
  - has an objective function that can be increased without bound
- A feasible region may be unbounded and yet there may be optimal solutions. This is common in minimization problems and is possible in maximization problems.

## Special Cases

- Alternative Optimal Solutions

  In the graphical method, if the objective function line is parallel to a boundary constraint in the direction of optimization, there are <u>alternate optimal solutions</u>, with all points on this line segment being optimal.
- Infeasibility

  A linear program which is overconstrained so that no point satisfies all the constraints is said to be <u>infeasible</u>.
- Unboundedness

  (See example on upcoming slide.)

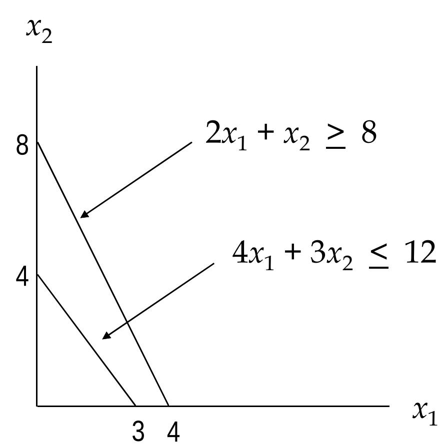
## Example: Infeasible Problem

- Solve graphically for the optimal solution:

$$\begin{array}{ll} \text{Max} & 2x_1 + 6x_2 \\ \text{s.t.} & 4x_1 + 3x_2 \le 12 \\ & 2x_1 + x_2 \ge 8 \\ & x_1, x_2 \ge 0 \end{array}$$

## Example: Infeasible Problem

- There are no points that satisfy both constraints, hence this problem has no feasible region, and no optimal solution.



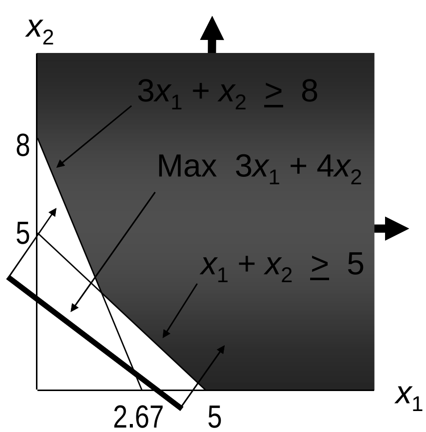
## Example: Unbounded Problem

- Solve graphically for the optimal solution:

$$\begin{aligned} \text{Max} \quad & 3x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 5 \\ & 3x_1 + x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

## Example: Unbounded Problem

- The feasible region is unbounded and the objective function line can be moved parallel to itself without bound so that $z$ can be increased infinitely.



## Linear Programming Applications

- Blending Problem
- Portfolio Planning Problem
- Product Mix Problem
- Transportation Problem
- Data Envelopment Analysis

## Blending Problem

Frederick's Feed Company receives four raw grains from which it blends its dry pet food. The pet food advertises that each 8-ounce can meets the minimum daily requirements for vitamin C, protein and iron. The cost of each raw grain as well as the vitamin C, protein, and iron units per pound of each grain are summarized on the next slide.

Frederick's is interested in producing the 8-ounce mixture at minimum cost while meeting the minimum daily requirements of 6 units of vitamin C, 5 units of protein, and 5 units of iron.

## Blending Problem

| Grain | Vitamin C Units/lb | Protein Units/lb | Iron Units/lb | Cost/lb |
|---|---|---|---|---|
| 1 | 9 | 12 | 0 | .75 |
| 2 | 16 | 10 | 14 | .90 |
| 3 | 8 | 10 | 15 | .80 |
| 4 | 10 | 8 | 7 | .70 |

## Blending Problem

- Define the decision variables

  $x_j$ = the pounds of grain $j$ ($j$ = 1,2,3,4) used in the 8-ounce mixture

- Define the objective function

  Minimize the total cost for an 8-ounce mixture:

  MIN $.75x_1 + .90x_2 + .80x_3 + .70x_4$

## Blending Problem

- Define the constraints

  Total weight of the mix is 8-ounces (.5 pounds):

  (1) $x_1 + x_2 + x_3 + x_4 = .5$

  Total amount of Vitamin C in the mix is at least 6 units:

  (2) $9x_1 + 16x_2 + 8x_3 + 10x_4 > 6$

  Total amount of protein in the mix is at least 5 units:

  (3) $12x_1 + 10x_2 + 10x_3 + 8x_4 > 5$

  Total amount of iron in the mix is at least 5 units:

  (4) $14x_2 + 15x_3 + 7x_4 > 5$

  Nonnegativity of variables: $x_j \geq 0$ for all $j$

## Blending Problem

- *The Management Scientist* Output

OBJECTIVE FUNCTION VALUE = 0.406

| VARIABLE | VALUE | REDUCED COSTS |
|---|---|---|
| X1 | 0.099 | 0.000 |
| X2 | 0.213 | 0.000 |
| X3 | 0.088 | 0.000 |
| X4 | 0.099 | 0.000 |

Thus, the optimal blend is about .10 lb. of grain 1, .21 lb. of grain 2, .09 lb. of grain 3, and .10 lb. of grain 4. The mixture costs Frederick's 40.6 cents.

## Portfolio Planning Problem

Winslow Savings has $20 million available for investment. It wishes to invest over the next four months in such a way that it will maximize the total interest earned over the four month period as well as have at least $10 million available at the start of the fifth month for a high rise building venture in which it will be participating.

## Portfolio Planning Problem

For the time being, Winslow wishes to invest only in 2-month government bonds (earning 2% over the 2-month period) and 3-month construction loans (earning 6% over the 3-month period). Each of these is available each month for investment. Funds not invested in these two investments are liquid and earn 3/4 of 1% per month when invested locally.

## Portfolio Planning Problem

Formulate a linear program that will help Winslow Savings determine how to invest over the next four months if at no time does it wish to have more than $8 million in either government bonds or construction loans.

## Portfolio Planning Problem

- Define the decision variables

  $g_j$ = amount of new investment in government bonds in month $j$

  $c_j$ = amount of new investment in construction loans in month $j$

  $l_j$ = amount invested locally in month $j$, where $j$ = 1,2,3,4

## Portfolio Planning Problem

- Define the objective function

Maximize total interest earned over the 4-month period.

MAX (interest rate on investment)(amount invested)

$$\text{MAX } .02g_1 + .02g_2 + .02g_3 + .02g_4 + .06c_1 + .06c_2 + .06c_3 + .06c_4 + .0075l_1 + .0075l_2 + .0075l_3 + .0075l_4$$

## Portfolio Planning Problem

- Define the constraints

Month 1's total investment limited to $20 million:

(1) $g_1 + c_1 + l_1 = 20{,}000{,}000$

Month 2's total investment limited to principle and interest invested locally in Month 1:

(2) $g_2 + c_2 + l_2 = 1.0075l_1$

or $g_2 + c_2 - 1.0075l_1 + l_2 = 0$

## Portfolio Planning Problem

- Define the constraints (continued)

Month 3's total investment amount limited to principle and interest invested in government bonds in Month 1 and locally invested in Month 2:

(3) $g_3 + c_3 + l_3 = 1.02g_1 + 1.0075l_2$

or $- 1.02g_1 + g_3 + c_3 - 1.0075l_2 + l_3 = 0$

## Portfolio Planning Problem

- Define the constraints (continued)

Month 4's total investment limited to principle and interest invested in construction loans in Month 1, goverment bonds in Month 2, and locally invested in Month 3:

(4) $g_4 + c_4 + l_4 = 1.06c_1 + 1.02g_2 + 1.0075l_3$

or $- 1.02g_2 + g_4 - 1.06c_1 + c_4 - 1.0075l_3 + l_4 = 0$

$10 million must be available at start of Month 5:

(5) $1.06c_2 + 1.02g_3 + 1.0075l_4 \geq 10{,}000{,}000$

## Portfolio Planning Problem

- Define the constraints (continued)

No more than \$8 million in government bonds at any time:

(6) $g_1 \leq 8{,}000{,}000$

(7) $g_1 + g_2 \leq 8{,}000{,}000$

(8) $g_2 + g_3 \leq 8{,}000{,}000$

(9) $g_3 + g_4 \leq 8{,}000{,}000$

## Portfolio Planning Problem

- Define the constraints (continued)

No more than \$8 million in construction loans at any time:

(10) $c_1 \leq 8{,}000{,}000$

(11) $c_1 + c_2 \leq 8{,}000{,}000$

(12) $c_1 + c_2 + c_3 \leq 8{,}000{,}000$

(13) $c_2 + c_3 + c_4 \leq 8{,}000{,}000$

Nonnegativity: $g_j, c_j, l_j \geq 0$ for $j = 1,2,3,4$

## Problem: Floataway Tours

Floataway Tours has $420,000 that may be used to purchase new rental boats for hire during the summer. The boats can be purchased from two different manufacturers. Floataway Tours would like to purchase at least 50 boats and would like to purchase the same number from Sleekboat as from Racer to maintain goodwill. At the same time, Floataway Tours wishes to have a total seating capacity of at least 200.

Pertinent data concerning the boats are summarized on the next slide. Formulate this problem as a linear program.

## Problem: Floataway Tours

- Data

| Boat | Builder | Cost | Maximum Seating | Expected Daily Profit |
|---|---|---|---|---|
| Speedhawk | Sleekboat | $6000 | 3 | $ 70 |
| Silverbird | Sleekboat | $7000 | 5 | $ 80 |
| Catman | Racer | $5000 | 2 | $ 50 |
| Classy | Racer | $9000 | 6 | $110 |

## Problem: Floataway Tours

- Define the decision variables

  $x_1$ = number of Speedhawks ordered

  $x_2$ = number of Silverbirds ordered

  $x_3$ = number of Catmans ordered

  $x_4$ = number of Classys ordered

- Define the objective function

  Maximize total expected daily profit:

  Max: (Expected daily profit per unit)

  x (Number of units)

  Max: $70x_1 + 80x_2 + 50x_3 + 110x_4$

## Problem: Floataway Tours

- Define the constraints

  (1) Spend no more than $420,000:

  $6000x_1 + 7000x_2 + 5000x_3 + 9000x_4 \leq 420{,}000$

  (2) Purchase at least 50 boats:

  $x_1 + x_2 + x_3 + x_4 \geq 50$

  (3) Number of boats from Sleekboat equals number of boats from Racer:

  $x_1 + x_2 = x_3 + x_4$ or $x_1 + x_2 - x_3 - x_4 = 0$

## Problem: Floataway Tours

- Define the constraints (continued)

  (4) Capacity at least 200:

  $3x_1 + 5x_2 + 2x_3 + 6x_4 \geq 200$

  Nonnegativity of variables:

  $x_j \geq 0$, for $j = 1,2,3,4$

## Problem: Floataway Tours

- Complete Formulation

$$
\begin{aligned}
\text{Max}\quad & 70x_1 + 80x_2 + 50x_3 + 110x_4 \\
\text{s.t.}\quad & \\
& 6000x_1 + 7000x_2 + 5000x_3 + 9000x_4 \leq 420{,}000 \\
& x_1 + x_2 + x_3 + x_4 \geq 50 \\
& x_1 + x_2 - x_3 - x_4 = 0 \\
& 3x_1 + 5x_2 + 2x_3 + 6x_4 \geq 200 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

## Problem: Floataway Tours

- Partial Spreadsheet Showing Problem Data

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | | LHS Coefficients | | | | |
| 2 | Constr. | X1 | X2 | X3 | X4 | RHS |
| 3 | #1 | 6 | 7 | 5 | 9 | 420 |
| 4 | #2 | 1 | 1 | 1 | 1 | 50 |
| 5 | #3 | 1 | 1 | -1 | -1 | 0 |
| 6 | #4 | 3 | 5 | 2 | 6 | 200 |
| 7 | Object. | 70 | 80 | 50 | 110 | |

## Problem: Floataway Tours

- Partial Spreadsheet Showing Solution

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 9 | | | Decision Variable Values | | | |
| 10 | | | X1 | X2 | X3 | X4 |
| 11 | No. of Boats | | 28 | 0 | 0 | 28 |
| 12 | | | | | | |
| 13 | Maximum Total Profit | | | 5040 | | |
| 14 | | | | | | |
| 15 | Constraints | | | LHS | | RHS |
| 16 | Spending Max. | | | 420.0 | <= | 420 |
| 17 | Min. # Boats | | | 56.0 | >= | 50 |
| 18 | Equal Sourcing | | | 0.0 | = | 0 |
| 19 | Min. Seating | | | 252.0 | >= | 200 |

## Problem: Floataway Tours

- The Management Science Output

OBJECTIVE FUNCTION VALUE = 5040.000

| Variable | Value | Reduced Cost |
|---|---|---|
| $x_1$ | 28.000 | 0.000 |
| $x_2$ | 0.000 | 2.000 |
| $x_3$ | 0.000 | 12.000 |
| $x_4$ | 28.000 | 0.000 |

| Constraint | Slack/Surplus | Dual Price |
|---|---|---|
| 1 | 0.000 | 0.012 |
| 2 | 6.000 | 0.000 |
| 3 | 0.000 | -2.000 |
| 4 | 52.000 | 0.000 |

## Problem: Floataway Tours

- Solution Summary
  - Purchase 28 Speedhawks from Sleekboat.
  - Purchase 28 Classy's from Racer.
  - Total expected daily profit is $5,040.00.
  - The minimum number of boats was exceeded by 6 (surplus for constraint #2).
  - The minimum seating capacity was exceeded by 52 (surplus for constraint #4).

## Problem: Floataway Tours

- Sensitivity Report

Adjustable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $D$12 | X1 | 28 | 0 | 70 | 45 | 1.875 |
| $E$12 | X2 | 0 | -2 | 80 | 2 | 1E+30 |
| $F$12 | X3 | 0 | -12 | 50 | 12 | 1E+30 |
| $G$12 | X4 | 28 | 0 | 110 | 1E+30 | 16.36363636 |

## Problem: Floataway Tours

- Sensitivity Report

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $E$17 | #1 | 420.0 | 12.0 | 420 | 1E+30 | 45 |
| $E$18 | #2 | 56.0 | 0.0 | 50 | 6 | 1E+30 |
| $E$19 | #3 | 0.0 | -2.0 | 0 | 70 | 30 |
| $E$20 | #4 | 252.0 | 0.0 | 200 | 52 | 1E+30 |

## Problem: U.S. Navy

The Navy has 9,000 pounds of material in Albany, Georgia which it wishes to ship to three installations: San Diego, Norfolk, and Pensacola. They require 4,000, 2,500, and 2,500 pounds, respectively. Government regulations require equal distribution of shipping among the three carriers.

The shipping costs per pound for truck, railroad, and airplane transit are shown on the next slide. Formulate and solve a linear program to determine the shipping arrangements (mode, destination, and quantity) that will minimize the total shipping cost.

## Problem: U.S. Navy

- Data

| | Destination | | |
|---|---|---|---|
| **Mode** | **San Diego** | **Norfolk** | **Pensacola** |
| Truck | $12 | $ 6 | $ 5 |
| Railroad | 20 | 11 | 9 |
| Airplane | 30 | 26 | 28 |

## Problem: U.S. Navy

- Define the Decision Variables

We want to determine the pounds of material, $x_{ij}$, to be shipped by mode $i$ to destination $j$. The following table summarizes the decision variables:

| | San Diego | Norfolk | Pensacola |
|---|---|---|---|
| **Truck** | $x_{11}$ | $x_{12}$ | $x_{13}$ |
| **Railroad** | $x_{21}$ | $x_{22}$ | $x_{23}$ |
| **Airplane** | $x_{31}$ | $x_{32}$ | $x_{33}$ |

## Problem: U.S. Navy

- Define the Objective Function

Minimize the total shipping cost.

Min: (shipping cost per pound for each mode per destination pairing) x (number of pounds shipped by mode per destination pairing).

Min: $12x_{11} + 6x_{12} + 5x_{13} + 20x_{21} + 11x_{22} + 9x_{23}$
$+ 30x_{31} + 26x_{32} + 28x_{33}$

## Problem: U.S. Navy

- Define the Constraints

  Equal use of transportation modes:

  (1) $x_{11} + x_{12} + x_{13} = 3000$

  (2) $x_{21} + x_{22} + x_{23} = 3000$

  (3) $x_{31} + x_{32} + x_{33} = 3000$

  Destination material requirements:

  (4) $x_{11} + x_{21} + x_{31} = 4000$

  (5) $x_{12} + x_{22} + x_{32} = 2500$

  (6) $x_{13} + x_{23} + x_{33} = 2500$

  Nonnegativity of variables:

  $x_{ij} \geq 0,\ i = 1,2,3$ and $j = 1,2,3$

## Problem: U.S. Navy

- Partial Spreadsheet Showing Problem Data

| | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | LHS Coefficients | | | | | | | | | |
| 2 | Con. | $X_{11}$ | $X_{12}$ | $X_{13}$ | $X_{21}$ | $X_{22}$ | $X_{23}$ | $X_{31}$ | $X_{32}$ | $X_{33}$ | RHS |
| 3 | #1 | 1 | 1 | 1 | | | | | | | 3000 |
| 4 | #2 | | | | 1 | 1 | 1 | | | | 3000 |
| 5 | #3 | | | | | | | 1 | 1 | 1 | 3000 |
| 6 | #4 | 1 | | | 1 | | | 1 | | | 4000 |
| 7 | #5 | | 1 | | | 1 | | | 1 | | 2500 |
| 8 | #6 | | | 1 | | | 1 | | | 1 | 2500 |
| 9 | Obj. | 12 | 6 | 5 | 20 | 11 | 9 | 30 | 26 | 28 | |

## Problem: U.S. Navy

- Partial Spreadsheet Showing Solution

| | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | | $X_{11}$ | $X_{12}$ | $X_{13}$ | $X_{21}$ | $X_{22}$ | $X_{23}$ | $X_{31}$ | $X_{32}$ | $X_{33}$ | |
| 13 | | 1000 | 2000 | 0 | 0 | 500 | 2500 | 3000 | 0 | 0 | |
| 14 | | Minimized Total Shipping Cost | | | | | | | 142000 | | |
| 15 | | | | | | | | | | | |
| 16 | Constraints | | LHS | | | | RHS | | | | |
| 17 | Truc | | 3000 | | = | | 3000 | | | | |
| 18 | Rail | | 3000 | | = | | 3000 | | | | |
| 19 | Air | | 3000 | | = | | 3000 | | | | |
| 20 | San | | 4000 | | = | | 4000 | | | | |
| 21 | Nor | | 2500 | | = | | 2500 | | | | |
| 22 | Pen | | 2500 | | = | | 2500 | | | | |

## Problem: U.S. Navy

- The Management Scientist Output

OBJECTIVE FUNCTION VALUE = 142000.000

| Variable | Value | Reduced Cost |
|---|---|---|
| $x_{11}$ | 1000.000 | 0.000 |
| $x_{12}$ | 2000.000 | 0.000 |
| $x_{13}$ | 0.000 | 1.000 |
| $x_{21}$ | 0.000 | 3.000 |
| $x_{22}$ | 500.000 | 0.000 |
| $x_{23}$ | 2500.000 | 0.000 |
| $x_{31}$ | 3000.000 | 0.000 |
| $x_{32}$ | 0.000 | 2.000 |
| $x_{33}$ | 0.000 | 6.000 |

## Problem: U.S. Navy

- Solution Summary
  - San Diego will receive 1000 lbs. by truck and 3000 lbs. by airplane.
  - Norfolk will receive 2000 lbs. by truck and 500 lbs. by railroad.
  - Pensacola will receive 2500 lbs. by railroad.
  - The total shipping cost will be $142,000.

## Data Envelopment Analysis

- Data envelopment analysis (DEA) is an LP application used to determine the relative operating efficiency of units with the same goals and objectives.
- DEA creates a fictitious composite unit made up of an optimal weighted average ($W_1$, $W_2$,…) of existing units.
- An individual unit, *k*, can be compared by determining *E*, the fraction of unit *k*'s input resources required by the optimal composite unit.
- If $E < 1$, unit *k* is less efficient than the composite unit and be deemed relatively inefficient.
- If $E = 1$, there is no evidence that unit *k* is inefficient, but one cannot conclude that *k* is absolutely efficient.

## Data Envelopment Analysis

- The DEA Model

MIN $E$

s.t. Weighted outputs $\geq$ Unit $k$'s output
(for each measured output)

Weighted inputs $\leq$ $E$ [Unit $k$'s input]
(for each measured input)

Sum of weights = 1

$E$, weights $\geq$ 0

## DEA Example: Roosevelt High

The Langley County School District is trying to determine the relative efficiency of its three high schools. In particular, it wants to evaluate Roosevelt High School.

The district is evaluating performances on SAT scores, the number of seniors finishing high school, and the number of students who enter college as a function of the number of teachers teaching senior classes, the prorated budget for senior instruction, and the number of students in the senior class.

## DEA Example:  Roosevelt High

- Input

| | Roosevelt | Lincoln | Washington |
|---|---|---|---|
| Senior Faculty | 37 | 25 | 23 |
| Budget ($100,000's) | 6.4 | 5.0 | 4.7 |
| Senior Enrollments | 850 | 700 | 600 |

## DEA Example:  Roosevelt High

- Output

| | Roosevelt | Lincoln | Washington |
|---|---|---|---|
| Average SAT Score | 800 | 830 | 900 |
| High School Graduates | 450 | 500 | 400 |
| College Admissions | 140 | 250 | 370 |

## DEA Example: Roosevelt High

- Decision Variables

  $E$ = Fraction of Roosevelt's input resources required by the composite high school

  $w_1$ = Weight applied to Roosevelt's input/output resources by the composite high school

  $w_2$ = Weight applied to Lincoln's input/output resources by the composite high school

  $w_3$ = Weight applied to Washington's input/output resources by the composite high school

## DEA Example: Roosevelt High

- Objective Function

  Minimize the fraction of Roosevelt High School's input resources required by the composite high school:

  MIN $E$

## DEA Example: Roosevelt High

- Constraints

Sum of the Weights is 1:

(1) $w_1 + w_2 + w_3 = 1$

Output Constraints:

Since $w_1 = 1$ is possible, each output of the composite school must be at least as great as that of Roosevelt:

(2) $800w_1 + 830w_2 + 900w_3 \geq 800$ (SAT Scores)

(3) $450w_1 + 500w_2 + 400w_3 \geq 450$ (Graduates)

(4) $140w_1 + 250w_2 + 370w_3 \geq 140$ (College Admissions)

## DEA Example: Roosevelt High

- Constraints

Input Constraints:

The input resources available to the composite school is a fractional multiple, $E$, of the resources available to Roosevelt. Since the composite high school cannot use more input than that available to it, the input constraints are:

(5) $37w_1 + 25w_2 + 23w_3 \leq 37E$ (Faculty)

(6) $6.4w_1 + 5.0w_2 + 4.7w_3 \leq 6.4E$ (Budget)

(7) $850w_1 + 700w_2 + 600w_3 \leq 850E$ (Seniors)

Nonnegativity of variables:

$E, w_1, w_2, w_3 \geq 0$

## DEA Example: Roosevelt High

- *Management Scientist* Output

OBJECTIVE FUNCTION VALUE = 0.765

| VARIABLE | VALUE | REDUCED COSTS |
|---|---|---|
| E | 0.765 | 0.000 |
| W1 | 0.000 | 0.235 |
| W2 | 0.500 | 0.000 |
| W3 | 0.500 | 0.000 |

## DEA Example: Roosevelt High

- *Management Scientist* Output

| CONSTRAINT | SLACK/SURPLUS | DUAL PRICES |
|---|---|---|
| 1 | 0.000 | -0.235 |
| 2 | 65.000 | 0.000 |
| 3 | 0.000 | -0.001 |
| 4 | 170.000 | 0.000 |
| 5 | 4.294 | 0.000 |
| 6 | 0.044 | 0.000 |
| 7 | 0.000 | 0.001 |

## DEA Example: Roosevelt High

- Conclusion

  The output shows that the composite school is made up of equal weights of Lincoln and Washington. Roosevelt is 76.5% efficient compared to this composite school when measured by college admissions (because of the 0 slack on this constraint (#4)). It is less than 76.5% efficient when using measures of SAT scores and high school graduates (there is positive slack in constraints 2 and 3.)