Newton’s Method

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Outline

1. Newton’s Method.
2. Example using Newton’s Method and Fixed point Iteration.
3. Practical application of Newton’s Method

Newton's Method

Context

Newton's (or the *Newton-Raphson*) method is one of the most powerful and well-known numerical methods for solving a root-finding problem.

Various ways of introducing Newton's method

- Graphically, as is often done in calculus.
- As a technique to obtain faster convergence than offered by other types of functional iteration.

Newton's Method

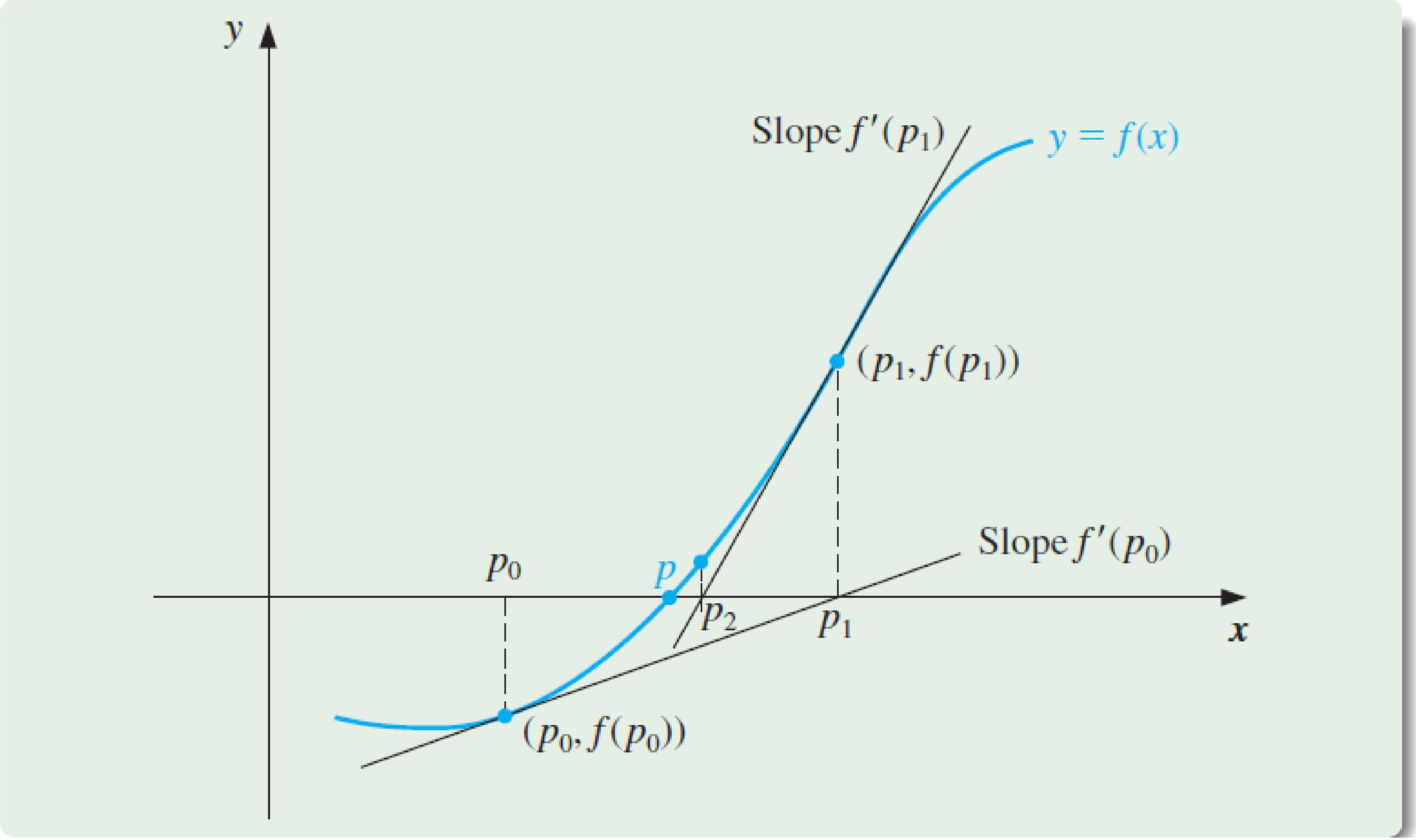
$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \equiv p_1.$$

Newton's Method

This sets the stage for Newton's method, which starts with an initial approximation p_0 and generates the sequence $\{p_n\}_{n=0}^{\infty}$, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \qquad \text{for } n \geq 1$$

Newton's Method: Using Successive Tangents



Newton's Method

Stopping Criteria for the Algorithm

- Various stopping procedures can be applied in Step 3.3.
- We can select a tolerance $\epsilon > 0$ and generate $p_1, \ldots, p_N$ until one of the following conditions is met:

$$|p_N - p_{N-1}| < \epsilon \quad (1)$$

$$\frac{|p_N - p_{N-1}|}{|p_N|} < \epsilon, \quad p_N \neq 0, \quad \text{or} \quad (2)$$

$$|f(p_N)| < \epsilon \quad (3)$$

- Note that none of these inequalities give precise information about the actual error $|p_N - p|$.

Newton's Method as a Functional Iteration Technique

Functional Iteration

- Newton's Method

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \qquad \text{for } n \geq 1$$

- can be written in the form

$$p_n = g\left(p_{n-1}\right)$$

- with

$$g\left(p_{n-1}\right) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \qquad \text{for } n \geq 1$$

Newton's Method

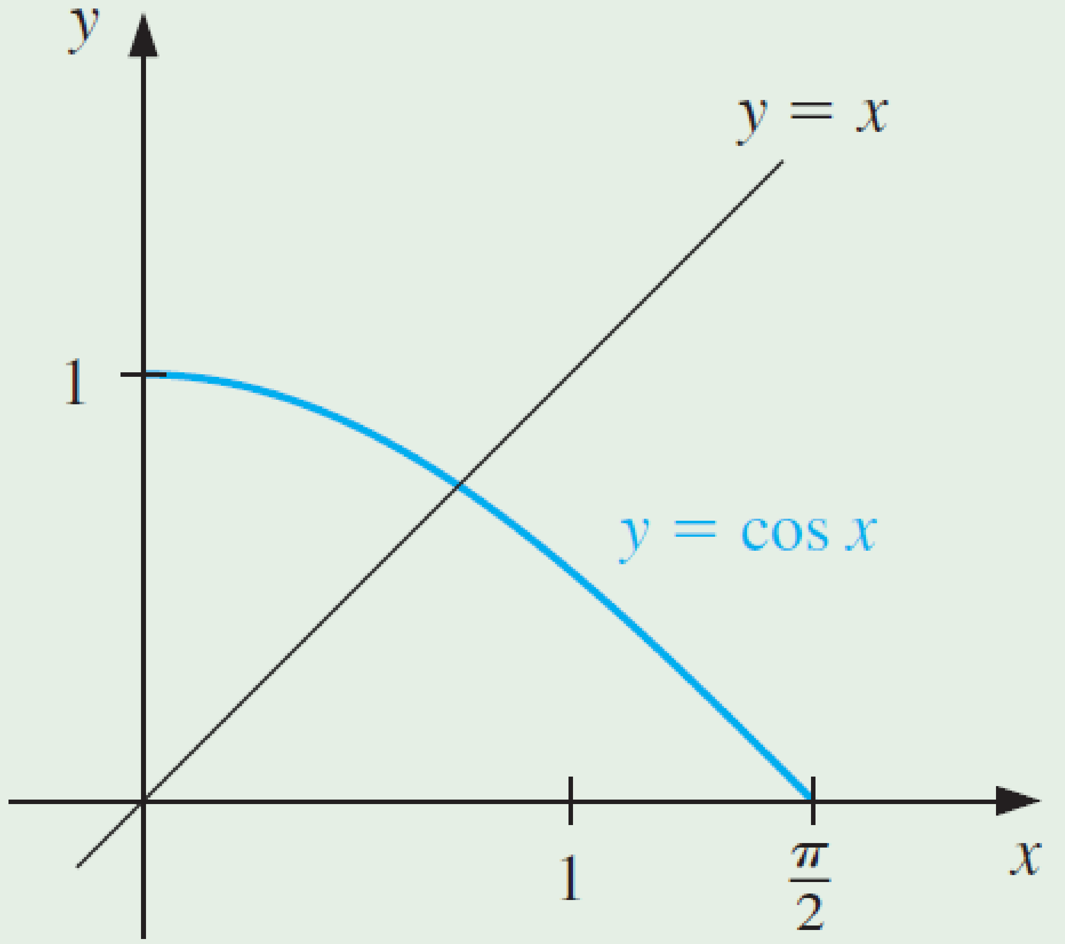
Example: Fixed-Point Iteration & Newton's Method

Consider the function

$$f(x) = \cos x - x = 0$$

Approximate a root of f using (a) a fixed-point method, and (b) Newton's Method

Newton's Method & Fixed-Point Iteration



Newton's Method & Fixed-Point Iteration

(a) Fixed-Point Iteration for $f(x) = \cos x - x$

- A solution to this root-finding problem is also a solution to the fixed-point problem
$$x = \cos x$$
and the graph implies that a single fixed-point p lies in $[0, \pi/2]$.
- The following table shows the results of fixed-point iteration with $p_0 = \pi/4$.
- The best conclusion from these results is that $p \approx 0.74$.

Newton's Method & Fixed-Point Iteration

Fixed-Point Iteration: $x = \cos(x)$, $x_0 = \frac{\pi}{4}$

n	p_{n-1}	p_n	$\lvert p_n - p_{n-1}\rvert$	e_n/e_{n-1}
1	0.7853982	0.7071068	0.0782914	—
2	0.707107	0.760245	0.053138	0.678719
3	0.760245	0.724667	0.035577	0.669525
4	0.724667	0.748720	0.024052	0.676064
5	0.748720	0.732561	0.016159	0.671826
6	0.732561	0.743464	0.010903	0.674753
7	0.743464	0.736128	0.007336	0.672816

Newton's Method

(b) Newton's Method for $f(x) = \cos x - x$

- To apply Newton's method to this problem we need

$$f'(x) = -\sin x - 1$$

- Starting again with $p_0 = \pi/4$, we generate the sequence defined, for $n \geq 1$, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f(p'_{n-1})} = p_{n-1} - \frac{\cos p_{n-1} - p_{n-1}}{-\sin p_{n-1} - 1}.$$

- This gives the approximations shown in the following table.

Newton's Method

Newton's Method for $f(x) = \cos(x) - x$, $x_0 = \frac{\pi}{4}$

n	p_{n-1}	$f(p_{n-1})$	$f'(p_{n-1})$	p_n	$\lvert p_n - p_{n-1} \rvert$
1	0.78539816	-0.078291	-1.707107	0.73953613	0.04586203
2	0.73953613	-0.000755	-1.673945	0.73908518	0.00045096
3	0.73908518	-0.000000	-1.673612	0.73908513	0.00000004
4	0.73908513	-0.000000	-1.673612	0.73908513	0.00000000

Newton's Method

Newton's Method for $f(x) = \cos(x) - x$, $x_0 = \frac{\pi}{4}$

n	p_{n-1}	$f(p_{n-1})$	$f'(p_{n-1})$	p_n	$\lvert p_n - p_{n-1} \rvert$
1	0.78539816	-0.078291	-1.707107	0.73953613	0.04586203
2	0.73953613	-0.000755	-1.673945	0.73908518	0.00045096
3	0.73908518	-0.000000	-1.673612	0.73908513	0.00000004
4	0.73908513	-0.000000	-1.673612	0.73908513	0.00000000

- An excellent approximation is obtained with $n = 3$.
- Because of the agreement of p_3 and p_4 we could reasonably expect this result to be accurate to the places listed.

Newton's Method in Practice

In a practical application . . .

- an initial approximation is selected
- and successive approximations are generated by Newton's method.
- These will generally either converge quickly to the root,
- or it will be clear that convergence is unlikely.