that each of these plays in connecting the new problem to integration. By doing so, you will see that the common thread running through so many diverse applications is the integral.

5.1 AREA BETWEEN CURVES

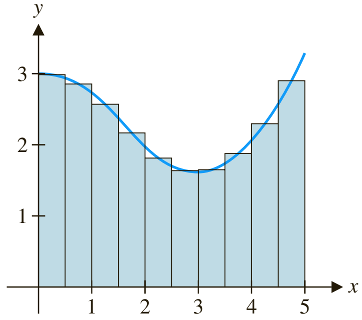


FIGURE 5.1
Approximation of area

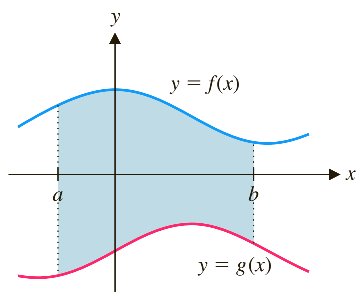


FIGURE 5.2
Area between two curves

We initially developed the definite integral (in Chapter 4) to compute the area under a curve. In particular, let f be a continuous function defined on $[a, b]$, where $f(x) \geq 0$ on $[a, b]$. To find the area under the curve $y = f(x)$ on the interval $[a, b]$, we begin by dividing (*partitioning*) $[a, b]$ into n subintervals of equal size, $\Delta x = \dfrac{b-a}{n}$. The points in the partition are then $x_0 = a$, $x_1 = x_0 + \Delta x$, $x_2 = x_1 + \Delta x$ and so on. That is,

$$x_i = a + i\Delta x, \quad \text{for } i = 0, 1, 2, \ldots, n.$$

On each subinterval $[x_{i-1}, x_i]$, we construct a rectangle of height $f(c_i)$, for some $c_i \in [x_{i-1}, x_i]$, as indicated in Figure 5.1 and take the sum of the areas of the n rectangles as an approximation of the area A under the curve:

$$A \approx \sum_{i=1}^{n} f(c_i)\,\Delta x.$$

As we take more and more rectangles, this sum approaches the exact area, which is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x = \int_a^b f(x)\,dx.$$

We now extend this notion to find the area bounded between the two curves $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ (see Figure 5.2), where f and g are continuous and $f(x) \geq g(x)$ on $[a, b]$. We first use rectangles to approximate the area. In this case, on each subinterval $[x_{i-1}, x_i]$, construct a rectangle, stretching from the lower curve $y = g(x)$ to the upper curve $y = f(x)$, as shown in Figure 5.3a. Referring to Figure 5.3b, the ith rectangle has height $h_i = f(c_i) - g(c_i)$, for some $c_i \in [x_{i-1}, x_i]$.

So, the area of the ith rectangle is

$$\text{Area} = \text{length} \times \text{width} = h_i\,\Delta x = [f(c_i) - g(c_i)]\,\Delta x.$$

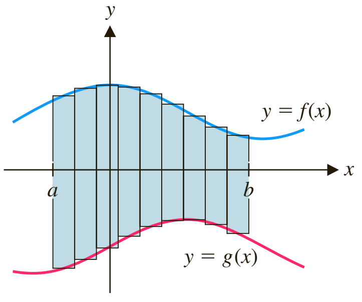


FIGURE 5.3a
Approximate area

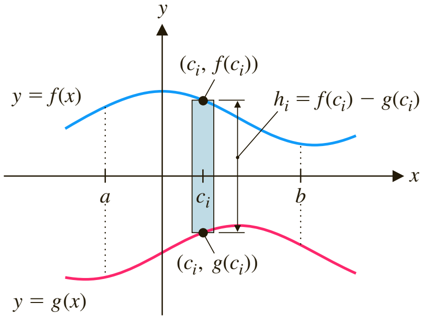


FIGURE 5.3b
Area of ith rectangle

The total area is then approximately equal to the sum of the areas of the n indicated rectangles,

$$A \approx \sum_{i=1}^{n} [f(c_i) - g(c_i)]\,\Delta x.$$

Finally, observe that if the limit as $n \to \infty$ exists, we will get the exact area, which we recognize as a definite integral:

AREA BETWEEN TWO CURVES

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(c_i) - g(c_i)]\,\Delta x = \int_a^b [f(x) - g(x)]\,dx. \tag{1.1}$$

REMARK 1.1

Formula (1.1) is valid only when $f(x) \geq g(x)$ on the interval $[a, b]$. In general, the area between $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$ is given by $\int_a^b |f(x) - g(x)|\,dx$. Notice that to evaluate this integral, you must evaluate $\int_c^d [f(x) - g(x)]\,dx$ on all subintervals where $f(x) \geq g(x)$, then evaluate $\int_c^d [g(x) - f(x)]\,dx$ on all subintervals where $g(x) \geq f(x)$ and finally, add the integrals together.

EXAMPLE 1.1 Finding the Area between Two Curves

Find the area bounded by the graphs of $y = 3 - x$ and $y = x^2 - 9$ (see Figure 5.4).

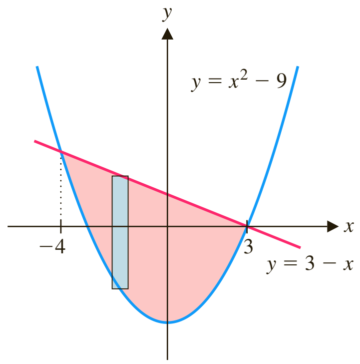


FIGURE 5.4
$y = 3 - x$ and $y = x^2 - 9$

Solution The region in Figure 5.4 is determined by the intersection of the two curves. The limits of integration will then correspond to the x-coordinates of the points of intersection. To find the limits, we set the two functions equal and solve for x. We have

$$3 - x = x^2 - 9 \quad \text{or} \quad 0 = x^2 + x - 12 = (x - 3)(x + 4).$$

Thus, the curves intersect at $x = -4$ and $x = 3$. Be careful to notice from the graph which curve forms the upper boundary of the region and which one forms the lower boundary. In this case, the upper boundary is formed by $y = 3 - x$. So, for each fixed value of x, the height of a rectangle (such as the one indicated in Figure 5.4) is

$$h(x) = (3 - x) - (x^2 - 9).$$

From (1.1), the area between the curves is then

$$\begin{aligned} A &= \int_{-4}^{3} [(3 - x) - (x^2 - 9)]\,dx \\ &= \int_{-4}^{3} (-x^2 - x + 12) = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 12x\right]_{-4}^{3} \\ &= \left[-\frac{3^3}{3} - \frac{3^2}{2} + 12(3)\right] - \left[-\frac{(-4)^3}{3} - \frac{(-4)^2}{2} + 12(-4)\right] = \frac{343}{6}. \end{aligned}$$

Sometimes, a given upper or lower boundary is not defined by a single function, as in the following case of intersecting graphs.

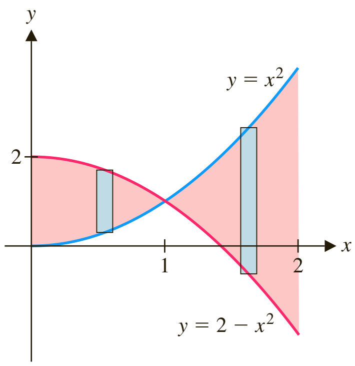


FIGURE 5.5
$y = x^2$ and $y = 2 - x^2$

EXAMPLE 1.2 Finding the Area between Two Curves That Cross

Find the area bounded by the graphs of $y = x^2$ and $y = 2 - x^2$ for $0 \le x \le 2$.

Solution Notice from Figure 5.5 that since the two curves intersect in the middle of the interval, we will need to compute two integrals, one on the interval where $2 - x^2 \ge x^2$ and one on the interval where $x^2 \ge 2 - x^2$. To find the point of intersection, we solve $x^2 = 2 - x^2$, so that $2x^2 = 2$ or $x^2 = 1$ or $x = \pm 1$. Since $x = -1$ is outside the interval of interest, the only intersection of note is at $x = 1$. From (1.1), the area is

$$\begin{aligned} A &= \int_0^1 [(2 - x^2) - x^2]\, dx + \int_1^2 [x^2 - (2 - x^2)]\, dx \\ &= \int_0^1 (2 - 2x^2)\, dx + \int_1^2 (2x^2 - 2)\, dx = \left[2x - \frac{2x^3}{3}\right]_0^1 + \left[\frac{2x^3}{3} - 2x\right]_1^2 \\ &= \left(2 - \frac{2}{3}\right) - (0 - 0) + \left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - 2\right) = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4. \end{aligned}$$

In example 1.2, the intersection point was easy to find. In example 1.3, the intersection points must be approximated numerically.

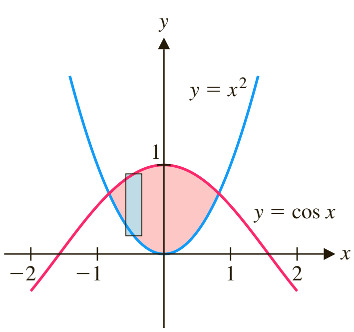


FIGURE 5.6
$y = \cos x$ and $y = x^2$

EXAMPLE 1.3 A Case Where the Intersection Points Are Known Only Approximately

Find the area bounded by the graphs of $y = \cos x$ and $y = x^2$.

Solution The graph of $y = \cos x$ and $y = x^2$ in Figure 5.6 indicates intersections at about $x = -1$ and $x = 1$, where $\cos x = x^2$. However, this equation cannot be solved algebraically. Instead, we use a rootfinding method to find the approximate solutions $x = \pm 0.824132$. [For instance, you can use Newton's method to find values of x for which $f(x) = \cos x - x^2 = 0$.] From the graph, we can see that between these two x-values, $\cos x \ge x^2$ and so, the desired area is given by

$$\begin{aligned} A &\approx \int_{-0.824132}^{0.824132} (\cos x - x^2)\, dx = \left[\sin x - \frac{1}{3}x^3\right]_{-0.824132}^{0.824132} \\ &= \sin 0.824132 - \frac{1}{3}(0.824132)^3 - \left[\sin(-0.824132) - \frac{1}{3}(-0.824132)^3\right] \\ &\approx 1.09475. \end{aligned}$$

Note that we have approximated both the limits of integration and the final calculations.

Finding the area of some regions may require breaking the region up into several pieces, each having different upper and/or lower boundaries.

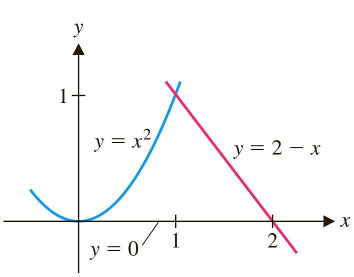


FIGURE 5.7a
$y = x^2$ and $y = 2 - x$

EXAMPLE 1.4 The Area of a Region Determined by Three Curves

Find the area bounded by the graphs of $y = x^2$, $y = 2 - x$ and $y = 0$.

Solution A sketch of the three defining curves is shown in Figure 5.7a. Notice that the top boundary of the region is the curve $y = x^2$ on the first portion of the interval and the

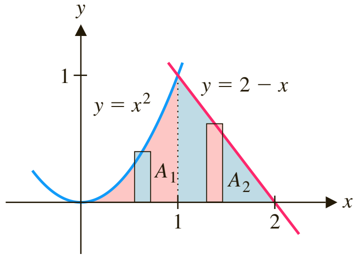


FIGURE 5.7b
$y = x^2$ and $y = 2 - x$

line $y = 2 - x$ on the second portion. To determine the point of intersection, we solve

$$2 - x = x^2 \quad \text{or} \quad 0 = x^2 + x - 2 = (x + 2)(x - 1).$$

Since $x = -2$ is to the left of the y-axis, the intersection we seek occurs at $x = 1$. We then break the region into two pieces, as shown in Figure 5.7b and find the area of each separately. The total area is then

$$\begin{aligned} A = A_1 + A_2 &= \int_0^1 (x^2 - 0)\,dx + \int_1^2 [(2 - x) - 0]\,dx \\ &= \frac{x^3}{3}\bigg|_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 = \frac{5}{6}. \end{aligned}$$

Although it was certainly not difficult to break up the region in example 1.4 into several pieces, we want to suggest an alternative that will prove to be surprisingly useful. Notice that if you turn the page sideways, Figure 5.7a will look like a region with a single curve determining each of the upper and lower boundaries. Of course, by turning the page sideways, you are essentially reversing the roles of x and y, which is the key to computing the area of this type of region: treat the left and right boundaries of the region as functions of y.

More generally, for two continuous functions, f and g of y, where $f(y) \geq g(y)$ for all y on the interval $c \leq y \leq d$, to find the area bounded between the two curves $x = f(y)$ and $x = g(y)$, we first partition the interval $[c, d]$ into n equal subintervals, each of width $\Delta y = \dfrac{d - c}{n}$ (see Figure 5.8a). We denote the points in the partition by $y_0 = c$, $y_1 = y_0 + \Delta y$, $y_2 = y_1 + \Delta y$ and so on. That is,

$$y_i = c + i\,\Delta y, \quad \text{for } i = 0, 1, 2, \ldots, n.$$

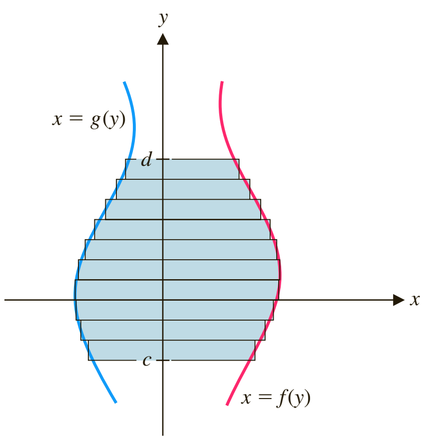


FIGURE 5.8a
Area between $x = g(y)$ and $x = f(y)$

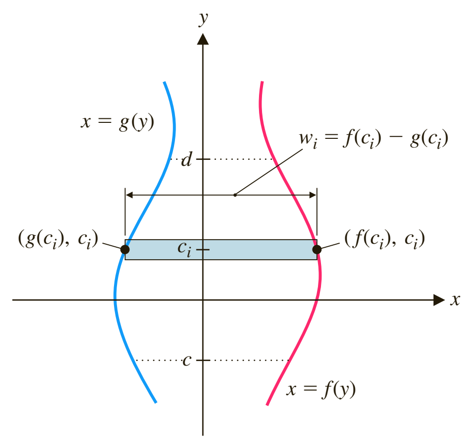


FIGURE 5.8b
Area of ith rectangle

On each subinterval $[y_{i-1}, y_i]$ (for $i = 1, 2, \ldots, n$), we then construct a rectangle of width $w_i = [f(c_i) - g(c_i)]$, for some $c_i \in [y_{i-1}, y_i]$, as shown in Figure 5.8b. The area of the ith rectangle is given by

$$\text{Area} = \text{length} \times \text{width} = [f(c_i) - g(c_i)]\,\Delta y.$$

The total area between the two curves is then given approximately by

$$A \approx \sum_{i=1}^{n} [f(c_i) - g(c_i)]\,\Delta y.$$

We get the exact area by taking the limit as $n \to \infty$ and recognizing the limit as a definite integral. We have

Area between two curves

$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(c_i) - g(c_i)]\,\Delta y = \int_c^d [f(y) - g(y)]\,dy.} \tag{1.2}$$

EXAMPLE 1.5 An Area Computed by Integrating with Respect to y

Repeat example 1.4, but integrate with respect to y instead.

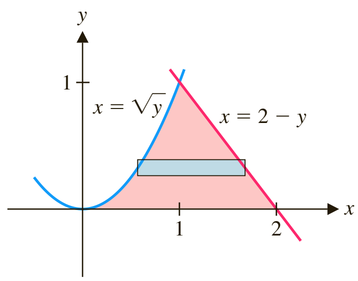


FIGURE 5.9
$y = x^2$ and $y = 2 - x$

Solution The area bounded by the graphs of $y = x^2$, $y = 2 - x$ and $y = 0$ is shown in Figure 5.9. Notice that the left-hand boundary of the region is formed by the graph of $y = x^2$ and the right-hand boundary of the region is formed by the line $y = 2 - x$ and so, a single integral with respect to y will suffice. To write these left- and right-hand boundaries as functions of y, solve the equation $y = x^2$ for x. We get $x = \sqrt{y}$ (since only the right half of the parabola forms the left boundary). Likewise, $y = 2 - x$ is equivalent to $x = 2 - y$. Finally, these curves intersect where $\sqrt{y} = 2 - y$. Squaring both sides gives us

$$y = (2 - y)^2 = 4 - 4y + y^2$$

or

$$0 = y^2 - 5y + 4 = (y - 1)(y - 4).$$

So, the curves intersect at $y = 1$ and $y = 4$. From Figure 5.9, it is clear that $y = 1$ is the solution we need. (What does the solution $y = 4$ correspond to?) From (1.2), the area is given by

$$A = \int_0^1 [(2 - y) - \sqrt{y}]\,dy = \left[2y - \frac{1}{2}y^2 - \frac{2}{3}y^{3/2}\right]_0^1 = 2 - \frac{1}{2} - \frac{2}{3} = \frac{5}{6}.$$

EXAMPLE 1.6 The Area of a Region Bounded by Functions of y

Find the area bounded by the graphs of $x = y^2$ and $x = 2 - y^2$.

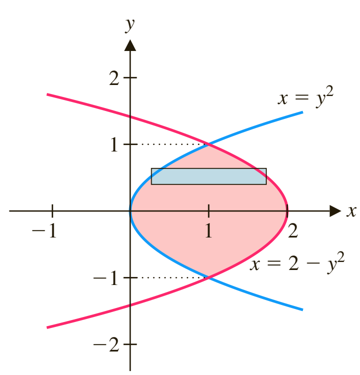


FIGURE 5.10
$x = y^2$ and $x = 2 - y^2$

Solution Of course, the graphs are parabolas opening to the right and left, respectively, as indicated in Figure 5.10. Notice that it's easiest to compute this area by integrating with respect to y, since integrating with respect to x would require us to break the region into two pieces. We must first find the two intersections of the curves. These occur where $y^2 = 2 - y^2$, or $y^2 = 1$, so that $y = \pm 1$. On the interval $[-1, 1]$, notice that $2 - y^2 \geq y^2$ (since the curve $x = 2 - y^2$ stays to the right of the curve $x = y^2$). So, from (1.2), the area is given by

$$A = \int_{-1}^{1} [(2 - y^2) - y^2]\,dy = \int_{-1}^{1} (2 - 2y^2)\,dy$$

$$= \left[2y - \frac{2}{3}y^3\right]_{-1}^{1} = \left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right) = \frac{8}{3}.$$

In collisions between a tennis racket and ball, the ball changes shape, first compressing and then expanding. Let x represent how far the ball is compressed, where $0 \leq x \leq m$, and let $f(x)$ represent the force exerted on the ball by the racket. Then, the energy transferred is proportional to the area under the curve $y = f(x)$. Suppose that $f_c(x)$ is the force during

compression of the ball and $f_e(x)$ is the force during expansion of the ball. Energy is transferred to the ball during compression and transferred away from the ball during expansion, so that the energy lost by the ball in the collision (due to friction) is proportional to $\int_0^m [f_c(x) - f_e(x)]\,dx$. The percentage of energy lost in the collision is then given by

$$100\,\frac{\int_0^m [f_c(x) - f_e(x)]\,dx}{\int_0^m f_c(x)\,dx}.$$

EXAMPLE 1.7 Estimating the Energy Lost by a Tennis Ball

Suppose that test measurements provide the following data on the collision of a tennis ball with a racket. Estimate the percentage of energy lost in the collision.

x (in.)	0.0	0.1	0.2	0.3	0.4
$f_c(x)$ (lb)	0	25	50	90	160
$f_e(x)$ (lb)	0	23	46	78	160

Solution The data are plotted in Figure 5.11, connected by line segments.

We need to estimate the area between the curves and the area under the top curve. Since we don't have a formula for either function, we must use a numerical method such as Simpson's Rule. For $\int_0^{0.4} f_c(x)\,dx$, we get

$$\int_0^{0.4} f_c(x)\,dx \approx \frac{0.1}{3}[0 + 4(25) + 2(50) + 4(90) + 160] = 24.$$

To use Simpson's Rule to approximate $\int_0^{0.4} [f_c(x) - f_e(x)]\,dx$, we need a table of function values for $f_c(x) - f_e(x)$. Subtraction gives us

x	0.0	0.1	0.2	0.3	0.4
$f_c(x) - f_e(x)$	0	2	4	12	0

from which Simpson's Rule gives us

$$\int_0^{0.4} [f_c(x) - f_e(x)]\,dx \approx \frac{0.1}{3}[0 + 4(2) + 2(4) + 4(12) + 0] = \frac{6.4}{3}.$$

The percentage of energy lost is then $\frac{100(6.4/3)}{24} = 8.9\%$. With over 90% of its energy retained in the collision, this is a lively tennis ball. ■

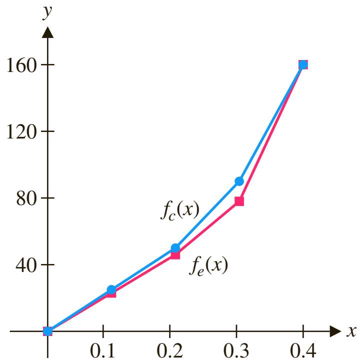


FIGURE 5.11
Force exerted on a tennis ball

BEYOND FORMULAS

In example 1.5, we viewed the given graphs as functions of y and set up the area as an integral of y. This idea indicates the direction that much of the rest of the course takes. The derivative and integral remain the two most important operations, but we diversify our options for working with them, often by changing variables. The flexible thinking that this promotes is key in calculus, as well as in other areas of mathematics and science. We develop some general techniques and often the first task in solving an application problem is to make the technique fit the problem at hand. In industry, do you expect that all problems have been encountered before or that new problems arise each day?

EXERCISES 5.1

WRITING EXERCISES

1. Suppose the functions f and g satisfy $f(x) \geq g(x) \geq 0$ for all x in the interval $[a, b]$. Explain in terms of the areas $A_1 = \int_a^b f(x)\,dx$ and $A_2 = \int_a^b g(x)\,dx$ why the area between the curves $y = f(x)$ and $y = g(x)$ is given by $\int_a^b |f(x) - g(x)|\,dx$.

2. Suppose the functions f and g satisfy $f(x) \leq g(x) \leq 0$ for all x in the interval $[a, b]$. Explain in terms of the areas $A_1 = \int_a^b f(x)\,dx$ and $A_2 = \int_a^b g(x)\,dx$ why the area between the curves $y = f(x)$ and $y = g(x)$ is given by $\int_a^b |f(x) - g(x)|\,dx$.

3. Suppose that the speeds of racing cars A and B are $v_A(t)$ and $v_B(t)$ mph, respectively. If $v_A(t) \geq v_B(t)$ for all t, $v_A(0) = v_B(0)$ and the race lasts from $t = 0$ to $t = 2$ hours, explain why car A will win the race by $\int_0^2 [v_A(t) - v_B(t)]\,dt$ miles.

4. Suppose that the speeds of racing cars A and B are $v_A(t)$ and $v_B(t)$ mph, respectively. If $v_A(t) \geq v_B(t)$ for $0 \leq t \leq 0.5$ and $1.1 \leq t \leq 1.6$ and $v_B(t) \geq v_A(t)$ for $0.5 \leq t \leq 1.1$ and $1.6 \leq t \leq 2$, describe the difference between $\int_0^2 |v_A(t) - v_B(t)|\,dt$ and $\int_0^2 [v_A(t) - v_B(t)]\,dt$. Which integral will tell you which car wins the race?

In exercises 1–4, find the area between the curves on the given interval.

1. $y = x^3$, $y = x^2 - 1$, $1 \leq x \leq 3$
2. $y = \cos x$, $y = x^2 + 2$, $0 \leq x \leq 2$
3. $y = e^x$, $y = x - 1$, $-2 \leq x \leq 0$
4. $y = e^{-x}$, $y = x^2$, $1 \leq x \leq 4$

In exercises 5–12, sketch and find the area of the region determined by the intersections of the curves.

5. $y = x^2 - 1$, $y = 7 - x^2$
6. $y = x^2 - 1$, $y = \frac{1}{2}x^2$
7. $y = x^3$, $y = 3x + 2$
8. $y = \sqrt{x}$, $y = x^2$
9. $y = 4xe^{-x^2}$, $y = |x|$
10. $y = \dfrac{2}{x^2 + 1}$, $y = |x|$
11. $y = \dfrac{5x}{x^2 + 1}$, $y = x$
12. $y = \sin x$ $(0 \leq x \leq 2\pi)$, $y = \cos x$

In exercises 13–18, sketch and estimate the area determined by the intersections of the curves.

13. $y = e^x$, $y = 1 - x^2$
14. $y = x^4$, $y = 1 - x$
15. $y = \sin x$, $y = x^2$
16. $y = \cos x$, $y = x^4$
17. $y = x^4$, $y = 2 + x$
18. $y = \ln x$, $y = x^2 - 2$

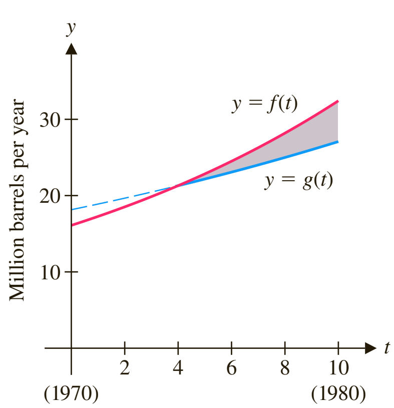
In exercises 19–26, sketch and find the area of the region bounded by the given curves. Choose the variable of integration so that the area is written as a single integral.

19. $y = x$, $y = 2 - x$, $y = 0$
20. $y = 2x$ $(x > 0)$, $y = 3 - x^2$, $x = 0$
21. $x = y$, $x = -y$, $x = 1$
22. $x = 3y$, $x = 2 + y^2$
23. $y = x$, $y = 2$, $y = 6 - x$, $y = 0$
24. $x = y^2$, $x = 4$
25. $y = e^x$, $y = 4e^{-x}$, $y = 1$
26. $y = \dfrac{\ln x}{x}$, $y = \dfrac{1 - x}{x^2 + 1}$, $x = 4$

27. The average value of a function $f(x)$ on the interval $[a, b]$ is $A = \dfrac{1}{b - a}\displaystyle\int_a^b f(x)\,dx$. Compute the average value of $f(x) = x^2$ on $[0, 3]$ and show that the area above $y = A$ and below $y = f(x)$ equals the area below $y = A$ and above $y = f(x)$.

28. Prove that the result of exercise 27 is always true by showing that $\int_a^b [f(x) - A]\,dx = 0$.

29. The United States oil consumption for the years 1970–1974 was approximately equal to $f(t) = 16.1e^{0.07t}$ million barrels per year, where $t = 0$ corresponds to 1970. Following an oil shortage in 1974, the country's consumption changed and was better modeled by $g(t) = 21.3e^{0.04(t-4)}$ million barrels per year, for $t \geq 4$. Show that $f(4) \approx g(4)$ and explain what this number represents. Compute the area between $f(t)$ and $g(t)$ for $4 \leq t \leq 10$. Use this number to estimate the number of barrels of oil saved by Americans' reduced oil consumption from 1974 to 1980.



30. Suppose that a nation's fuelwood consumption is given by $76e^{0.03t}$ m^3/yr and new tree growth is $50 - 6e^{0.09t}$ m^3/yr. Compute and interpret the area between the curves for $0 \le t \le 10$.

31. Suppose that the birthrate for a certain population is $b(t) = 2e^{0.04t}$ million people per year and the death rate for the same population is $d(t) = 2e^{0.02t}$ million people per year. Show that $b(t) \ge d(t)$ for $t \ge 0$ and explain why the area between the curves represents the increase in population. Compute the increase in population for $0 \le t \le 10$.

32. Suppose that the birthrate for a population is $b(t) = 2e^{0.04t}$ million people per year and the death rate for the same population is $d(t) = 3e^{0.02t}$ million people per year. Find the intersection T of the curves ($T > 0$). Interpret the area between the curves for $0 \le t \le T$ and the area between the curves for $T \le t \le 30$. Compute the net change in population for $0 \le t \le 30$.

33. In collisions between a ball and a striking object (e.g., a baseball bat or tennis racket), the ball changes shape, first compressing and then expanding. If x represents the change in diameter of the ball (e.g., in inches) for $0 \le x \le m$ and $f(x)$ represents the force between the ball and striking object (e.g., in pounds), then the area under the curve $y = f(x)$ is proportional to the energy transferred. Suppose that $f_c(x)$ is the force during compression and $f_e(x)$ is the force during expansion. Explain why $\int_0^m [f_c(x) - f_e(x)]\,dx$ is proportional to the energy lost by the ball (due to friction) and thus $\int_0^m [f_c(x) - f_e(x)]\,dx / \int_0^m f_c(x)\,dx$ is the proportion of energy lost in the collision. For a baseball and bat, reasonable values are shown (see Adair's book *The Physics of Baseball*):

x (in.)	0	0.1	0.2	0.3	0.4
$f_c(x)$ (lb)	0	250	600	1200	1750
$f_e(x)$ (lb)	0	10	100	270	1750

Use Simpson's Rule to estimate the proportion of energy retained by the baseball.

34. Using the same notation as in exercise 33, values for the force $f_c(x)$ during compression and force $f_e(x)$ during expansion of a golf ball are given by

x (in.)	0	0.045	0.09	0.135	0.18
$f_c(x)$ (lb)	0	200	500	1000	1800
$f_e(x)$ (lb)	0	125	350	700	1800

Use Simpson's Rule to estimate the proportion of energy retained by the golf ball.

35. Much like the compression and expansion of a ball discussed in exercises 33 and 34, the force exerted by a tendon as a function of its extension determines the loss of energy (see the chapter introduction). Suppose that x is the extension of the tendon, $f_s(x)$ is the force during stretching of the tendon and $f_r(x)$ is the force during recoil of the tendon. The data given are for a hind leg tendon of a wallaby (see Alexander's book *Exploring Biomechanics*):

x (mm)	0	0.75	1.5	2.25	3.0
$f_s(x)$ (N)	0	110	250	450	700
$f_r(x)$ (N)	0	100	230	410	700

Use Simpson's Rule to estimate the proportion of energy returned by the tendon.

36. The arch of a human foot acts like a spring during walking and jumping, storing energy as the foot stretches (i.e., the arch flattens) and returning energy as the foot recoils. In the data, x is the vertical displacement of the arch, $f_s(x)$ is the force on the foot during stretching and $f_r(x)$ is the force during recoil (see Alexander's book *Exploring Biomechanics*):

x (mm)	0	2.0	4.0	6.0	8.0
$f_s(x)$ (N)	0	300	1000	1800	3500
$f_r(x)$ (N)	0	150	700	1300	3500

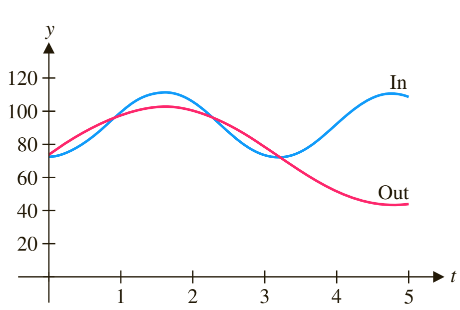
Use Simpson's Rule to estimate the proportion of energy returned by the arch.

37. The velocities of two runners are given by $f(t) = 10$ mph and $g(t) = 10 - \sin t$ mph. Find and interpret the integrals $\int_0^{\pi} [f(t) - g(t)]\,dt$ and $\int_0^{2\pi} [f(t) - g(t)]\,dt$.

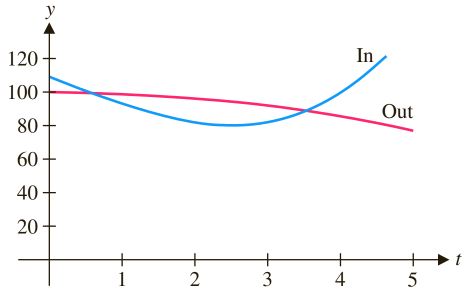
38. The velocities of two racing cars A and B are given by $f(t) = 40(1 - e^{-t})$ mph and $g(t) = 20t$ mph, respectively. The cars start at the same place at time $t = 0$. Estimate (a) the largest lead for car A and (b) the time at which car B catches up.

In exercises 39 and 40, the graph shows the rate of flow of water in gallons per hour into and out of a tank. Assuming that the tank starts with 400 gallons, estimate the amount of water in the tank at hours 1, 2, 3, 4 and 5 and sketch a graph of the amount of water in the tank.

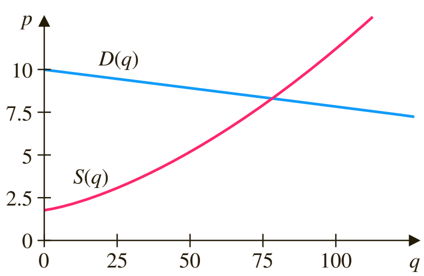
39.



40.

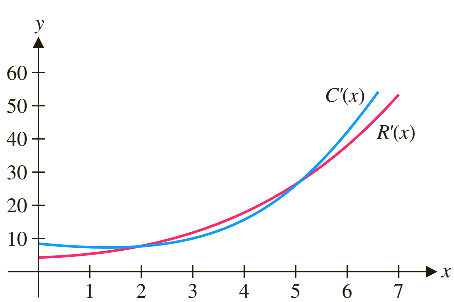


41. The graph shows the supply and demand curves for a product. The point of intersection (q^*, p^*) gives the equilibrium quantity and equilibrium price for the product. The **consumer surplus** is defined to be $\text{CS} = \int_0^{q^*} D(q)\,dq - p^*q^*$. Shade in the area of the graph that represents the consumer surplus, and compute this in the case where $D(q) = 10 - \frac{1}{40}q$ and $S(t) = 2 + \frac{1}{120}q + \frac{1}{1200}q^2$.



42. Repeat exercise 41 for the **producer surplus** defined by $\text{PS} = p^*q^* - \int_0^{q^*} S(q)\,dq$.

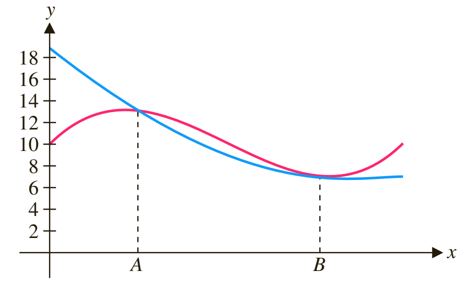
43. Let $C'(x)$ be the marginal cost of producing x thousand copies of an item and let $R'(x)$ be the marginal revenue from the sale of that item, with graphs as shown. Assume that $R'(x) = C'(x)$ at $x = 2$ and $x = 5$. Interpret the area between the curves for each interval: (a) $0 \le x \le 2$, (b) $2 \le x \le 5$, (c) $0 \le x \le 5$ and (d) $5 \le x \le 6$.



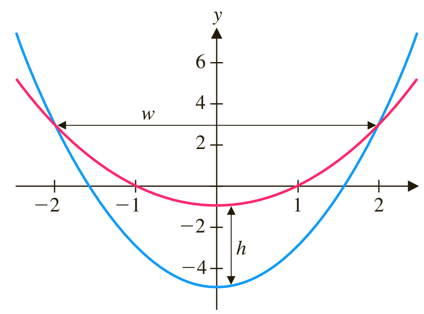
44. A basic principle of economics is that profit is maximized when marginal cost equals marginal revenue. At which intersection is profit maximized in exercise 43? Explain your answer. In terms of profit, what does the other intersection point represent?

45. Suppose that the parabola $y = ax^2 + bx + c$ and the line $y = mx + n$ intersect at $x = A$ and $x = B$ with $A < B$. Show that the area between the curves equals $\frac{|a|}{6}(B - A)^3$. (Hint: Use A and B to rewrite the integrand and then integrate.)

46. Suppose that the cubic $y = ax^3 + bx^2 + cx + d$ and the parabola $y = kx^2 + mx + n$ intersect at $x = A$ and $x = B$ with B repeated (that is, the curves are tangent at B; see the figure). Show that the area between the curves equals $\frac{|a|}{12}(B - A)^4$.

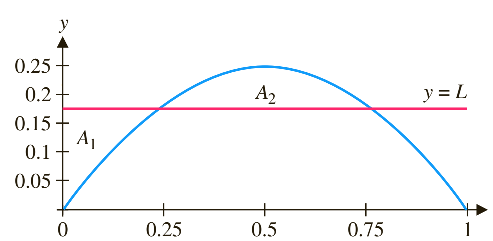


47. Consider two parabolas, each of which has its vertex at $x = 0$, but with different concavities. Let h be the difference in y-coordinates of the vertices, and let w be the difference in the x-coordinates of the intersection points. Show that the area between the curves is $\frac{2}{3}hw$.

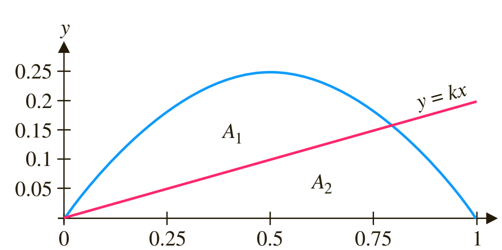


48. Show that for any constant m, the area between $y = 2 - x^2$ and $y = mx$ is $\frac{1}{6}(m^2 + 8)^{3/2}$. Find the minimum such area.

49. For $y = x - x^2$ as shown, find the value of L such that $A_1 = A_2$.



50. For $y = x - x^2$ and $y = kx$ as shown, find k such that $A_1 = A_2$.



EXPLORATORY EXERCISES

1. At this stage, you can compute the area of any "simple" planar region. For a general figure bounded on the left by a function $x = l(y)$, on the right by a function $x = r(y)$, on top by a function $y = t(x)$ and on the bottom by a function $y = b(x)$, write the area as a sum of integrals. (Hint: Divide the region into subregions whose area can be written as the integral of $r(y) - l(y)$ or $t(x) - b(x)$.)

2. Find the area between $y = x^2$ and $y = mx$ for any constant $m > 0$. Without doing further calculations, use this area to find the area between $y = \sqrt{x}$ and $y = mx$.

3. For $x > 0$, let $f(x)$ be the area between $y = 1$ and $y = \sin^2 t$ for $0 \le t \le x$. Without calculating $f(x)$, find as many relationships as possible between the graphical features (zeros, extrema, inflection points) of $y = f(x)$ and the graphical features of $y = \sin^2 x$.

5.2 VOLUME: SLICING, DISKS AND WASHERS

As we shall see throughout this chapter, the integral is an amazingly versatile tool. In section 5.1, we used definite integrals to compute area. In this section, we use integrals to compute the volume of a three-dimensional solid. We begin with a simple problem.

When designing a building, architects must perform numerous detailed calculations. For instance, in order to analyze a building's heating and cooling systems, engineers must calculate the volume of air being processed.

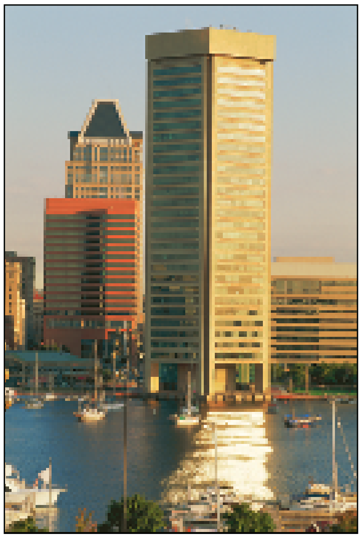


FIGURE 5.12a

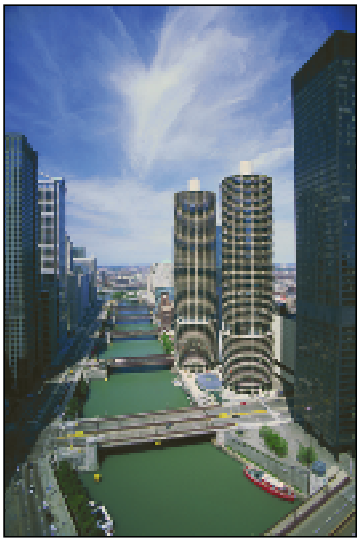


FIGURE 5.12b

You already know how to compute some volumes. For instance, the building shown in Figure 5.12a is essentially a rectangular box, whose volume is given by lwh, where l is the length, w is the width and h is the height. The right circular cylinders seen in the building in Figure 5.12b have volume given by $\pi r^2 h$, where h is the height and r is the radius of the circular cross section. Notice in each case that the building has a familiar cross section (a rectangle in Figure 5.12a and a circle in Figure 5.12b) that is extended vertically. We call any such solid a **cylinder** (*any* solid whose cross sections perpendicular to some axis running through the solid are all the same). Now, notice the connection between the volume

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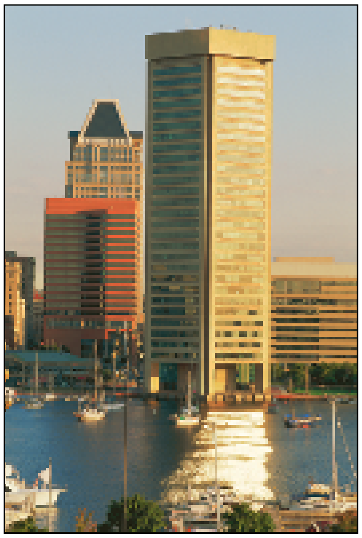
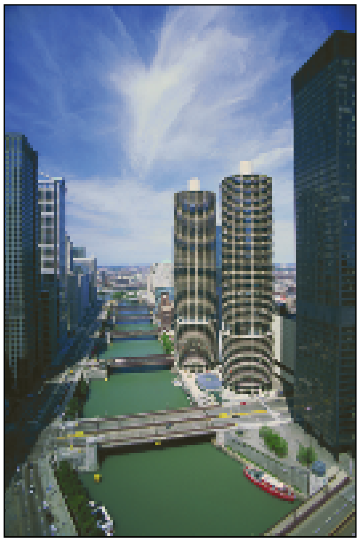


FIGURE 5.12a



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HISTORICAL NOTES

Archimedes (ca. 287–212 B.C.) A Greek mathematician and scientist who was among the first to derive formulas for volumes and areas. Archimedes is known for discovering the basic laws of hydrostatics (he reportedly leapt from his bathtub, shouting "Eureka!" and ran into the streets to share his discovery) and levers ("Give me a place to stand on and I can move the earth."). An ingenious engineer, his catapults, grappling cranes and reflecting mirrors terrorized a massive Roman army that eventually conquered his hometown of Syracuse. On the day the Romans' 3-year siege ended, Archimedes was studying diagrams drawn in the dirt when a soldier tried to arrest him. Archimedes' last words were reportedly, "Do not disturb my circles."

formulas for these two cylinders. The volume of a right circular cylinder is

$$V = \underbrace{(\pi r^2)}_{\text{cross-sectional area}} \times \underbrace{h}_{\text{height}}.$$

Likewise, in the case of a box, we have

$$V = \underbrace{(\text{length} \times \text{width})}_{\text{cross-sectional area}} \times \text{height}.$$

In general, the volume of any cylinder is found by

$$V = (\text{cross-sectional area}) \times (\text{height}).$$

Volumes by Slicing

If either the cross-sectional area or width of a solid is not constant, we will need to modify our approach somewhat. For instance, pyramids and domes do not have constant cross-sectional area, as seen in Figures 5.13a and 5.13b. Since we don't know how to find the volume, we take the approach we've used a number of times now: first approximate the volume and then improve the approximation.

For any solid that extends from $x = a$ to $x = b$, we start by partitioning the interval $[a, b]$ on the x-axis into n subintervals, each of width $\Delta x = \frac{b-a}{n}$. As usual, we denote $x_0 = a$, $x_1 = a + \Delta x$ and so on, so that

$$x_i = a + i\Delta x, \quad \text{for } i = 0, 1, 2, \ldots, n.$$

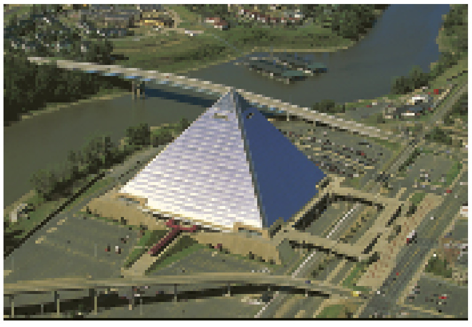


FIGURE 5.13a
Pyramid Arena in Memphis

FIGURE 5.13b
U.S. Capitol Building

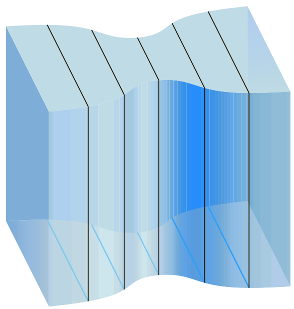


FIGURE 5.14a
Sliced solid

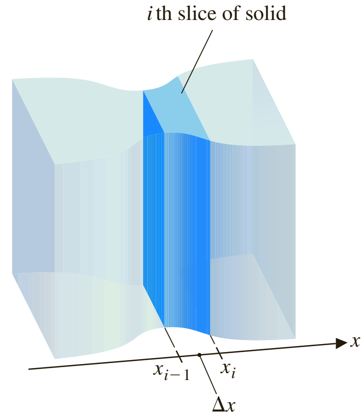


FIGURE 5.14b
*i*th slice of solid

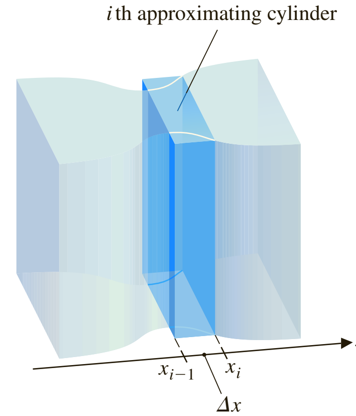


FIGURE 5.14c
*i*th approximating cylinder

We then slice the solid perpendicular to the x-axis at each of the $(n-1)$ points $x_1, x_2, \ldots, x_{n-1}$ (see Figure 5.14a). We next need to approximate the volume of each of the n slices. First, notice that if n is large, then each slice of the solid will be thin and so, the cross-sectional area is nearly constant throughout any given slice. Suppose that the area of the cross section corresponding to any particular value of x is given by $A(x)$. Observe that the slice between $x = x_{i-1}$ and $x = x_i$ is nearly a cylinder (see Figure 5.14b). So, for any point c_i in the interval $[x_{i-1}, x_i]$, the area of the cross sections on that interval are all approximately $A(c_i)$. The volume V_i of the ith slice is approximately the volume of the cylinder lying along the interval $[x_{i-1}, x_i]$, with constant cross-sectional area $A(c_i)$ (see Figure 5.14c), so that

$$V_i \approx \underbrace{A(c_i)}_{\text{cross-sectional area}} \; \underbrace{\Delta x}_{\text{width}},$$

where Δx is the width of the slice.

Repeating this process for each of the n slices, we find that the total volume V of the solid is approximately

$$V \approx \sum_{i=1}^{n} A(c_i)\,\Delta x.$$

Notice that as the number of slices increases, the volume approximation should improve and we get the exact volume by computing

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(c_i)\,\Delta x,$$

assuming the limit exists. You should recognize this limit as the definite integral

Volume of a solid with cross-sectional area $A(x)$

$$\boxed{V = \int_a^b A(x)\,dx.} \tag{2.1}$$

REMARK 2.1

We use the same process followed here to derive many important formulas. In each case, we divide an object into n smaller pieces, approximate the quantity of interest for each of the small pieces, sum the approximations and then take a limit, ultimately recognizing that we have derived a definite integral. For this reason, it is essential that you understand the concept behind formula (2.1). Memorization will not do this for you. However, if you understand how the various pieces of this puzzle fit together, then the rest of this chapter should fall into place for you nicely.

EXAMPLE 2.1 Computing Volume from Cross-Sectional Areas

The Pyramid Arena in Memphis

The Pyramid Arena in Memphis (pictured in the margin) has a square base of side approximately 600 feet and a height of approximately 320 feet. Find the volume of the pyramid with these measurements.

Solution To use (2.1), we need to have a formula for the cross-sectional area. The square horizontal cross sections of the pyramid make this easy, but we need a formula for the size of the square at each height. Orient the x-axis upward through the point at the top of the pyramid. At $x = 0$, the cross section is a square of side 600 feet. At $x = 320$, the cross section can be thought of as a square of side 0 feet. If $f(x)$ represents the side length of the square cross section at height x, we know that $f(0) = 600$, $f(320) = 0$ and $f(x)$ is a linear function. (Think about this; the sides of the pyramid do not curve.) The slope of the line is $m = \frac{600-0}{0-320} = -\frac{15}{8}$ and we use the y-intercept of 600 to get

$$f(x) = -\frac{15}{8}x + 600.$$

Since this is the length of a side of a square, the cross-sectional area is simply the square of this quantity. Then from (2.1), we have

$$V = \int_0^{320} A(x)\,dx = \int_0^{320} \left(-\frac{15}{8}x + 600\right)^2 dx.$$

Observe that we can evaluate this integral by substitution, by taking $u = -\frac{15}{8}x + 600$, so that $du = -\frac{15}{8}\,dx$. This gives us

$$V = \int_0^{320} \left(-\frac{15}{8}x + 600\right)^2 dx = -\frac{8}{15}\int_{600}^{0} u^2\,du$$

$$= \frac{8}{15}\int_0^{600} u^2\,du = \frac{8}{15}\,\frac{u^3}{3}\bigg|_0^{600} = 38{,}400{,}000\ \text{ft}^2.$$

In example 2.1, we knew how to compute the cross-sectional area exactly. In many important applications, the cross-sectional area is not known exactly, but must be approximated using measurements. In such cases, we can still find the volume (approximately), but we'll need to use numerical integration.

EXAMPLE 2.2 Estimating Volume from Cross-Sectional Data

In medical imaging, such as CT (computerized tomography) and MRI (magnetic resonance imaging) processes, numerous measurements are taken and processed by a

computer to construct a three-dimensional image of the tissue the physician wishes to study. The process is similar to the slicing process we have used to find the volume of a solid. In this case, however, mathematical representations of various slices of the tissue are combined to produce a three-dimensional image that can be displayed and sliced back apart for a physician to determine the health of the tissue. Suppose that an MRI scan indicates that the cross-sectional areas of adjacent slices of a tumor are given by the values in the table.

x (cm)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$A(x)$ (cm^2)	0.0	0.1	0.4	0.3	0.6	0.9	1.2	0.8	0.6	0.2	0.1

Estimate the volume of the tumor.

Solution To find the volume of the tumor, we would compute [following (2.1)]

$$V = \int_0^1 A(x)\,dx,$$

except that we only know $A(x)$ at a finite number of points. Does this ring any bells? Notice that we can use Simpson's Rule (see section 4.7) with $\Delta x = 0.1$ to estimate the value of this integral:

$$\begin{aligned} V &= \int_0^1 A(x)\,dx \\ &\approx \frac{b-a}{3n}\left[\begin{array}{l} A(0) + 4A(0.1) + 2A(0.2) + 4A(0.3) + 2A(0.4) + 4A(0.5) \\ \quad + 2A(0.6) + 4A(0.7) + 2A(0.8) + 4A(0.9) + A(1) \end{array}\right] \\ &= \frac{0.1}{3}(0 + 0.4 + 0.8 + 1.2 + 1.2 + 3.6 + 2.4 + 3.2 + 1.2 + 0.8 + 0.1) \\ &= 0.49667\ \text{cm}^3. \quad \blacksquare \end{aligned}$$

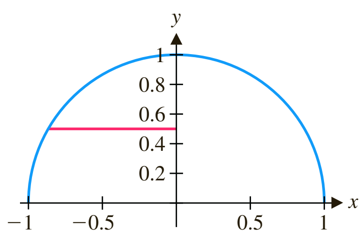


FIGURE 5.15a

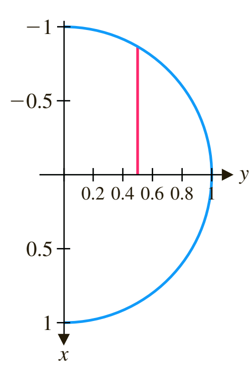


FIGURE 5.15b

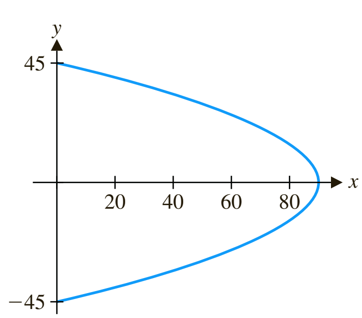


FIGURE 5.15c

We now turn to the problem of finding the volume of the dome in Figure 5.13b. Since the horizontal cross sections are circles, we need only to determine the radius of each circle. Starting with the vertical cross section of the dome in Figure 5.15a, we rotate the dome to get Figure 5.15b. Here, observe that the radius equals the height of the function defining the outline of the dome. We use this insight in example 2.3.

EXAMPLE 2.3 Computing the Volume of a Dome

Suppose that a dome has circular cross sections, with outline $y = \pm\sqrt{\frac{45}{2}(90-x)}$ for $0 \le x \le 90$ (in units of feet, this gives dimensions similar to the Capitol Dome in Figure 5.13b. A graph of this sideways parabola is shown in Figure 5.15c). Find the volume of the dome.

Solution Following our previous discussion, we know that the radius of a circular cross section corresponds to the distance from the x-axis to the top half of the parabola $y = \sqrt{\frac{45}{2}(90-x)}$. That is, the radius is given by $r(x) = \sqrt{\frac{45}{2}(90-x)}$. Each cross section is a circle with this radius, so the cross-sectional areas are given by

$$A(x) = \pi\left(\sqrt{\frac{45}{2}(90-x)}\right)^2,$$

for $0 \le x \le 90$. The volume is then given by

$$V = \int_0^{90} A(x)\,dx = \int_0^{90} \pi \left(\sqrt{\frac{45}{2}(90-x)}\right)^2 dx = \int_0^{90} \pi \left(2025 - \frac{45}{2}x\right) dx$$

$$= \pi \left[2025x - \frac{45}{4}x^2\right]_0^{90} = 91{,}125\pi \approx 286{,}278 \text{ ft}^3.$$

Observe that an alternative way of stating the problem in example 2.3 is to say: Find the volume formed by revolving the region bounded by the curve $y = \sqrt{\frac{45}{2}(90-x)}$ and the x-axis, for $0 \le x \le 90$, about the x-axis.

Example 2.3 can be generalized to the *method of disks* used to compute the volume of a solid formed by revolving a two-dimensional region about a vertical or horizontal line. We consider this general method next.

The Method of Disks

Suppose that $f(x) \ge 0$ and f is continuous on the interval $[a, b]$. Take the region bounded by the curve $y = f(x)$ and the x-axis, for $a \le x \le b$, and revolve it about the x-axis, generating a solid (see Figures 5.16a and 5.16b). We can find the volume of this solid by slicing it perpendicular to the x-axis and recognizing that each cross section is a circular disk of radius $r = f(x)$ (see Figure 5.16b). From (2.1), we then have that the volume of the solid is

Volume of a solid of revolution (Method of disks)

$$\boxed{V = \int_a^b \underbrace{\pi[f(x)]^2}_{\text{cross-sectional area } = \pi r^2} dx.} \tag{2.2}$$

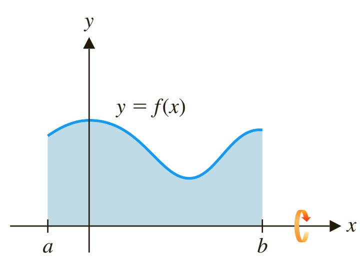


FIGURE 5.16a
$y = f(x) \ge 0$

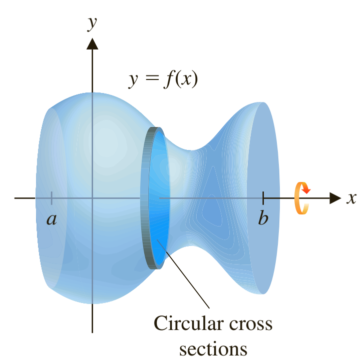


FIGURE 5.16b
Solid of revolution

Since the cross sections of such a solid of revolution are all disks, we refer to this method of finding volume as the **method of disks.**

EXAMPLE 2.4 Using the Method of Disks to Compute Volume

Revolve the region under the curve $y = \sqrt{x}$ on the interval $[0, 4]$ about the x-axis and find the volume of the resulting solid of revolution.

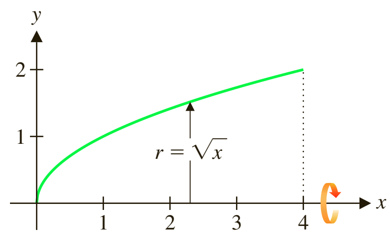


FIGURE 5.17a
$y = \sqrt{x}$

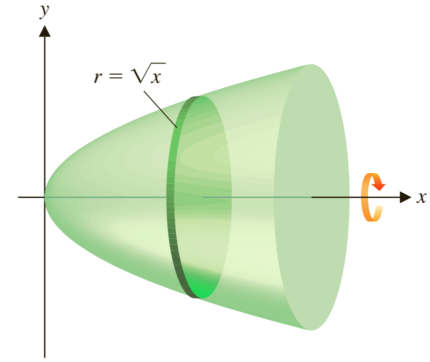


FIGURE 5.17b
Solid of revolution

Solution It's critical to draw a picture of the region and the solid of revolution, so that you get a clear idea of the radii of the circular cross sections. You can see from Figures 5.17a and 5.17b that the radius of each cross section is given by $r = \sqrt{x}$. From (2.2), we then get the volume:

$$V = \int_0^4 \underbrace{\pi\left[\sqrt{x}\right]^2}_{\text{cross-sectional area } = \pi r^2} dx = \pi \int_0^4 x\,dx = \pi \left.\frac{x^2}{2}\right|_0^4 = 8\pi.$$

In a similar way, suppose that $g(y) \geq 0$ and g is continuous on the interval $[c, d]$. Then, revolving the region bounded by the curve $x = g(y)$ and the y-axis, for $c \leq y \leq d$, about the y-axis generates a solid (see Figures 5.18a and 5.18b). Once again, notice from Figure 5.18b that the cross sections of the resulting solid of revolution are circular disks of radius $r = g(y)$. All that has changed here is that we have interchanged the roles of the variables x and y. The volume of the solid is then given by

Volume of a solid of revolution (Method of disks)

$$\boxed{V = \int_c^d \underbrace{\pi[g(y)]^2}_{\text{cross-sectional area } = \pi r^2} dy.} \tag{2.3}$$

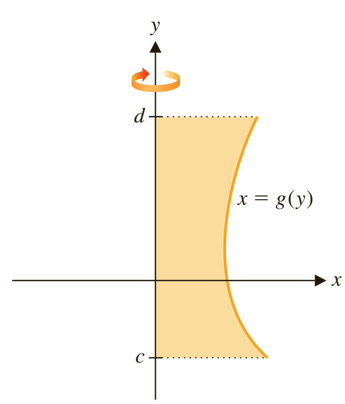


FIGURE 5.18a
Revolve about the y-axis

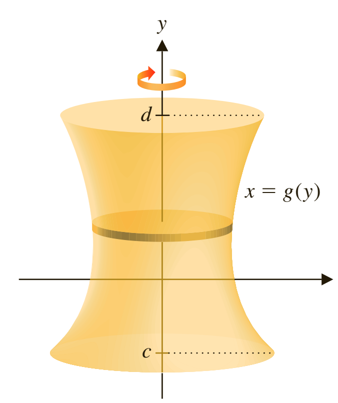


FIGURE 5.18b
Solid of revolution

REMARK 2.2

When using the method of disks, the variable of integration depends solely on the axis about which you revolve the two-dimensional region: revolving about the x-axis requires integration with respect to x, while revolving about the y-axis requires integration with respect to y. This is easily determined by looking at a sketch of the solid. Don't make the mistake of simply looking for what you can plug in where. This is a recipe for disaster, for the rest of this chapter will require you to make similar choices, each based on distinctive requirements of the problem at hand.

EXAMPLE 2.5 Using the Method of Disks with y as the Independent Variable

Find the volume of the solid resulting from revolving the region bounded by the curves $y = 4 - x^2$ and $y = 1$ from $x = 0$ to $x = \sqrt{3}$ about the y-axis.

Solution You will find a graph of the curve in Figure 5.19a and of the solid in Figure 5.19b.

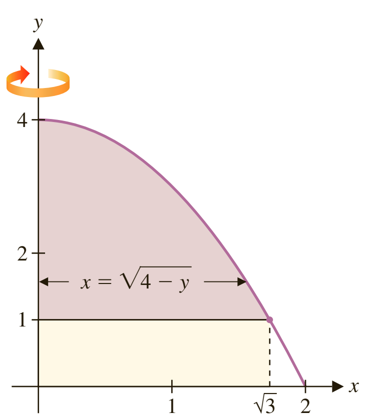


FIGURE 5.19a
$y = 4 - x^2$

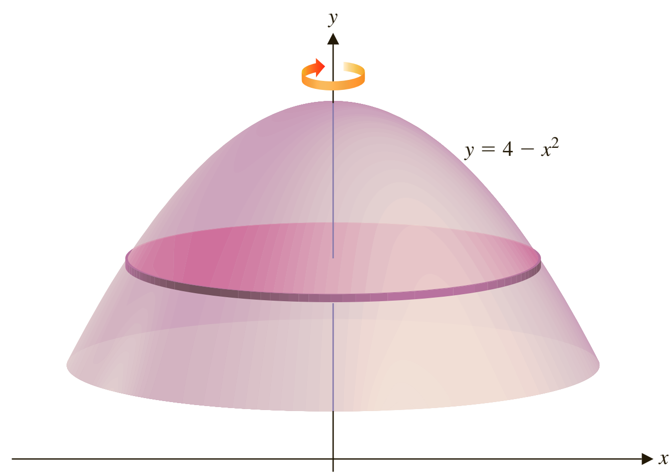


FIGURE 5.19b
Solid of revolution

Notice from Figures 5.19a and 5.19b that the radius of any of the circular cross sections is given by x. So, we must solve the equation $y = 4 - x^2$ for x, to get $x = \sqrt{4-y}$. Since the surface extends from $y = 1$ to $y = 4$, the volume is given by (2.3) to be

$$V = \int_1^4 \underbrace{\pi\left(\sqrt{4-y}\right)^2}_{\pi r^2} dy = \int_1^4 \pi(4-y)\,dy$$

$$= \pi\left[4y - \frac{y^2}{2}\right]_1^4 = \pi\left[(16-8) - \left(4 - \frac{1}{2}\right)\right] = \frac{9\pi}{2}.$$

The Method of Washers

One complication that occurs in computing volumes is that the solid may have a cavity or "hole" in it. Another occurs when a region is revolved about a line other than the x-axis

or the y-axis. Neither case will present you with any significant difficulties, if you look carefully at the figures. We illustrate these ideas in examples 2.6 and 2.7.

EXAMPLE 2.6 Computing Volumes of Solids with and without Cavities

Let R be the region bounded by the graphs of $y = \frac{1}{4}x^2$, $x = 0$ and $y = 1$. Compute the volume of the solid formed by revolving R about (a) the y-axis, (b) the x-axis and (c) the line $y = 2$.

Solution (a) The region R is shown in Figure 5.20a and the solid formed by revolving it about the y-axis is shown in Figure 5.20b. Notice that this part of the problem is similar to example 2.5.

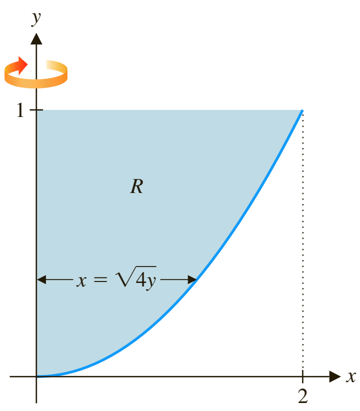


FIGURE 5.20a
$x = \sqrt{4y}$

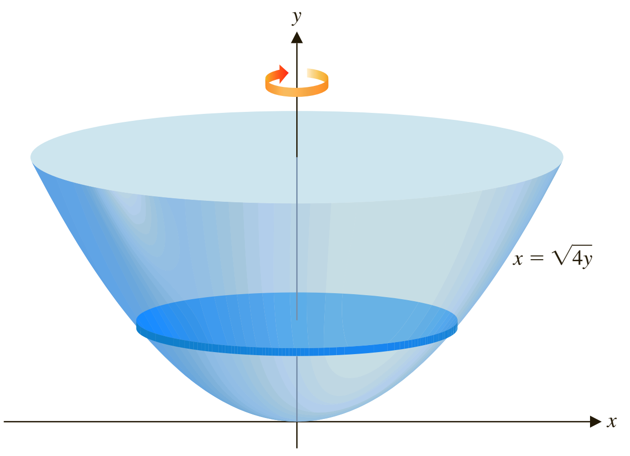


FIGURE 5.20b
Solid of revolution

From (2.3), the volume is given by

$$V = \int_0^1 \underbrace{\pi\left(\sqrt{4y}\right)^2}_{\pi r^2} dy = \pi \frac{4}{2} y^2 \Big|_0^1 = 2\pi.$$

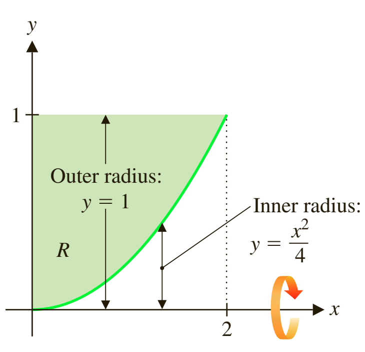


FIGURE 5.21a
$y = \frac{1}{4}x^2$

(b) Revolving the region R about the x-axis produces a cavity in the middle of the solid. See Figure 5.21a for a graph of the region R and Figure 5.21b for a picture of the solid. Our strategy is to compute the volume of the outside of the object (as if it were filled in) and then subtract the volume of the cavity. Before diving into a computation, be sure to visualize the geometry behind this. For the present example, the outside surface of the solid is formed by revolving the line $y = 1$ about the x-axis. The cavity is formed by revolving the curve $y = \frac{1}{4}x^2$ about the x-axis. Look carefully at Figures 5.21a and 5.21b and make certain that you see this. The outer radius, r_O, is the distance from the x-axis to the line $y = 1$, or $r_O = 1$. The inner radius, r_I, is the distance from the x-axis to the curve $y = \frac{1}{4}x^2$, or $r_I = \frac{1}{4}x^2$. Applying (2.2) twice, we see that the volume is given by

$$V = \int_0^2 \underbrace{\pi(1)^2}_{\pi\ (\text{outer radius})^2} dx - \int_0^2 \underbrace{\pi\left(\frac{1}{4}x^2\right)^2}_{\pi\ (\text{inner radius})^2} dx$$

$$= \pi \int_0^2 \left(1 - \frac{x^4}{16}\right) dx = \pi\left(x - \frac{1}{80}x^5\right)\Big|_0^2 = \pi\left(2 - \frac{32}{80}\right) = \frac{8}{5}\pi.$$

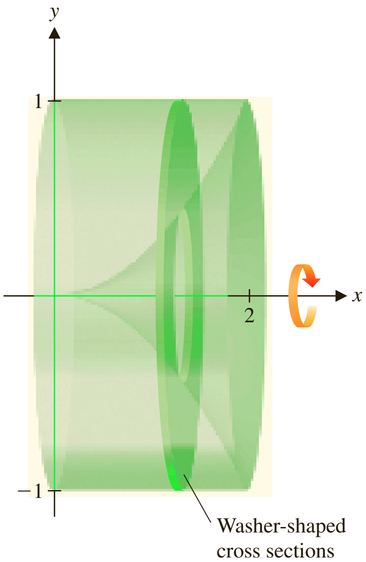


FIGURE 5.21b
Solid with cavity

The solid in part (c) is formed by revolving the region R about the line $y = 2$. This time, the rotation produces a washer-like solid with a cylindrical hole in the middle. The region R is shown in Figure 5.22a and the solid is shown in Figure 5.22b.

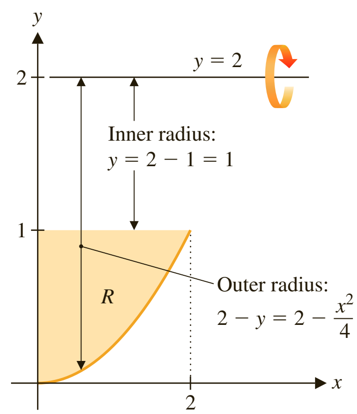


FIGURE 5.22a
Revolve about $y = 2$

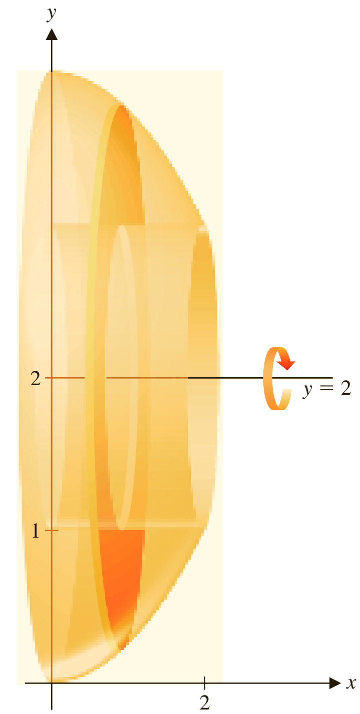


FIGURE 5.22b
Solid of revolution

The volume is computed in the same way as in part (b), by subtracting the volume of the cavity from the volume of the outside solid. From Figures 5.22a and 5.22b, notice that the radius of the outer surface is the distance from the line $y = 2$ to the curve $y = \frac{1}{4}x^2$. This outer radius is then $r_O = 2 - \frac{1}{4}x^2$. The radius of the inner hole is the distance from the line $y = 2$ to the line $y = 1$. This inner radius is then $r_I = 2 - 1 = 1$. From (2.2), the volume is given by

$$V = \int_0^2 \underbrace{\pi\left(2 - \frac{1}{4}x^2\right)^2}_{\pi\,(\text{outer radius})^2} dx - \int_0^2 \underbrace{\pi(2-1)^2}_{\pi\,(\text{inner radius})^2} dx$$

$$= \pi \int_0^2 \left[\left(4 - x^2 + \frac{x^4}{16}\right) - 1\right] dx = \pi \left[3x - \frac{1}{3}x^3 + \frac{1}{80}x^5\right]_0^2$$

$$= \pi\left(6 - \frac{8}{3} + \frac{32}{80}\right) = \frac{56}{15}\pi. \blacksquare$$

In parts (b) and (c) of example 2.6, the volume was computed by subtracting an inner volume from an outer volume in order to compensate for a cavity inside the solid. This technique is a slight generalization of the method of disks and is referred to as the **method of washers,** since the cross sections of the solids look like washers.

EXAMPLE 2.7 Revolving a Region about Different Lines

Let R be the region bounded by $y = 4 - x^2$ and $y = 0$. Find the volume of the solids obtained by revolving R about each of the following: (a) the y-axis, (b) the line $y = -3$, (c) the line $y = 7$ and (d) the line $x = 3$.

Solution For part (a), we draw the region R in Figure 5.23a and the solid of revolution in Figure 5.23b.

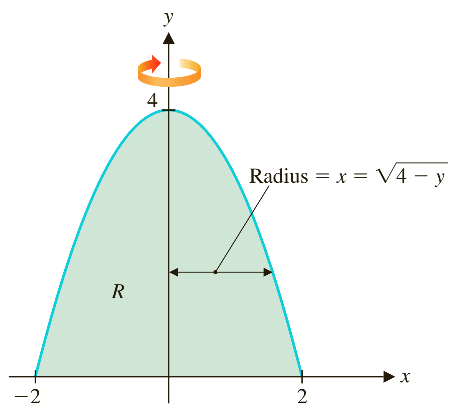


FIGURE 5.23a
Revolve about y-axis

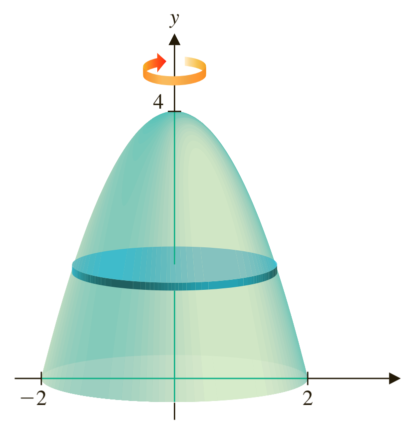


FIGURE 5.23b
Solid of revolution

From Figure 5.23b, notice that each cross section of the solid is a circular disk, whose radius is simply x. Solving for x, we get $x = \sqrt{4-y}$, where we have selected x to be positive, since in this context, x represents a distance. From (2.3), the volume of the solid of revolution is given by

$$V = \int_0^4 \underbrace{\pi\left(\sqrt{4-y}\right)^2}_{\pi(\text{radius})^2} dy = \pi\int_0^4 (4-y)\,dy = \pi\left[4y - \frac{y^2}{2}\right]_0^4 = 8\pi.$$

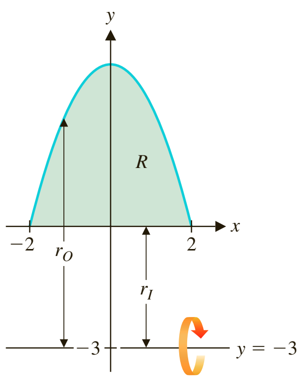


FIGURE 5.24a
Revolve about $y = -3$

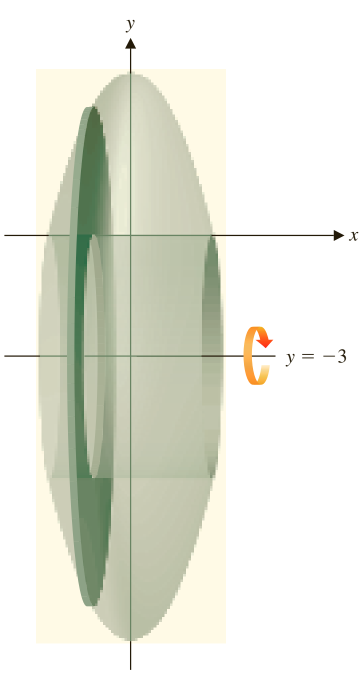


FIGURE 5.24b
Solid of revolution

For part (b), we have sketched the region R in Figure 5.24a and the solid of revolution in Figure 5.24b. Notice from Figure 5.24b that the cross sections of the solid are shaped like washers and the outer radius r_O is the distance from the axis of revolution $y = -3$ to the curve $y = 4 - x^2$. That is,

$$r_O = y - (-3) = (4 - x^2) - (-3) = 7 - x^2,$$

while the inner radius is the distance from the x-axis to the line $y = -3$. That is,

$$r_I = 0 - (-3) = 3.$$

From (2.2), the volume is

$$V = \int_{-2}^{2} \underbrace{\pi(7-x^2)^2}_{\pi(\text{outer radius})^2} dx - \int_{-2}^{2} \underbrace{\pi(3)^2}_{\pi(\text{inner radius})^2} dx = \frac{1472}{15}\pi,$$

where we have left the details of the computation as an exercise.

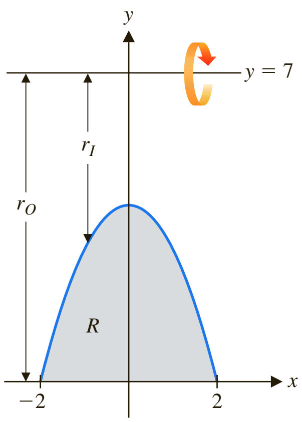


FIGURE 5.25a
Revolve about $y = 7$

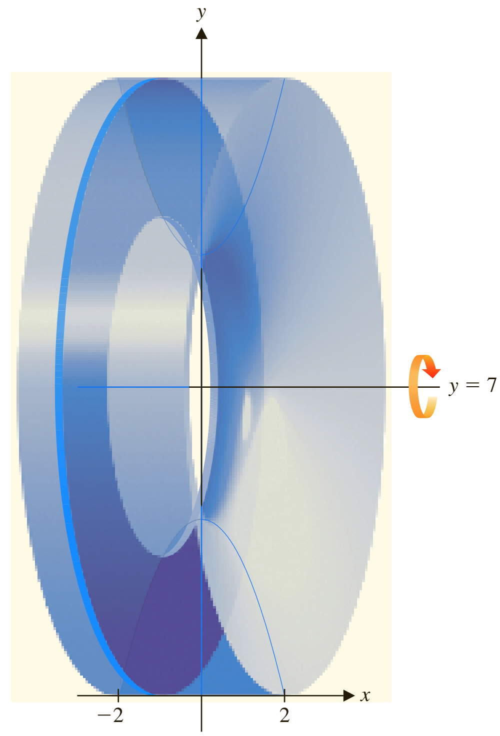


FIGURE 5.25b
Solid of revolution

Part (c) (revolving about the line $y = 7$) is very similar to part (b). You can see the region R in Figure 5.25a and the solid in Figure 5.25b.

The cross sections of the solid are again shaped like washers, but this time, the outer radius is the distance from the line $y = 7$ to the x-axis, that is, $r_O = 7$. The inner radius is the distance from the line $y = 7$ to the curve $y = 4 - x^2$,

$$r_I = 7 - (4 - x^2) = 3 + x^2.$$

From (2.2), the volume of the solid is then

$$V = \int_{-2}^{2} \underbrace{\pi(7)^2}_{\pi\,(\text{outer radius})^2} dx - \int_{-2}^{2} \underbrace{\pi(3 + x^2)^2}_{\pi\,(\text{inner radius})^2} dx = \frac{576}{5}\pi,$$

where we again leave the details of the calculation as an exercise.

Finally, for part (d) (revolving about the line $x = 3$), we show the region R in Figure 5.26a and the solid of revolution in Figure 5.26b. In this case, the cross sections of the solid are washers, but the inner and outer radii are a bit trickier to determine than in the previous parts. The outer radius is the distance between the line $x = 3$ and the *left* half of the parabola, while the inner radius is the distance between the line $x = 3$ and the *right* half of the parabola. The parabola is given by $y = 4 - x^2$, so that $x = \pm\sqrt{4 - y}$. Notice that $x = \sqrt{4 - y}$ corresponds to the right half of the parabola, while $x = -\sqrt{4 - y}$ describes the left half of the parabola. This gives us

$$r_I = 3 - \sqrt{4 - y} \quad \text{and} \quad r_O = 3 - \left(-\sqrt{4 - y}\right) = 3 + \sqrt{4 - y}.$$

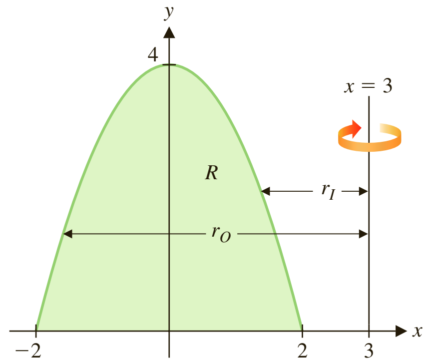


FIGURE 5.26a
Revolve about $x = 3$

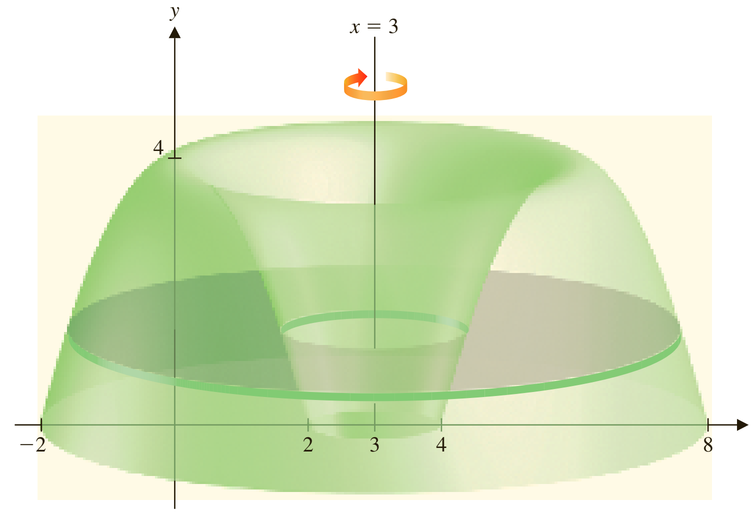


FIGURE 5.26b
Solid of revolution

Consequently, we get the volume

$$V = \int_0^4 \underbrace{\pi\left(3 + \sqrt{4-y}\right)^2}_{\pi\,(\text{outer radius})^2} dy - \int_0^4 \underbrace{\pi\left(3 - \sqrt{4-y}\right)^2}_{\pi\,(\text{inner radius})^2} dy = 64\pi,$$

where we leave the details of this rather messy calculation to you. In section 5.3, we present an alternative method for finding the volume of a solid of revolution that, for the present problem, will produce much simpler integrals. ■

REMARK 2.3

You will be most successful in finding volumes of solids of revolution if you draw reasonable figures and label them carefully. Don't simply look for what to plug in where. You only need to keep in mind how to find the area of a cross section of the solid. Integration does the rest.

EXERCISES 5.2

WRITING EXERCISES

1. Discuss the relationships (e.g., perpendicular or parallel) to the x-axis and y-axis of the disks in examples 2.4 and 2.5. Explain how this relationship enables you to correctly determine the variable of integration.

2. The methods of disks and washers were developed separately in the text, but each is a special case of the general volume formula. Discuss the advantages of learning separate formulas versus deriving each example separately from the general formula. For example, would you prefer to learn the extra formulas or have to work each problem from basic principles? How many formulas is too many to learn?

3. To find the area of a triangle of the form Δ in section 5.1, explain why you would use y-integration. In this section, would it be easier to compute the volume of this triangle revolved about the x-axis or y-axis? Explain your preference.

4. In part (a) of example 2.7, Figure 5.23a extends from $x = -\sqrt{4-y}$ to $x = \sqrt{4-y}$, but we used $\sqrt{4-y}$ as the radius. Explain why this is the correct radius and not $2\sqrt{4-y}$.

In exercises 1–4, find the volume of the solid with cross-sectional area $A(x)$.

1. $A(x) = x + 2, -1 \le x \le 3$

2. $A(x) = 10e^{0.01x}, 0 \le x \le 10$

3. $A(x) = \pi(4-x)^2, 0 \le x \le 2$

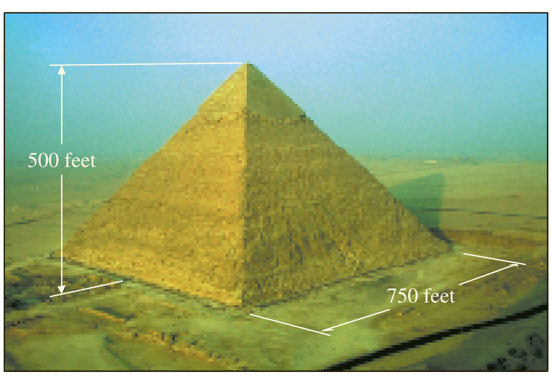
4. $A(x) = 2(x+1)^2, 1 \le x \le 4$

In exercises 5–12, set up an integral and compute the volume.

5. The outline of a dome is given by $y = 60 - \frac{x^2}{60}$ for $-60 \le x \le 60$ (units of feet), with circular cross-sections perpendicular to the y-axis. Find its volume.

6. Find the volume of a pyramid of height 160 feet that has a square base of side 300 feet. These dimensions are half those of the pyramid in example 2.1. How does the volume compare?

7. The great pyramid at Gizeh is 500 feet high, rising from a square base of side 750 feet. Compute its volume using integration. Does your answer agree with the geometric formula?



8. Suppose that instead of completing a pyramid, the builders at Gizeh had stopped at height 250 feet (with a square plateau top of side 375 feet). Compute the volume of this structure. Explain why the volume is greater than half the volume of the pyramid in exercise 7.

9. A church steeple is 30 feet tall with square cross sections. The square at the base has side 3 feet, the square at the top has side 6 inches and the side varies linearly in between. Compute the volume.

10. A house attic has rectangular cross sections parallel to the ground and triangular cross sections perpendicular to the ground. The rectangle is 30 feet by 60 feet at the bottom of the attic and the triangles have base 30 feet and height 10 feet. Compute the volume of the attic.

11. A pottery jar has circular cross sections of radius $4 + \sin\frac{x}{2}$ inches for $0 \le x \le 2\pi$. Sketch a picture of the jar and compute its volume.

12. A pottery jar has circular cross sections of radius $4 - \sin\frac{x}{2}$ inches for $0 \le x \le 2\pi$. Sketch a picture of the jar and compute its volume.

13. Suppose an MRI scan indicates that cross-sectional areas of adjacent slices of a tumor are as given in the table. Use Simpson's Rule to estimate the volume.

x (cm)	0.0	0.1	0.2	0.3	0.4	0.5
$A(x)$ (cm^2)	0.0	0.1	0.2	0.4	0.6	0.4

x (cm)	0.6	0.7	0.8	0.9	1.0
$A(x)$ (cm^2)	0.3	0.2	0.2	0.1	0.0

14. Suppose an MRI scan indicates that cross-sectional areas of adjacent slices of a tumor are as given in the table. Use Simpson's Rule to estimate the volume.

x (cm)	0.0	0.2	0.4	0.6	0.8	1.0	1.2
$A(x)$ (cm^2)	0.0	0.2	0.3	0.2	0.4	0.2	0.0

15. Estimate the volume from the cross-sectional areas.

x (ft)	0.0	0.5	1.0	1.5	2.0
$A(x)$ (ft^2)	1.0	1.2	1.4	1.3	1.2

16. Estimate the volume from the cross-sectional areas.

x (m)	0.0	0.1	0.2	0.3	0.4
$A(x)$ (m^2)	2.0	1.8	1.7	1.6	1.8

x (m)	0.5	0.6	0.7	0.8
$A(x)$ (m^2)	2.0	2.1	2.2	2.4

In exercises 17–20, compute the volume of the solid formed by revolving the given region about the given line.

17. Region bounded by $y = 2 - x$, $y = 0$ and $x = 0$ about (a) the x-axis; (b) $y = 3$

18. Region bounded by $y = x^2$, $y = 0$ and $x = 2$ about (a) the x-axis; (b) $y = 4$

19. Region bounded by $y = \sqrt{x}$, $y = 2$ and $x = 0$ about (a) the y-axis; (b) $x = 4$

20. Region bounded by $y = 2x$, $y = 2$ and $x = 0$ about (a) the y-axis; (b) $x = 1$

In exercises 21–24, a solid is formed by revolving the given region about the given line. Compute the volume exactly if possible and estimate if necessary.

21. Region bounded by $y = e^x$, $x = 0$, $x = 2$ and $y = 0$ about (a) the y-axis; (b) $y = -2$

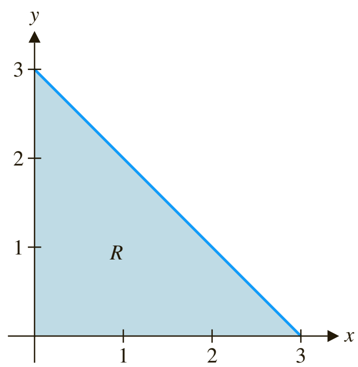
22. Region bounded by $y = \sec x$, $y = 0$, $x = -\pi/4$ and $x = \pi/4$ about (a) $y = 1$; (b) the x-axis

23. Region bounded by $y = \sqrt{\frac{x}{x^2+2}}$, the x-axis and $x = 1$ about (a) the x-axis; (b) $y = 3$

24. Region bounded by $y = e^{-x^2}$ and $y = x^2$ about (a) the x-axis; (b) $y = -1$

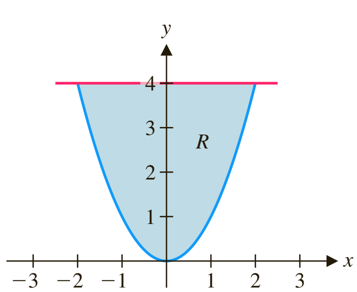
25. Let R be the region bounded by $y = 3 - x$, the x-axis and the y-axis. Compute the volume of the solid formed by revolving R about the given line.

(a) the y-axis (b) the x-axis (c) $y = 3$
(d) $y = -3$ (e) $x = 3$ (f) $x = -3$



26. Let R be the region bounded by $y = x^2$ and $y = 4$. Compute the volume of the solid formed by revolving R about the given line.

(a) $y = 4$ (b) the y-axis (c) $y = 6$
(d) $y = -2$ (e) $x = 2$ (f) $x = -4$



27. Let R be the region bounded by $y = x^2$, $y = 0$ and $x = 1$. Compute the volume of the solid formed by revolving R about the given line.

(a) the y-axis (b) the x-axis (c) $x = 1$
(d) $y = 1$ (e) $x = -1$ (f) $y = -1$

28. Let R be the region bounded by $y = x$, $y = -x$ and $x = 1$. Compute the volume of the solid formed by revolving R about the given line.

(a) the x-axis (b) the y-axis
(c) $y = 1$ (d) $y = -1$

29. Let R be the region bounded by $y = ax^2$, $y = h$ and the y-axis (where a and h are positive constants). Compute the volume of the solid formed by revolving this region about the y-axis. Show that your answer equals half the volume of a cylinder of height h and radius $\sqrt{h/a}$. Sketch a picture to illustrate this.

30. Use the result of exercise 29 to immediately write down the volume of the solid formed by revolving the region bounded by $y = ax^2$, $x = \sqrt{h/a}$ and the x-axis about the y-axis.

31. Suppose that the square consisting of all points (x, y) with $-1 \le x \le 1$ and $-1 \le y \le 1$ is revolved about the y-axis. Show that the volume of the resulting solid is 2π.

32. Suppose that the circle $x^2 + y^2 = 1$ is revolved about the y-axis. Show that the volume of the resulting solid is $\frac{4}{3}\pi$.

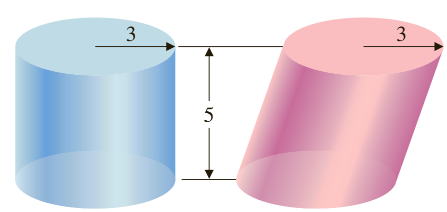
33. Suppose that the triangle with vertices $(-1, -1)$, $(0, 1)$ and $(1, -1)$ is revolved about the y-axis. Show that the volume of the resulting solid is $\frac{2}{3}\pi$.

34. Sketch the square, circle and triangle of exercises 31–33 on the same axes. Show that the relative volumes of the revolved regions (cylinder, sphere and cone, respectively) are 3:2:1.

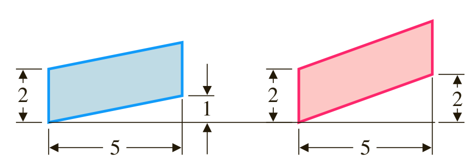
35. Verify the formula for the volume of a sphere by revolving the circle $x^2 + y^2 = r^2$ about the y-axis.

36. Verify the formula for the volume of a cone by revolving the line segment $y = -\frac{h}{r}x + h$, $0 \le x \le r$, about the y-axis.

37. Let A be a right circular cyclinder with radius 3 and height 5. Let B be the tilted circular cylinder with radius 3 and height 5. Determine whether A and B enclose the same volume.



38. Determine whether the two indicated parallelograms have the same area. (Exercises 37 and 38 illustrate Cavalieri's Theorem.)



39. The base of a solid V is the circle $x^2 + y^2 = 1$. Find the volume if V has (a) square cross sections and (b) semicircular cross sections perpendicular to the x-axis.

40. The base of a solid V is the triangle with vertices $(-1, 0)$, $(0, 1)$ and $(1, 0)$. Find the volume if V has (a) square cross sections and (b) semicircular cross sections perpendicular to the x-axis.

41. The base of a solid V is the region bounded by $y = x^2$ and $y = 2 - x^2$. Find the volume if V has (a) square cross sections, (b) semicircular cross sections and (c) equilateral triangle cross sections perpendicular to the x-axis.

42. The base of a solid V is the region bounded by $y = \ln x$, $x = 2$ and $y = 0$. Find the volume if V has (a) square cross sections, (b) semicircular cross sections and (c) equilateral triangle cross sections perpendicular to the x-axis.

43. The base of a solid V is the region bounded by $y = e^{-2x}$, $y = 0$, $x = 0$ and $x = \ln 5$. Find the volume if V has (a) square cross sections and (b) semicircular cross sections perpendicular to the x-axis.

44. The base of solid V is the region bounded by $y = x^2$ and $y = \sqrt{x}$. Find the volume if V has (a) square cross sections and (b) semicircular cross sections perpendicular to the x-axis.

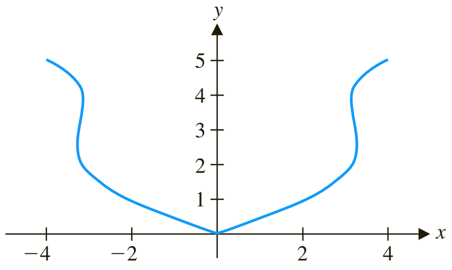
45. Use the given table of values to estimate the volume of the solid formed by revolving $y = f(x)$, $0 \le x \le 3$, about the x-axis.

x	0	0.5	1.0	1.5	2.0	2.5	3.0
$f(x)$	2.0	1.2	0.9	0.4	1.0	1.4	1.6

46. Use the given table of values to estimate the volume of the solid formed by revolving $y = f(x)$, $0 \le x \le 2$, about the x-axis.

x	0	0.25	0.50	0.75	1.0	1.25	1.50	1.75	2.0
$f(x)$	4.0	3.6	3.4	3.2	3.5	3.8	4.2	4.6	5.0

47. Water is poured at a constant rate into the vase with outline as shown and circular cross sections. Sketch a graph showing the height of the water in the vase as a function of time.



48. Sketch a graph of the rate of flow versus time if you poured water into the vase of exercise 47 in such a way that the height of the water in the vase increased at a constant rate.

EXPLORATORY EXERCISES

1. Generalize the result of exercise 34 to any rectangle. That is, sketch the rectangle with $-a \le x \le a$ and $-b \le y \le b$, the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the triangle with vertices $(-a, -b)$, $(0, b)$ and $(a, -b)$. Show that the relative volumes of the solid formed by revolving these regions about the y-axis are 3:2:1.

2. Take the circle $(x - 2)^2 + y^2 = 1$ and revolve it about the y-axis. The resulting donut-shaped solid is called a **torus.** Compute its volume. Show that the volume equals the area of the circle times the distance travelled by the center of the circle. This is an example of **Pappus' Theorem,** dating from the fourth century B.C. Verify that the result also holds for the triangle in exercise 25, parts (c) and (d).

5.3 VOLUMES BY CYLINDRICAL SHELLS

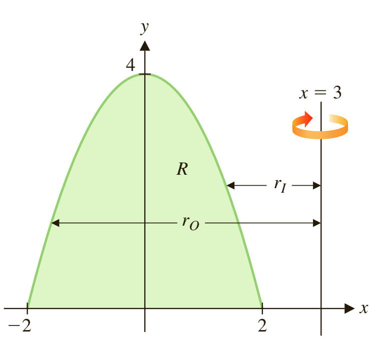


FIGURE 5.27
Revolve about $x = 3$

In this section, we present an alternative to the method of washers discussed in section 5.2. This method will help with solving some problems such as example 2.7, part (d), where the method of washers led to a rather awkward integral. There, we let R be the region bounded by the graphs of $y = 4 - x^2$ and $y = 0$ (see Figure 5.27). If R is revolved about the line $x = 3$, as indicated in Figure 5.27, how would you compute the volume of the resulting solid?

The geometry of the region R makes it awkward to integrate with respect to y, since the left-hand and right-hand boundaries of R are the left and right halves of the parabola, respectively. On the other hand, since R is nicely defined on top by $y = 4 - x^2$ and on bottom by $y = 0$, it might be easier to integrate with respect to x. Unfortunately, in this case, the method of washers requires the y-integration. The solution lies in an alternative method of computing volumes that uses the opposite variable of integration.

Before returning to this example, we consider the general case for a region revolved about the y-axis. Let R denote the region bounded by the graph of $y = f(x)$ and the x-axis

41. The base of a solid V is the region bounded by $y = x^2$ and $y = 2 - x^2$. Find the volume if V has (a) square cross sections, (b) semicircular cross sections and (c) equilateral triangle cross sections perpendicular to the x-axis.

42. The base of a solid V is the region bounded by $y = \ln x$, $x = 2$ and $y = 0$. Find the volume if V has (a) square cross sections, (b) semicircular cross sections and (c) equilateral triangle cross sections perpendicular to the x-axis.

43. The base of a solid V is the region bounded by $y = e^{-2x}$, $y = 0$, $x = 0$ and $x = \ln 5$. Find the volume if V has (a) square cross sections and (b) semicircular cross sections perpendicular to the x-axis.

44. The base of solid V is the region bounded by $y = x^2$ and $y = \sqrt{x}$. Find the volume if V has (a) square cross sections and (b) semicircular cross sections perpendicular to the x-axis.

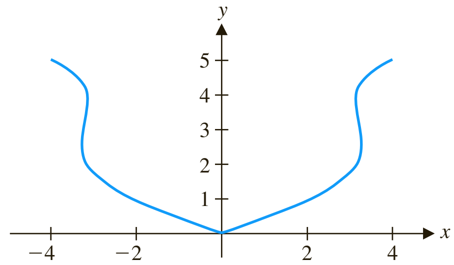
45. Use the given table of values to estimate the volume of the solid formed by revolving $y = f(x)$, $0 \le x \le 3$, about the x-axis.

x	0	0.5	1.0	1.5	2.0	2.5	3.0
$f(x)$	2.0	1.2	0.9	0.4	1.0	1.4	1.6

46. Use the given table of values to estimate the volume of the solid formed by revolving $y = f(x)$, $0 \le x \le 2$, about the x-axis.

x	0	0.25	0.50	0.75	1.0	1.25	1.50	1.75	2.0
$f(x)$	4.0	3.6	3.4	3.2	3.5	3.8	4.2	4.6	5.0

47. Water is poured at a constant rate into the vase with outline as shown and circular cross sections. Sketch a graph showing the height of the water in the vase as a function of time.



48. Sketch a graph of the rate of flow versus time if you poured water into the vase of exercise 47 in such a way that the height of the water in the vase increased at a constant rate.

EXPLORATORY EXERCISES

1. Generalize the result of exercise 34 to any rectangle. That is, sketch the rectangle with $-a \le x \le a$ and $-b \le y \le b$, the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the triangle with vertices $(-a, -b)$, $(0, b)$ and $(a, -b)$. Show that the relative volumes of the solid formed by revolving these regions about the y-axis are 3:2:1.

2. Take the circle $(x - 2)^2 + y^2 = 1$ and revolve it about the y-axis. The resulting donut-shaped solid is called a **torus.** Compute its volume. Show that the volume equals the area of the circle times the distance travelled by the center of the circle. This is an example of **Pappus' Theorem,** dating from the fourth century B.C. Verify that the result also holds for the triangle in exercise 25, parts (c) and (d).

5.3 VOLUMES BY CYLINDRICAL SHELLS

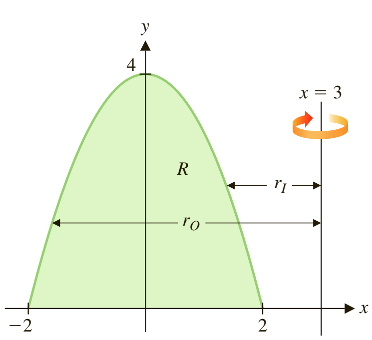


FIGURE 5.27
Revolve about $x = 3$

In this section, we present an alternative to the method of washers discussed in section 5.2. This method will help with solving some problems such as example 2.7, part (d), where the method of washers led to a rather awkward integral. There, we let R be the region bounded by the graphs of $y = 4 - x^2$ and $y = 0$ (see Figure 5.27). If R is revolved about the line $x = 3$, as indicated in Figure 5.27, how would you compute the volume of the resulting solid?

The geometry of the region R makes it awkward to integrate with respect to y, since the left-hand and right-hand boundaries of R are the left and right halves of the parabola, respectively. On the other hand, since R is nicely defined on top by $y = 4 - x^2$ and on bottom by $y = 0$, it might be easier to integrate with respect to x. Unfortunately, in this case, the method of washers requires the y-integration. The solution lies in an alternative method of computing volumes that uses the opposite variable of integration.

Before returning to this example, we consider the general case for a region revolved about the y-axis. Let R denote the region bounded by the graph of $y = f(x)$ and the x-axis

on the interval $[a, b]$, where $0 < a < b$ and $f(x) \geq 0$ on $[a, b]$ (see Figure 5.28a). If we revolve this region about the y-axis, we get the solid shown in Figure 5.28b.

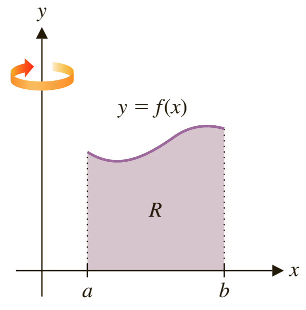


FIGURE 5.28a
Revolve about y-axis

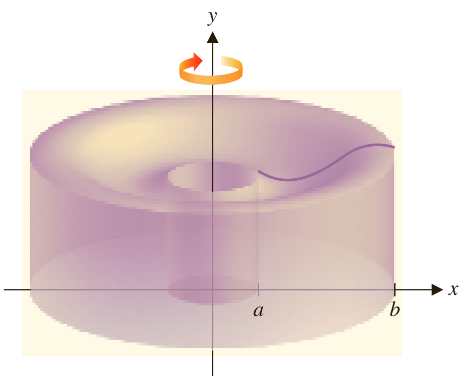


FIGURE 5.28b
Solid of revolution

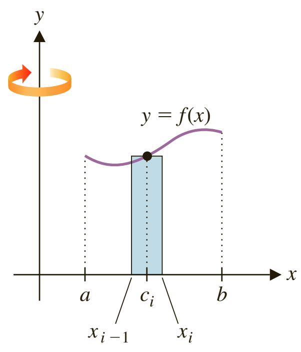


FIGURE 5.29a
ith rectangle

We first partition the interval $[a, b]$ into n subintervals of equal width $\Delta x = \dfrac{b-a}{n}$. On each subinterval $[x_{i-1}, x_i]$, pick a point c_i and construct the rectangle of height $f(c_i)$ as indicated in Figure 5.29a. Revolving this rectangle about the y-axis forms a thin cylindrical shell (i.e., a hollow cylinder, like a pipe), as in Figure 5.29b.

To find the volume of this thin cylindrical shell, imagine cutting the cylinder from top to bottom and then flattening out the shell. After doing this, you should have essentially a thin rectangular sheet, as seen in Figure 5.29c.

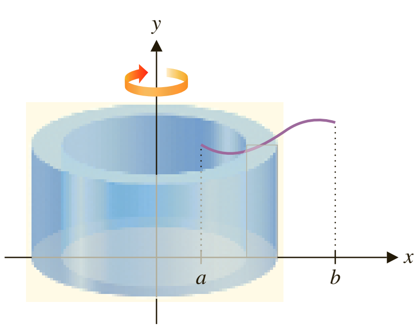


FIGURE 5.29b
Cylindrical shell

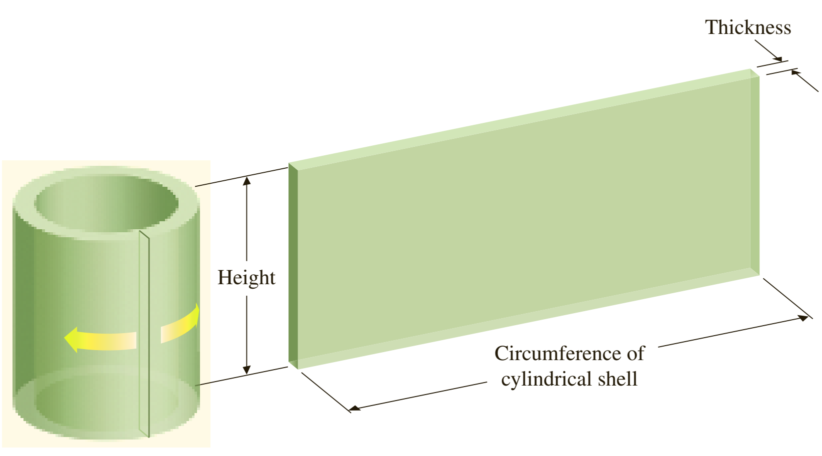


FIGURE 5.29c
Flattened cylindrical shell

Notice that the length of such a thin sheet corresponds to the circumference of the cylindrical shell, which is $2\pi \cdot \text{radius} = 2\pi c_i$. So, the volume V_i of the ith cylindrical shell is approximately

$$\begin{aligned} V_i &\approx \text{length} \times \text{width} \times \text{height} \\ &= (2\pi \times \text{radius}) \times \text{thickness} \times \text{height} \\ &= (2\pi c_i)\, \Delta x\, f(c_i). \end{aligned}$$

The total volume V of the solid can then be approximated by the sum of the volumes of the n cylindrical shells:

$$V \approx \sum_{i=1}^{n} 2\pi \underbrace{c_i}_{\text{radius}} \underbrace{f(c_i)}_{\text{height}} \underbrace{\Delta x}_{\text{thickness}}.$$

As we have done many times now, we can get the exact volume of the solid by taking the limit as $n \to \infty$ and recognizing the resulting definite integral. We have

Volume of a solid of revolution (cylindrical shells)

$$\boxed{V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi c_i f(c_i)\,\Delta x = \int_a^b 2\pi \underbrace{x}_{\text{radius}} \underbrace{f(x)}_{\text{height}} \underbrace{dx}_{\text{thickness}}.} \tag{3.1}$$

REMARK 3.1

Do not rely on simply memorizing formula (3.1). You must strive to understand the meaning of the components. It's simple to do if you just think of how they correspond to the volume of a cylindrical shell:

2π(radius) (height) (thickness).

If you think of volumes in this way, you will be able to solve any problem you encounter.

EXAMPLE 3.1 Using the Method of Cylindrical Shells

Revolve the region bounded by the graphs of $y = x$ and $y = x^2$ in the first quadrant about the y-axis.

Solution From Figure 5.30a, notice that the region has an upper boundary of $y = x$ and a lower boundary of $y = x^2$ and runs from $x = 0$ to $x = 1$. Here, we have drawn a sample rectangle that generates a cylindrical shell. The resulting solid of revolution can be seen in Figure 5.30b. We can write down an integral for the volume by analyzing the various components of the solid in Figures 5.30a and 5.30b. From (3.1), we have

$$V = \int_0^1 2\pi \underbrace{x}_{\text{radius}} \underbrace{(x - x^2)}_{\text{height}} \underbrace{dx}_{\text{thickness}}$$

$$= 2\pi \int_0^1 (x^2 - x^3)\,dx = 2\pi \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^1 = \frac{\pi}{6}.$$

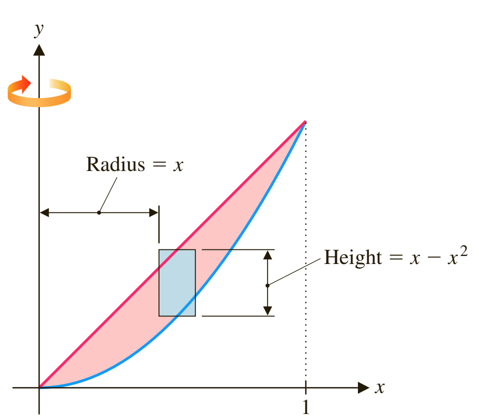


FIGURE 5.30a
Sample rectangle generating a cylindrical shell

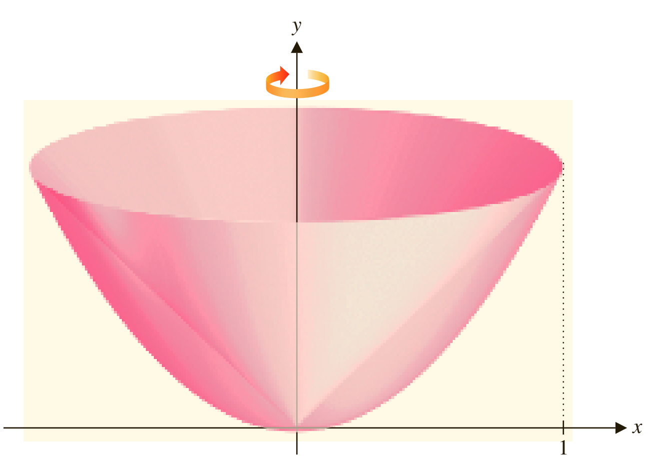


FIGURE 5.30b
Solid of revolution

We can now generalize this method to solve the introductory example.

EXAMPLE 3.2 A Volume Where Shells Are Simpler Than Washers

Find the volume of the solid formed by revolving the region bounded by the graph of $y = 4 - x^2$ and the x-axis about the line $x = 3$.

Solution Look carefully at Figure 5.31a, where we have drawn a sample rectangle that generates a cylindrical shell and at the solid shown in Figure 5.31b. Notice that the radius of a cylindrical shell is the distance from the line $x = 3$ to the shell:

$$r = 3 - x.$$

This gives us the volume

$$V = \int_{-2}^{2} 2\pi \underbrace{(3 - x)}_{\text{radius}} \underbrace{(4 - x^2)}_{\text{height}} \underbrace{dx}_{\text{thickness}}$$

$$= 2\pi \int_{-2}^{2} (x^3 - 3x^2 - 4x + 12)\, dx = 64\pi,$$

where the routine details of the calculation of the integral are left to the reader.

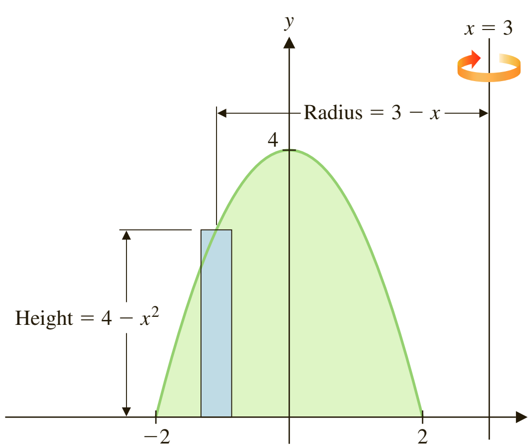


FIGURE 5.31a
Typical rectangle generating a cylindrical shell

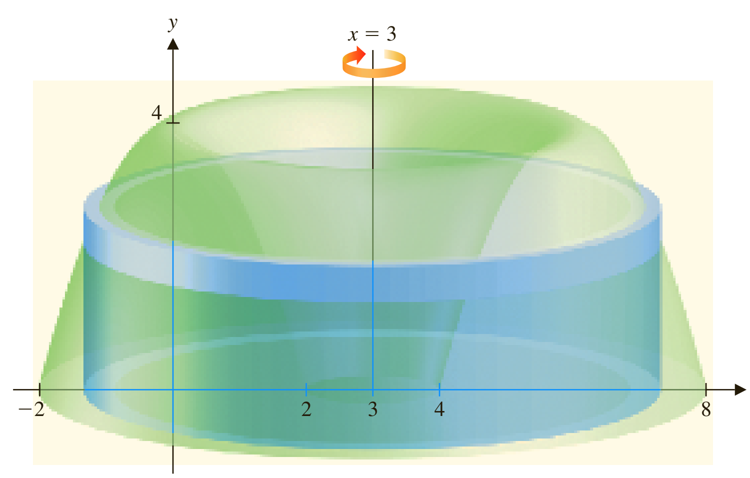


FIGURE 5.31b
Solid of revolution

Your first step in a volume calculation should be to analyze the geometry of the solid to decide whether it's easier to integrate with respect to x or y. Note that for a given solid, the variable of integration in the method of shells is exactly *opposite* that of the method of washers. So, your choice of integration variable will determine which method you use.

EXAMPLE 3.3 Computing Volumes Using Shells and Washers

Let R be the region bounded by the graphs of $y = x$, $y = 2 - x$ and $y = 0$. Compute the volume of the solid formed by revolving R about the lines (a) $y = 2$, (b) $y = -1$ and (c) $x = 3$.

Solution The region R is shown in Figure 5.32a. The geometry of the region suggests that we should consider y as the variable of integration. Look carefully at the differences among the following three volumes.

(a) Revolving R about the line $y = 2$, observe that the radius of a cylindrical shell is the distance from the line $y = 2$ to the shell: $2 - y$, for $0 \le y \le 1$ (see Figure 5.32b). The height is the difference in the x-values on the two curves: solving for x, we have $x = y$ and $x = 2 - y$. Following (3.1), we get the volume

$$V = \int_0^1 2\pi \underbrace{(2-y)}_{\text{radius}} \underbrace{[(2-y)-y]}_{\text{height}} \underbrace{dy}_{\text{thickness}} = \frac{10}{3}\pi,$$

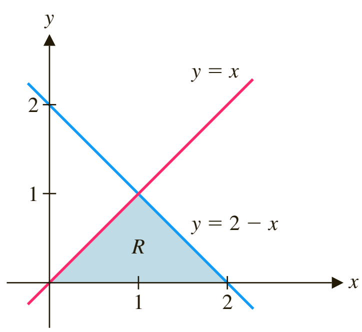


FIGURE 5.32a
$y = x$ and $y = 2 - x$

where we leave the routine details of the calculation to you.

(b) Revolving R about the line $y = -1$, notice that the height of the cylindrical shells is the same as in part (a), but the radius r is the distance from the line $y = -1$ to the shell: $r = y - (-1) = y + 1$ (see Figure 5.32c). This gives us the volume

$$V = \int_0^1 2\pi \underbrace{[y-(-1)]}_{\text{radius}} \underbrace{[(2-y)-y]}_{\text{height}} \underbrace{dy}_{\text{thickness}} = \frac{8}{3}\pi.$$

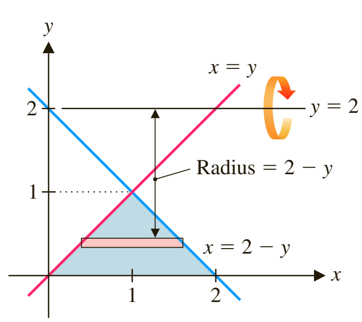


FIGURE 5.32b
Revolve about $y = 2$

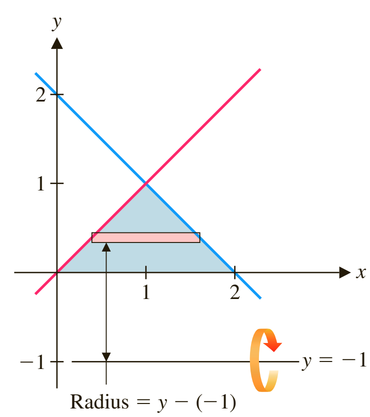


FIGURE 5.32c
Revolve about $y = -1$

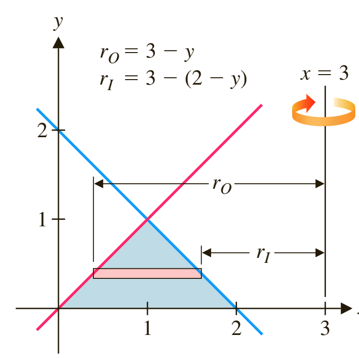


FIGURE 5.32d
Solid of revolution

(c) Finally, revolving R about the line $x = 3$, notice that to find the volume using cylindrical shells, we would need to break the calculation into two pieces, since the height of the cylindrical shells would be different for $x \in [0, 1]$ than for $x \in [1, 2]$. (Think about this some.) On the other hand, this is done easily by the method of washers. Observe that the outer radius is the distance from the line $x = 3$ to the line $x = y$: $r_O = 3 - y$, while the inner radius is the distance from the line $x = 3$ to the line $x = 2 - y$: $r_I = 3 - (2 - y)$ (see Figure 5.32d). This gives us the volume

$$V = \int_0^1 \pi \left\{ \underbrace{(3-y)^2}_{\text{outer radius}^2} - \underbrace{[3-(2-y)]^2}_{\text{inner radius}^2} \right\} dy = 4\pi.$$ ■

Note the importance of sketching a picture and visualizing the solid. The most challenging aspect of these problems is to figure out how to set up the integral. Look carefully at your picture and determine which variable you will use in the integration (i.e., determine whether you will use washers or shells). Then, determine the components of the appropriate integral [i.e., the radius (or radii) and possibly the height], again by looking at the picture. Finally, do whatever it takes to evaluate the integral. If you don't know how to evaluate it, you can fall back on your CAS or approximate it numerically (e.g., by Simpson's Rule).

EXAMPLE 3.4 Approximating Volumes Using Shells and Washers

Let R be the region bounded by the graphs of $y = \cos x$ and $y = x^2$. Compute the volume of the solid formed by revolving R about the lines (a) $x = 2$ and (b) $y = 2$.

Solution First, we sketch the region R (see Figure 5.33a). Since the top and bottom of R are each defined by a curve of the form $y = f(x)$, we will want to integrate with respect to x. We next look for the points of intersection of the two curves, by solving the equation $\cos x = x^2$. Since we can't solve this exactly, we must use an approximate method (e.g., Newton's method) to obtain the approximate intersections at $x = \pm 0.824132$.

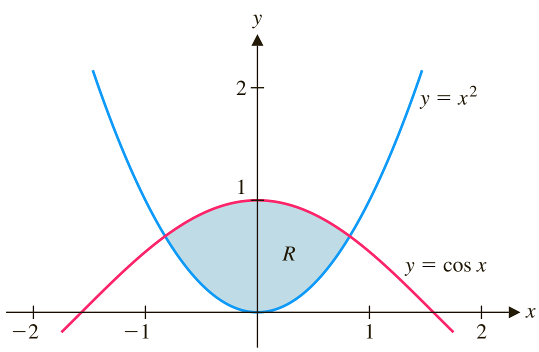


FIGURE 5.33a
$y = \cos x,\ y = x^2$

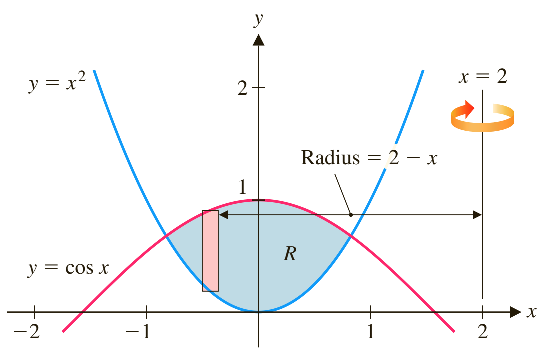


FIGURE 5.33b
Revolve about $x = 2$

(a) If we revolve the region about the line $x = 2$, we should use cylindrical shells (see Figure 5.33b). In this case, observe that the radius r of a cylindrical shell is the distance from the line $x = 2$ to the shell: $r = 2 - x$, while the height of a shell is $\cos x - x^2$. We get the volume

$$V \approx \int_{-0.824132}^{0.824132} 2\pi \underbrace{(2 - x)}_{\text{radius}} \underbrace{(\cos x - x^2)}_{\text{height}}\, dx \approx 13.757,$$

where we have approximated the value of the integral numerically. (We will see how to find an antiderivative for this integrand in Chapter 6.)

(b) If we revolve the region about the line $y = 2$ (see Figure 5.33c on the following page), we use the method of washers. In this case, observe that the outer radius of a washer is the distance from the line $y = 2$ to the curve $y = x^2$: $r_O = 2 - x^2$, while the inner radius is the distance from the line $y = 2$ to the curve $y = \cos x$: $r_I = 2 - \cos x$

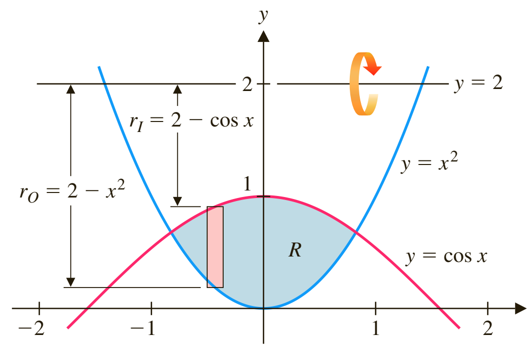


FIGURE 5.33c
Revolve about $y = 2$

(again, see Figure 5.33c). This gives us the volume

$$V \approx \int_{-0.824132}^{0.824132} \pi \left[\underbrace{(2 - x^2)^2}_{\text{other radius}^2} - \underbrace{(2 - \cos x)^2}_{\text{inner radius}^2} \right] dx \approx 10.08,$$

where we have approximated the value of the integral numerically. ■

We close this section with a summary of strategies for computing volumes of solids of revolution.

VOLUME OF A SOLID OF REVOLUTION

- Sketch the region to be revolved.
- Determine the variable of integration (x if the region has a well-defined top and bottom, y if the region has well-defined left and right boundaries).
- Based on the axis of revolution and the variable of integration, determine the method (disks or washers for x-integration about a horizontal axis or y-integration about a vertical axis, shells for x-integration about a vertical axis or y-integration about a horizontal axis).
- Label your picture with the inner and outer radii for disks or washers; label the radius and height for cylindrical shells.
- Set up the integral(s) and evaluate.

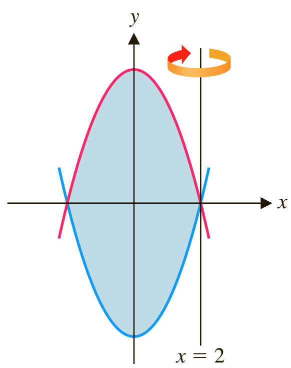
EXERCISES 5.3

WRITING EXERCISES

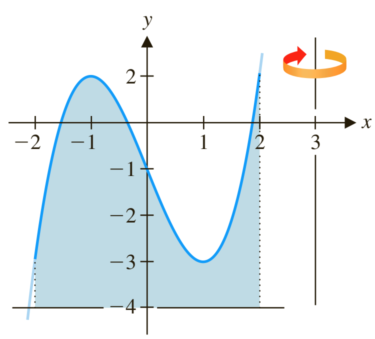
1. Explain why the method of cylindrical shells produces an integral with x as the variable of integration when revolving about a vertical axis. (Describe where the shells are and which direction to move in to go from shell to shell.)

2. Explain why the method of cylindrical shells has the same form whether or not the solid has a hole or cavity. That is, there is no need for separate methods analogous to disks and washers.

3. Suppose that the region bounded by $y = x^2 - 4$ and $y = 4 - x^2$ is revolved about the line $x = 2$. Carefully explain which method (disks, washers or shells) would be easiest to use to compute the volume.



4. Suppose that the region bounded by $y = x^3 - 3x - 1$ and $y = -4$, $-2 \le x \le 2$, is revolved about $x = 3$. Explain what would be necessary to compute the volume using the method of washers and what would be necessary to use the method of cylindrical shells. Which method would you prefer and why?



In exercises 1–8, sketch the region, draw in a typical shell, identify the radius and height of each shell and compute the volume.

1. The region bounded by $y = x^2$ and the x-axis, $-1 \le x \le 1$, revolved about $x = 2$

2. The region bounded by $y = x^2$ and the x-axis, $-1 \le x \le 1$, revolved about $x = -2$

3. The region bounded by $y = x$, $y = -x$ and $x = 1$ revolved about the y-axis

4. The region bounded by $y = x$, $y = -x$ and $x = 1$ revolved about $x = 1$

5. The region bounded by $y = x$, $y = -x$ and $y = 2$ revolved about $y = 3$

6. The region bounded by $y = x$, $y = -x$ and $y = 2$ revolved about $y = -2$

7. The right half of $x^2 + (y - 1)^2 = 1$ revolved about the x-axis

8. The right half of $x^2 + (y - 1)^2 = 1$ revolved about $y = 2$

In exercises 9–16, use cylindrical shells to compute the volume.

9. The region bounded by $y = x^2$ and $y = 2 - x^2$, revolved about $x = -2$

10. The region bounded by $y = x^2$ and $y = 2 - x^2$, revolved about $x = 2$

11. The region bounded by $x = y^2$ and $x = 4$ revolved about $y = -2$

12. The region bounded by $x = y^2$ and $x = 4$ revolved about $y = 2$

13. The region bounded by $y = x$ and $y = x^2 - 2$ revolved about $x = 2$

14. The region bounded by $y = x$ and $y = x^2 - 2$ revolved about $x = 3$

15. The region bounded by $x = (y - 1)^2$ and $x = 9$ revolved about $y = 5$

16. The region bounded by $x = (y - 1)^2$ and $x = 9$ revolved about $y = -3$

In exercises 17–26, use the best method available to find each volume.

17. The region bounded by $y = 4 - x$, $y = 4$ and $y = x$ revolved about
(a) the x-axis (b) the y-axis (c) $x = 4$ (d) $y = 4$

18. The region bounded by $y = x + 2$, $y = -x - 2$ and $x = 0$ revolved about
(a) $y = -2$ (b) $x = -2$ (c) the y-axis (d) the x-axis

19. The region bounded by $y = x$ and $y = x^2 - 6$ revolved about
(a) $x = 3$ (b) $y = 3$ (c) $x = -3$ (d) $y = -6$

20. The region bounded by $x = y^2$ and $x = 2 + y$ revolved about
(a) $x = -1$ (b) $y = -1$ (c) $x = -2$ (d) $y = -2$

21. The region bounded by $y = \cos x$ and $y = x^4$ revolved about
(a) $x = 2$ (b) $y = 2$ (c) the x-axis (d) the y-axis

22. The region bounded by $y = \sin x$ and $y = x^2$ revolved about
(a) $y = 1$ (b) $x = 1$ (c) the y-axis (d) the x-axis

23. The region bounded by $y = x^2$ $(x \ge 0)$, $y = 2 - x$ and $x = 0$ revolved about
(a) the x-axis (b) the y-axis (c) $x = 1$ (d) $y = 2$

24. The region bounded by $y = 2 - x^2$, $y = x$ $(x > 0)$ and the y-axis revolved about
(a) the x-axis (b) the y-axis (c) $x = -1$ (d) $y = -1$

25. The region to the right of $x = y^2$ and to the left of $y = 2 - x$ and $y = x - 2$ revolved about
(a) the x-axis (b) the y-axis

26. The region bounded by $y = e^x - 1$, $y = 2 - x$ and the x-axis revolved about

(a) x-axis (b) y-axis

In exercises 27–32, the integral represents the volume of a solid. Sketch the region and axis of revolution that produce the solid.

27. $\int_0^2 \pi(2x - x^2)^2 dx$

28. $\int_{-2}^2 \pi[(4 - x^2 + 4)^2 - (x^2 - 4 + 4)^2]\,dx$

29. $\int_0^1 \pi[(\sqrt{y})^2 - y^2]\,dy$

30. $\int_0^2 \pi(4 - y^2)^2 dy$

31. $\int_0^1 2\pi x(x - x^2)\,dx$

32. $\int_0^2 2\pi(4 - y)(y + y)\,dy$

33. Use a method similar to our derivation of equation (3.1) to derive the following fact about a circle of radius R. Area $= \pi R^2 = \int_0^R c(r)\,dr$, where $c(r) = 2\pi r$ is the circumference of a circle of radius r.

34. You have probably noticed that the circumference of a circle ($2\pi r$) equals the derivative with respect to r of the area of the circle (πr^2). Use exercise 33 to explain why this is not a coincidence.

35. A jewelry bead is formed by drilling a $\frac{1}{2}$-cm radius hole from the center of a 1-cm radius sphere. Explain why the volume is given by $\int_{1/2}^1 4\pi x\sqrt{1 - x^2}\,dx$. Evaluate this integral or compute the volume in some easier way.

36. Find the size of the hole in exercise 35 such that exactly half the volume is removed.

37. An anthill is in the shape formed by revolving the region bounded by $y = 1 - x^2$ and the x-axis about the y-axis. A researcher removes a cylindrical core from the center of the hill. What should the radius be to give the researcher 10% of the dirt?

38. The outline of a rugby ball has the shape of $\frac{x^2}{30} + \frac{y^2}{16} = 1$. The ball itself is the revolution of this ellipse about either the x-axis or y-axis. Find the volume of the ball.

EXPLORATORY EXERCISES

1. From a sphere of radius R, a hole of radius r is drilled out of the center. Compute the volume removed in terms of R and r. Compute the length L of the hole in terms of R and r. Rewrite the volume in terms of L. Is it reasonable to say that the volume removed depends on L and not on R?

2. In each case, sketch the solid and find the volume formed by revolving the region about (i) the x-axis and (ii) the y-axis. Compute the volume exactly if possible and estimate numerically if necessary. (a) Region bounded by $y = \sec x\sqrt{\tan x + 1}$, $y = 0$, $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$. (b) Region bounded by $x = \sqrt{y^2 + 1}$, $x = 0$, $y = -1$ and $y = 1$. (c) Region bounded by $y = \frac{\sin x}{x}$, $y = 0$, $x = \pi$ and $x = 0$. (d) Region bounded by $y = x^3 - 3x^2 + 2x$ and $y = 0$. (e) Region bounded by $y = e^{-x^2}$ and $y = (x - 1)^2$.

5.4 ARC LENGTH AND SURFACE AREA

Length and area are quantities you already understand intuitively. But, as you have learned with area, the calculation of these quantities can be surprisingly challenging for many geometric shapes. In this section, we compute the length of a curve in two dimensions and the area of a surface in three dimensions. As always, pay particular attention to the derivations. As we have done a number of times now, we start with an approximation and then proceed to the exact solution, using the notion of limit.

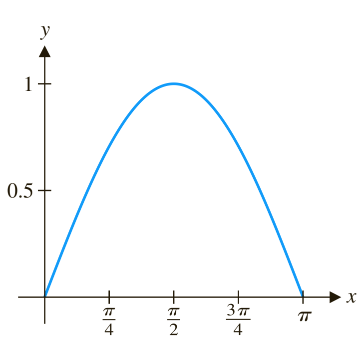


FIGURE 5.34a
$y = \sin x$

Arc Length

What could we mean by the *length* of the portion of the sine curve shown in Figure 5.34a? (We call the length of a curve its **arc length.**) If the curve represented a road, you could measure the length on your car's odometer by driving along that section of road. If the curve were actually a piece of string, you could straighten out the string and then measure its length with a ruler. Both of these ideas are very helpful intuitively. They both involve turning the problem of measuring length in two dimensions into the (much easier) problem of measuring the length in one dimension.

26. The region bounded by $y = e^x - 1$, $y = 2 - x$ and the x-axis revolved about

(a) x-axis (b) y-axis

In exercises 27–32, the integral represents the volume of a solid. Sketch the region and axis of revolution that produce the solid.

27. $\int_0^2 \pi(2x - x^2)^2 dx$

28. $\int_{-2}^2 \pi[(4 - x^2 + 4)^2 - (x^2 - 4 + 4)^2]\,dx$

29. $\int_0^1 \pi[(\sqrt{y})^2 - y^2]\,dy$

30. $\int_0^2 \pi(4 - y^2)^2 dy$

31. $\int_0^1 2\pi x(x - x^2)\,dx$

32. $\int_0^2 2\pi(4 - y)(y + y)\,dy$

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5.4 ARC LENGTH AND SURFACE AREA

Length and area are quantities you already understand intuitively. But, as you have learned with area, the calculation of these quantities can be surprisingly challenging for many geometric shapes. In this section, we compute the length of a curve in two dimensions and the area of a surface in three dimensions. As always, pay particular attention to the derivations. As we have done a number of times now, we start with an approximation and then proceed to the exact solution, using the notion of limit.

Arc Length

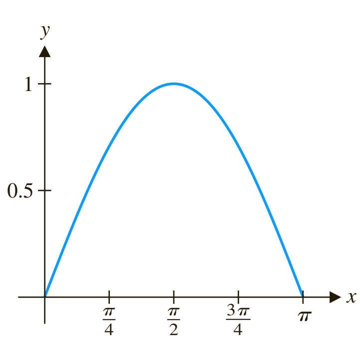


FIGURE 5.34a
$y = \sin x$

What could we mean by the *length* of the portion of the sine curve shown in Figure 5.34a? (We call the length of a curve its **arc length.**) If the curve represented a road, you could measure the length on your car's odometer by driving along that section of road. If the curve were actually a piece of string, you could straighten out the string and then measure its length with a ruler. Both of these ideas are very helpful intuitively. They both involve turning the problem of measuring length in two dimensions into the (much easier) problem of measuring the length in one dimension.

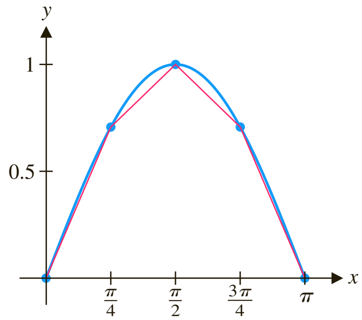


FIGURE 5.34b
Four line segments approximating $y = \sin x$

n	*Length*
8	3.8125
16	3.8183
32	3.8197
64	3.8201
128	3.8202

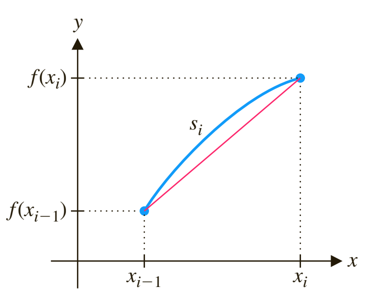


FIGURE 5.35
Straight-line approximation of arc length

To accomplish this mathematically, we first approximate the curve with several line segments joined together. In Figure 5.34b, the line segments connect the points (0, 0), $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{\pi}{2}, 1\right)$, $\left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and $(\pi, 0)$ on the curve $y = \sin x$. An approximation of the arc length s of the curve is given by the sum of the lengths of these line segments:

$$s \approx \sqrt{\left(\frac{\pi}{4}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} + \sqrt{\left(\frac{\pi}{4}\right)^2 + \left(1 - \frac{1}{\sqrt{2}}\right)^2} + \sqrt{\left(\frac{\pi}{4}\right)^2 + \left(\frac{1}{\sqrt{2}} - 1\right)^2} + \sqrt{\left(\frac{\pi}{4}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \approx 3.79.$$

You might notice that this estimate is too small. (Why is that?) We will improve our approximation by using more than four line segments. In the table at left, we show estimates of the length of the curve using n line segments for larger values of n. As you would expect, the approximation of length will get closer to the actual length of the curve, as the number of line segments increases. This general idea should sound familiar.

We develop this notion further now for the more general problem of finding the arc length of the curve $y = f(x)$ on the interval $[a, b]$. Here, we'll assume that f is continuous on $[a, b]$ and differentiable on (a, b). (Where have you seen hypotheses like these before?) As usual, we begin by partitioning the interval $[a, b]$ into n equal pieces: $a = x_0 < x_1 < \cdots < x_n = b$, where $x_i - x_{i-1} = \Delta x = \frac{b - a}{n}$, for each $i = 1, 2, \ldots, n$.

Between each pair of adjacent points on the curve, $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$, we approximate the arc length s_i by the straight-line distance between the two points (see Figure 5.35). From the usual distance formula, we have

$$s_i \approx d\{(x_{i-1}, f(x_{i-1})), (x_i, f(x_i))\} = \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}.$$

Since f is continuous on all of $[a, b]$ and differentiable on (a, b), f is also continuous on the subinterval $[x_{i-1}, x_i]$ and is differentiable on (x_{i-1}, x_i). By the Mean Value Theorem, we then have

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}),$$

for some number $c_i \in (x_{i-1}, x_i)$. This gives us the approximation

$$\begin{aligned} s_i &\approx \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2} \\ &= \sqrt{(x_i - x_{i-1})^2 + [f'(c_i)(x_i - x_{i-1})]^2} \\ &= \sqrt{1 + [f'(c_i)]^2}\,\underbrace{(x_i - x_{i-1})}_{\Delta x} = \sqrt{1 + [f'(c_i)]^2}\,\Delta x. \end{aligned}$$

Adding together the lengths of these n line segments, we get an approximation of the total arc length,

$$s \approx \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}\,\Delta x.$$

Notice that as n gets larger, this approximation should approach the exact arc length, that is,

$$s = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}\,\Delta x.$$

You should recognize this as the limit of a Riemann sum for $\sqrt{1+[f'(x)]^2}$, so that the arc length is given exactly by the definite integral:

Arc length of $y = f(x)$ on the interval $[a, b]$

$$\boxed{s = \int_a^b \sqrt{1+[f'(x)]^2}\,dx,} \tag{4.1}$$

whenever the limit exists.

REMARK 4.1

The formula for arc length is very simple. Unfortunately, very few functions produce arc length integrals that can be evaluated exactly. You should expect to use a numerical integration method on your calculator or computer to compute most arc lengths.

EXAMPLE 4.1 Using the Arc Length Formula

Find the arc length of the portion of the curve $y = \sin x$ with $0 \le x \le \pi$. (We estimated this as 3.79 in our introductory example.)

Solution From (4.1), the arc length is

$$s = \int_0^\pi \sqrt{1+(\cos x)^2}\,dx.$$

Try to find an antiderivative of $\sqrt{1+\cos^2 x}$, but don't try for too long. (The best our CAS can do is $\sqrt{2}$ EllipticE$[x, \frac{1}{2}]$, which doesn't seem especially helpful.) Using a numerical integration method, the arc length is

$$s = \int_0^\pi \sqrt{1+(\cos x)^2}\,dx \approx 3.8202.$$

Even for very simple curves, evaluating the arc length integral exactly can be quite challenging.

EXAMPLE 4.2 Estimating an Arc Length

Find the arc length of the portion of the curve $y = x^2$ with $0 \le x \le 1$.

Solution Using the arc length formula (4.1), we get

$$s = \int_0^1 \sqrt{1+(2x)^2}\,dx = \int_0^1 \sqrt{1+4x^2}\,dx \approx 1.4789,$$

where we have again evaluated the integral numerically. (In this case, you can find an antiderivative using a clever substitution or a CAS.)

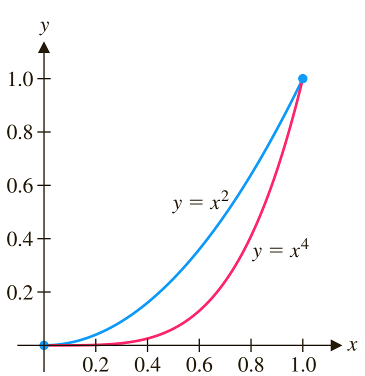


FIGURE 5.36
$y = x^2$ and $y = x^4$

The graphs of $y = x^2$ and $y = x^4$ look surprisingly similar on the interval [0, 1] (see Figure 5.36). They both connect the points (0, 0) and (1, 1), are increasing and are concave up. If you graph them simultaneously, you will note that $y = x^4$ starts out flatter and then becomes steeper from about $x = 0.7$ on. (Try *proving* that this is true!) Arc length gives us one way to quantify the difference between the two graphs.

EXAMPLE 4.3 A Comparison of Arc Lengths of Power Functions

Find the arc length of the portion of the curve $y = x^4$ with $0 \le x \le 1$ and compare to the arc length of the portion of the curve $y = x^2$ on the same interval.

Solution From (4.1), the arc length for $y = x^4$ is given by

$$\int_0^1 \sqrt{1+(4x^3)^2}\,dx = \int_0^1 \sqrt{1+16x^6}\,dx \approx 1.6002.$$

Notice that this arc length is about 8% larger than that of $y = x^2$, as found in example 4.2.

In the exercises, you will be asked to explore the trend in the lengths of the portion of the curves $y = x^6$, $y = x^8$ and so on, on the interval [0, 1]. Can you guess now what happens to the arc length of the portion of $y = x^n$, on the interval [0, 1], as $n \to \infty$?

As with many words, everyday usage of the word *length* can be ambiguous and misleading. For instance, when somebody says that the *length* of a frisbee throw was 100 yards, the length refers to the horizontal distance covered, not to the arc length of the frisbee's flight path. In this case, reporting the horizontal distance is more meaningful than the arc length (and is easier to measure). In many other cases, arc length is the quantity of interest. For example, suppose you need to hang a banner between two poles that are 20 feet apart. If you have only 20 feet of rope to work with, you are going to be in trouble. The length of rope required is determined by the arc length, rather than the horizontal distance.

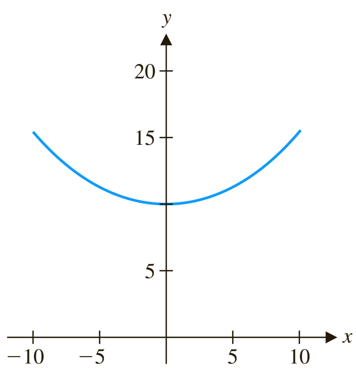


FIGURE 5.37
$y = 5(e^{x/10} + e^{-x/10})$

EXAMPLE 4.4 Computing the Length of Cable Hanging between Two Poles

A cable is to be hung between two poles of equal height that are 20 feet apart. It can be shown that such a hanging cable assumes the shape of a *catenary,* the general form of which is $y = a\cosh x/a = \frac{a}{2}(e^{x/a} + e^{-x/a})$. In this case, suppose that the cable takes the shape of $y = 5(e^{x/10} + e^{-x/10})$, for $-10 \le x \le 10$, as seen in Figure 5.37. How long is the cable?

Solution From (4.1), the arc length of the curve is given by

$$\begin{aligned} s &= \int_{-10}^{10} \sqrt{1 + \left(\frac{e^{x/10}}{2} - \frac{e^{-x/10}}{2}\right)^2}\,dx \\ &= \int_{-10}^{10} \sqrt{1 + \frac{1}{4}(e^{x/5} - 2 + e^{-x/5})}\,dx \\ &= \int_{-10}^{10} \sqrt{\frac{1}{4}(e^{x/5} + 2 + e^{-x/5})}\,dx \\ &= \int_{-10}^{10} \sqrt{\frac{1}{4}(e^{x/10} + e^{-x/10})^2}\,dx \\ &= \int_{-10}^{10} \frac{1}{2}(e^{x/10} + e^{-x/10})\,dx \\ &= 5(e^{x/10} - e^{-x/10})\Big|_{x=-10}^{x=10} \\ &= 10(e - e^{-1}) \\ &\approx 23.504 \text{ feet,} \end{aligned}$$

which corresponds to the horizontal distance of 20 feet plus about $3\frac{1}{2}$ feet of slack. ■

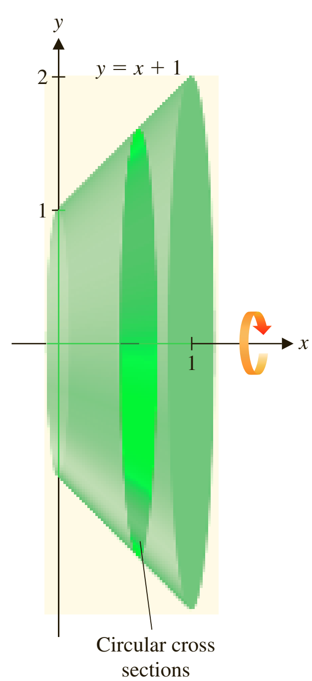


FIGURE 5.38
Surface of revolution

○ Surface Area

In sections 5.2 and 5.3, we saw how to compute the volume of a solid formed by revolving a two-dimensional region about a fixed axis. In addition, we often want to determine the area of the *surface* that is generated by the revolution. For instance, when revolving the line $y = x + 1$, for $0 \le x \le 1$, about the x-axis, the surface generated looks like a megaphone with two open ends, as shown in Figure 5.38.

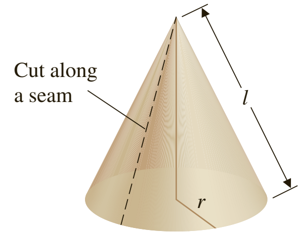


FIGURE 5.39a
Right circular cone

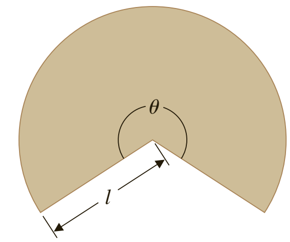


FIGURE 5.39b
Flattened cone

Notice that this is the bottom portion of a right circular cone that has had its top cut off by a plane parallel to its base. Before we go any further, we pause to find the curved surface area of a right circular cone. In Figure 5.39a, we show a right circular cone of base radius r and **slant height** l. (As you'll see later, it is more convenient in this context to specify the slant height than the altitude.) If we cut the cone along a seam and flatten it out, we get the circular sector shown in Figure 5.39b. Notice that the curved surface area of the cone is the same as the area A of the circular sector. This is the area of a circle of radius l multiplied by the fraction of the circle included: θ out of a possible 2π radians, or

$$A = \pi(\text{radius})^2 \frac{\theta}{2\pi} = \pi l^2 \frac{\theta}{2\pi} = \frac{\theta}{2} l^2. \tag{4.2}$$

The only problem with this is that we don't know θ. However, notice that by the way we constructed the sector (i.e., by flattening the cone), the circumference of the sector is the same as the circumference of the base of the cone. That is,

$$2\pi r = 2\pi l \frac{\theta}{2\pi} = l\theta.$$

Dividing by l gives us
$$\theta = \frac{2\pi r}{l}.$$

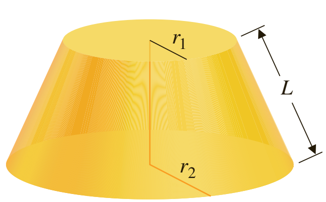


FIGURE 5.40
Frustum of a cone

From (4.2), the curved surface area of the cone is now

$$A = \frac{\theta}{2} l^2 = \frac{\pi r}{l} l^2 = \pi r l.$$

Recall that we were originally interested in finding the surface area of only a portion of a right circular cone (look back at Figure 5.38). For the **frustum** of a cone shown in Figure 5.40, the curved surface area is given by

$$A = \pi(r_1 + r_2)L.$$

You can verify this by subtracting the curved surface area of two cones, where you must use similar triangles to find the height of the larger cone from which the frustum is cut. We leave the details of this as an exercise.

Returning to the original problem of revolving the line $y = x + 1$ on the interval $[0, 1]$ about the x-axis (seen in Figure 5.38), we have $r_1 = 1, r_2 = 2$ and $L = \sqrt{2}$ (from the Pythagorean Theorem). The curved surface area is then

$$A = \pi(1 + 2)\sqrt{2} = 3\pi\sqrt{2} \approx 13.329.$$

For the general problem of finding the curved surface area of a surface of revolution, consider the case where $f(x) \geq 0$ and where f is continuous on the interval $[a, b]$ and differentiable

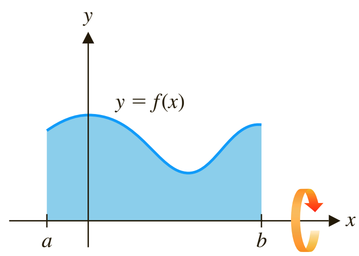


FIGURE 5.41a
Revolve about x-axis

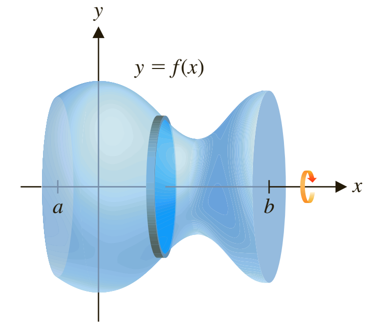


FIGURE 5.41b
Surface of revolution

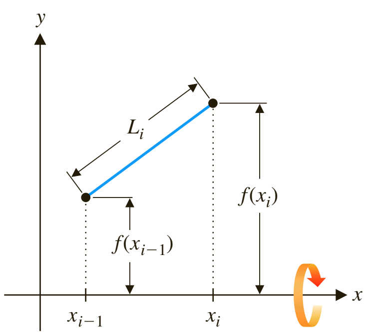


FIGURE 5.42
Revolve about x-axis

on (a, b). If we revolve the graph of $y = f(x)$ about the x-axis on the interval $[a, b]$ (see Figure 5.41a), we get the surface of revolution seen in Figure 5.41b.

As we have done many times now, we first partition the interval $[a, b]$ into n pieces of equal size: $a = x_0 < x_1 < \cdots < x_n = b$, where $x_i - x_{i-1} = \Delta x = \frac{b-a}{n}$, for each $i = 1, 2, \ldots, n$. On each subinterval $[x_{i-1}, x_i]$, we can approximate the curve by the straight line segment joining the points $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$, as in Figure 5.42. Notice that revolving this line segment around the x-axis generates the frustum of a cone. The surface area of this frustum will give us an approximation to the actual surface area on the interval $[x_{i-1}, x_i]$. First, observe that the slant height of this frustum is

$$L_i = d\{(x_{i-1}, f(x_{i-1})), (x_i, f(x_i))\} = \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2},$$

from the usual distance formula. Because of our assumptions on f, we can apply the Mean Value Theorem, to obtain

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}),$$

for some number $c_i \in (x_{i-1}, x_i)$. This gives us

$$L_i = \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2} = \sqrt{1 + [f'(c_i)]^2}\,\underbrace{(x_i - x_{i-1})}_{\Delta x}.$$

The surface area S_i of that portion of the surface on the interval $[x_{i-1}, x_i]$ is approximately the surface area of the frustum of the cone,

$$\begin{aligned} S_i &\approx \pi[f(x_i) + f(x_{i-1})]\sqrt{1 + [f'(c_i)]^2}\,\Delta x \\ &\approx 2\pi f(c_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x, \end{aligned}$$

since if Δx is small, $\qquad f(x_i) + f(x_{i-1}) \approx 2f(c_i).$

Repeating this argument for each subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, gives us an approximation to the total surface area S,

$$S \approx \sum_{i=1}^{n} 2\pi f(c_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x.$$

As n gets larger, this approximation approaches the actual surface area,

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(c_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x.$$

Recognizing this as the limit of a Riemann sum gives us the integral

SURFACE AREA OF A SOLID OF REVOLUTION

$$S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx, \tag{4.3}$$

whenever the integral exists.

You should notice that the factor of $\sqrt{1+[f'(x)]^2}\,dx$ in the integrand in (4.3) corresponds to the arc length of a small section of the curve $y = f(x)$, while the factor $2\pi f(x)$ corresponds to the circumference of the solid of revolution. This should make sense to you, as follows. For any small segment of the curve, if we approximate the surface area by revolving a small segment of the curve of radius $f(x)$ around the x-axis, the surface area generated is simply the surface area of a cylinder,

$$S = 2\pi rh = 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx,$$

since the radius of such a small cylindrical segment is $f(x)$ and the height of the cylinder is $h = \sqrt{1+[f'(x)]^2}\,dx$. It is far better to think about the surface area formula in this way than to simply memorize the formula.

REMARK 4.2

There are exceptionally few functions f for which the integral in (4.3) can be computed exactly. Don't worry; we have numerical integration for just such occasions.

EXAMPLE 4.5 Using the Surface Area Formula

Find the surface area of the surface generated by revolving $y = x^4$, for $0 \le x \le 1$, about the x-axis.

Solution Using the surface area formula (4.3), we have

$$S = \int_0^1 2\pi x^4\sqrt{1+(4x^3)^2}\,dx = \int_0^1 2\pi x^4\sqrt{1+16x^6}\,dx \approx 3.4365,$$

where we have used a numerical method to approximate the value of the integral. ■

EXERCISES 5.4

WRITING EXERCISES

1. Explain in words how the arc length integral is derived from the lengths of the approximating secant line segments.

2. Explain why the sum of the lengths of the line segments in Figure 5.34b is less than the arc length of the curve in Figure 5.34a.

3. Discuss whether the arc length integral is more accurately called a formula or a definition (i.e., can you precisely define the length of a curve without using the integral?).

4. Suppose you graph the trapezoid bounded by $y = x + 1$, $y = -x - 1$, $x = 0$ and $x = 1$, cut it out and roll it up. Explain why you would not get Figure 5.38. (Hint: Compare areas and carefully consider Figures 5.39a and 5.39b.)

In exercises 1–4, approximate the length of the curve using n secant lines for $n = 2$; $n = 4$.

1. $y = x^2, 0 \le x \le 1$
2. $y = x^4, 0 \le x \le 1$
3. $y = \cos x, 0 \le x \le \pi$
4. $y = \ln x, 1 \le x \le 3$

In exercises 5–14, compute the arc length exactly.

5. $y = 2x + 1, 0 \le x \le 2$
6. $y = \sqrt{1 - x^2}, -1 \le x \le 1$
7. $y = 4x^{3/2} + 1, 1 \le x \le 2$
8. $y = \frac{1}{4}(e^{2x} + e^{-2x}), 0 \le x \le 1$
9. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x, 1 \le x \le 2$

10. $y = \frac{1}{6}x^3 + \frac{1}{2x}$, $1 \le x \le 3$

11. $y = \frac{1}{8}x^4 + \frac{1}{4x^2}$, $-2 \le x \le -1$

12. $y = e^{x/2} + e^{-x/2}$, $-1 \le x \le 1$

13. $y = \frac{1}{3}x^{3/2} - x^{1/2}$, $1 \le x \le 4$

14. $y = \frac{3}{4}\left(\frac{x^{4/3}}{2} - x^{2/3}\right)$, $1 \le x \le 8$

In exercises 15–22, set up the integral for arc length and then approximate the integral with a numerical method.

15. $y = x^3$, $-1 \le x \le 1$

16. $y = x^3$, $-2 \le x \le 2$

17. $y = 2x - x^2$, $0 \le x \le 2$

18. $y = \tan x$, $0 \le x \le \pi/4$

19. $y = \cos x$, $0 \le x \le \pi$

20. $y = \ln x$, $1 \le x \le 3$

21. $y = \int_0^x u \sin u \, du$, $0 \le x \le \pi$

22. $y = \int_0^x e^{-u} \sin u \, du$, $0 \le x \le \pi$

23. A rope is to be hung between two poles 20 feet apart. If the rope assumes the shape of the catenary $y = 5(e^{x/10} + e^{-x/10})$, $-10 \le x \le 10$, compute the length of the rope.

24. A rope is to be hung between two poles 60 feet apart. If the rope assumes the shape of the catenary $y = 15(e^{x/30} + e^{-x/30})$, $-30 \le x \le 30$, compute the length of the rope.

25. In example 4.4, compute the "sag" in the cable—that is, the difference between the y-values in the middle ($x = 0$) and at the poles ($x = 10$). Given this, is the arc length calculation surprising?

26. Sketch and compute the length of the **astroid** defined by $x^{2/3} + y^{2/3} = 1$.

27. A football punt follows the path $y = \frac{1}{15}x(60 - x)$ yards. Sketch a graph. How far did the punt go horizontally? How high did it go? Compute the arc length. If the ball was in the air for 4 seconds, what was the ball's average velocity?

28. A baseball outfielder's throw follows the path $y = \dfrac{1}{300}x(100 - x)$ yards. Sketch a graph. How far did the ball go horizontally? How high did it go? Compute the arc length. Explain why the baseball player would want a small arc length, while the football player in exercise 27 would want a large arc length.

29. In example 4.4, show that the arc length is exactly $20\sinh(1)$, where $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

30. Evaluate the integral in example 4.4 with your CAS. Compare its answer to that of exercise 29.

31. The **elliptic integral of the second kind** is defined by $\text{EllipticE}(\phi, m) = \int_0^\phi \sqrt{1 - m\sin^2 u}\, du$. Referring to example 4.1, many CASs report $\sqrt{2}\,\text{EllipticE}(x, \frac{1}{2})$ as an antiderivative of $\sqrt{1 + \cos^2 x}$. Verify that this is an antiderivative.

32. Many CASs report the antiderivative
$$\int \sqrt{1 + 16x^6}\, dx = \frac{1}{4}x\sqrt{1 + 16x^6} + \int \frac{3/4}{\sqrt{1 + 16x^6}}\, dx.$$
Verify that this is an antiderivative.

33. For example 4.2, find an antiderivative with your CAS, evaluate the antiderivative at the endpoints and compare the difference in values to the value your CAS gives using numerical integration. Try to do this for example 4.3.

34. Briefly explain what it means when your CAS returns $\int f(x)\, dx$ when asked to evaluate the indefinite integral $\int f(x)\, dx$.

In exercises 35–42, set up the integral for the surface area of the surface of revolution and approximate the integral with a numerical method.

35. $y = x^2$, $0 \le x \le 1$, revolved about the x-axis

36. $y = \sin x$, $0 \le x \le \pi$, revolved about the x-axis

37. $y = 2x - x^2$, $0 \le x \le 2$, revolved about the x-axis

38. $y = x^3 - 4x$, $-2 \le x \le 0$, revolved about the x-axis

39. $y = e^x$, $0 \le x \le 1$, revolved about the x-axis

40. $y = \ln x$, $1 \le x \le 2$, revolved about the x-axis

41. $y = \cos x$, $0 \le x \le \pi/2$, revolved about the x-axis

42. $y = \sqrt{x}$, $1 \le x \le 2$, revolved about the x-axis

In exercises 43–46, compute the arc length L_1 of the curve and the length L_2 of the secant line connecting the endpoints of the curve. Compute the ratio L_2/L_1; the closer this number is to 1, the straighter the curve is.

43. $y = \sin x$, $-\frac{\pi}{6} \le x \le \frac{\pi}{6}$

44. $y = \cos x$, $-\frac{\pi}{6} \le x \le \frac{\pi}{6}$

45. $y = e^x$, $3 \le x \le 5$

46. $y = e^x$, $-5 \le x \le -3$

47. For $y = x^6$, $y = x^8$ and $y = x^{10}$, compute the arc length for $0 \le x \le 1$. Using results from examples 4.2 and 4.3, identify the pattern for the length of $y = x^n$, $0 \le x \le 1$, as n increases. Conjecture the limit as $n \to \infty$.

48. To help understand the result of exercise 47, determine $\lim_{n\to\infty} x^n$ for each x such that $0 \le x < 1$. Compute the length of this limiting curve. Connecting this curve to the endpoint $(1, 1)$, what is the total length?

49. Prove that $y = x^4$ is flatter than $y = x^2$ for $0 < x < \sqrt{1/2}$ and steeper for $x > \sqrt{1/2}$. Compare the flatness and steepness of $y = x^6$ and $y = x^4$.

50. Suppose that the square consisting of all (x, y) with $-1 \le x \le 1$ and $-1 \le y \le 1$ is revolved about the y-axis. Compute the surface area.

51. Suppose that the circle $x^2 + y^2 = 1$ is revolved about the y-axis. Compute the surface area.

52. Suppose that the triangle with vertices $(-1, -1)$, $(0, 1)$ and $(1, -1)$ is revolved about the y-axis. Compute the surface area.

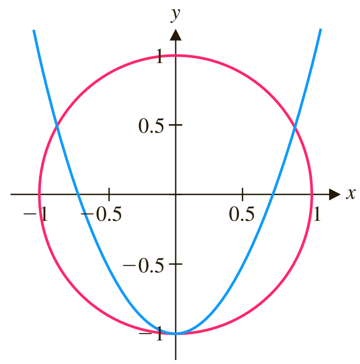
53. Sketch the square, circle and triangle of exercises 50–52 on the same axes. Show that the relative surface areas of the solids of revolution (cylinder, sphere and cone, respectively) are $3:2:\tau$, where τ is the **golden mean** defined by $\tau = \frac{1+\sqrt{5}}{2}$.

EXPLORATORY EXERCISES

1. In exercises 47 and 48, you explored the length of $y = x^n$ on the interval $[0, 1]$. In this exercise, you will look at more general polynomials on the same interval. First, guess what the largest possible arc length would be for a polynomial on a given interval. Now, check the arc length for some selected parabolas. Explain why $f_c(x) = cx(1-x)$ is a downward-opening parabola (for $c > 0$) with x-intercepts 0 and 1 and vertex at $x = 1/2$. What will happen to the arc length as c gets larger? Suppose we limit ourselves to polynomials that have function values lying between -1 and 1. Verify that on the interval $[0, 1]$, the functions $g_2(x) = 4x(1-x)$, $g_3(x) = 20x(1/2 - x)(1-x)$ and $g_4(x) = 80x(1/3 - x)(2/3 - x)(1-x)$ all have this property. Compare the arc lengths of $g_2(x)$, $g_3(x)$ and $g_4(x)$. What will be the limiting value of such arc lengths? Construct a function $g_5(x)$ that continues this pattern.

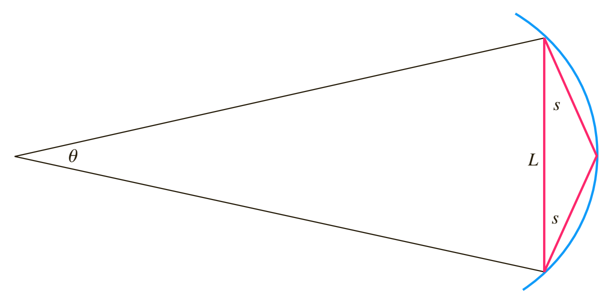
2. In this exercise, you will explore a famous paradox (often called Gabriel's horn). Suppose that the curve $y = 1/x$, for $1 \le x \le R$ (where R is a large positive constant), is revolved about the x-axis. Compute the enclosed volume and the surface area of the resulting surface. (In both cases, antiderivatives can be found, although you may need help from your CAS to get the surface area.) Determine the limit of the volume and surface area as $R \to \infty$. Now for the paradox. Based on your answers, you should have a solid with finite volume, but infinite surface area. Thus, the three-dimensional solid could be completely filled with a finite amount of paint but the outside surface could never be completely painted.

3. Let C be the portion of the parabola $y = ax^2 - 1$ inside the circle $x^2 + y^2 = 1$.



Find the value of $a > 0$ that maximizes the arc length of C.

4. The figure shows an arc of a circle subtended by an angle θ, with a chord of length L and two chords of length s. Show that $2s = \frac{L}{\cos(\theta/4)}$.



Start with a quarter-circle and use this formula repeatedly to derive the infinite product

$$\cos\frac{\pi}{4}\cos\frac{\pi}{8}\cos\frac{\pi}{16}\cos\frac{\pi}{32}\cdots = \frac{2}{\pi}$$

where the left-hand side represents $\lim_{n\to\infty}\left(\cos\frac{\pi}{2^n}\cos\frac{\pi}{2^{n-1}}\cdots\cos\frac{\pi}{4}\right)$.

5.5 PROJECTILE MOTION

In sections 2.1, 2.3 and 4.1, we discussed aspects of the motion of an object moving in a straight line path (rectilinear motion). We saw that if we know a function describing the position of an object at any time t, then we can determine its velocity and acceleration by differentiation. A much more important problem is to go backward, that is, to find the

51. Suppose that the circle $x^2 + y^2 = 1$ is revolved about the y-axis. Compute the surface area.

52. Suppose that the triangle with vertices $(-1, -1)$, $(0, 1)$ and $(1, -1)$ is revolved about the y-axis. Compute the surface area.

53. Sketch the square, circle and triangle of exercises 50–52 on the same axes. Show that the relative surface areas of the solids of revolution (cylinder, sphere and cone, respectively) are $3{:}2{:}\tau$, where τ is the **golden mean** defined by $\tau = \frac{1+\sqrt{5}}{2}$.

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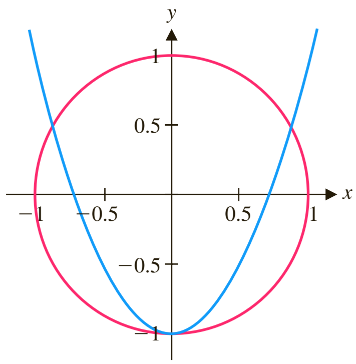

Find the value of $a > 0$ that maximizes the arc length of C.

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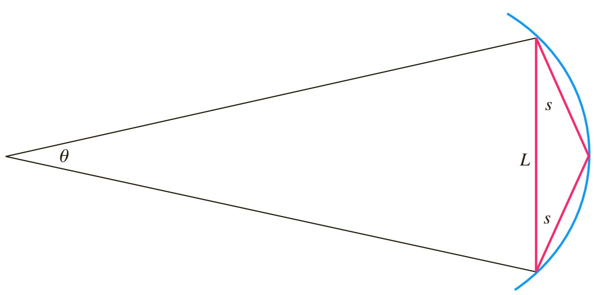


Start with a quarter-circle and use this formula repeatedly to derive the infinite product

$$\cos\frac{\pi}{4}\cos\frac{\pi}{8}\cos\frac{\pi}{16}\cos\frac{\pi}{32}\cdots = \frac{2}{\pi}$$

where the left-hand side represents $\lim_{n\to\infty}(\cos\frac{\pi}{2^n}\cos\frac{\pi}{2^{n-1}}\ldots\cos\frac{\pi}{4})$.

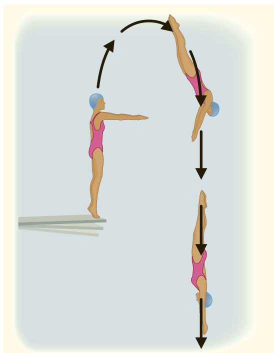
5.5 PROJECTILE MOTION

In sections 2.1, 2.3 and 4.1, we discussed aspects of the motion of an object moving in a straight line path (rectilinear motion). We saw that if we know a function describing the position of an object at any time t, then we can determine its velocity and acceleration by differentiation. A much more important problem is to go backward, that is, to find the

position and velocity of an object, given its acceleration. Mathematically, this means that, starting with the derivative of a function, we must find the original function. Now that we have integration at our disposal, we can accomplish this with ease.

You may already be familiar with **Newton's second law of motion,** which says that

$$F = ma,$$



where F is the sum of the forces acting on an object, m is the mass of the object and a is the acceleration of the object.

Start by imagining that you are diving. The primary force acting on you throughout the dive is gravity. The force due to gravity is your own weight, which is related to mass by $W = mg$, where g is the gravitational constant. (Common approximations of g, accurate near sea level, are 32 ft/s^2 and 9.8 m/s^2.) To keep the problem simple mathematically, we will ignore any other forces, such as air resistance.

Let $h(t)$ represent your height above the water t seconds after starting your dive. Then the force due to gravity is $F = -mg$, where the minus sign indicates that the force is acting downward, in the negative direction. From our earlier work, we know that the acceleration is $a(t) = h''(t)$. Newton's second law then gives us $-mg = mh''(t)$ or

$$h''(t) = -g.$$

Notice that the position function of *any* object (regardless of its mass) subject to gravity and no other forces will satisfy the same equation. The only differences from situation to situation are the initial conditions (the initial velocity and initial position) and the questions being asked.

EXAMPLE 5.1 Finding the Velocity of a Diver at Impact

If a diving board is 15 feet above the surface of the water and a diver starts with initial velocity 8 ft/s (in the upward direction), what is the diver's velocity at impact (assuming no air resistance)?

Solution If the height (in feet) at time t is given by $h(t)$, Newton's second law gives us $h''(t) = -32$. Since the diver starts 15 feet above the water with initial velocity of 8 ft/s, we have the initial conditions $h(0) = 15$ and $h'(0) = 8$. Finding $h(t)$ now takes little more than elementary integration. We have

$$\int h''(t)\,dt = \int -32\,dt$$

or

$$h'(t) = -32t + c.$$

From the initial velocity, we have

$$8 = h'(0) = -32(0) + c = c,$$

so that the velocity at any time t is given by

$$h'(t) = -32t + 8.$$

To find the velocity at impact, you first need to find the *time* of impact. Notice that the diver will hit the water when $h(t) = 0$ (i.e., when the height above the water is 0).

Integrating the velocity function gives us the height function:

$$\int h'(t)\,dt = \int (-32t + 8)\,dt$$

or

$$h(t) = -16t^2 + 8t + c.$$

From the initial height, we have

$$15 = h(0) = -16(0)^2 + 8(0) + c = c,$$

so that the height above the water at any time t is given by

$$h(t) = -16t^2 + 8t + 15.$$

Impact then occurs when

$$\begin{aligned} 0 = h(t) &= -16t^2 + 8t + 15 \\ &= -(4t + 3)(4t - 5), \end{aligned}$$

so that $t = \frac{5}{4}$ is the time of impact. (Ignore the extraneous solution $t = -\frac{3}{4}$.) When $t = \frac{5}{4}$, the velocity is $h'\left(\frac{5}{4}\right) = -32\left(\frac{5}{4}\right) + 8 = -32$ ft/s (impact velocity). To put this in more familiar units of velocity, multiply by 3600/5280 to convert to miles per hour. In this case, the impact velocity is about 22 mph. (You probably don't want to come down in the wrong position at that speed!) ■

In example 5.1, the negative sign of the velocity indicated that the diver was coming down. In many situations, both upward and downward motions are important.

EXAMPLE 5.2 An Equation for the Vertical Motion of a Ball

A ball is propelled straight upward from the ground with initial velocity 64 ft/s. Ignoring air resistance, find an equation for the height of the ball at any time t. Also, determine the maximum height and the amount of time the ball spends in the air.

Solution With gravity as the only force, the height $h(t)$ satisfies $h''(t) = -32$. The initial conditions are $h'(0) = 64$ and $h(0) = 0$. We then have

$$\int h''(t)\,dt = \int -32\,dt$$

or

$$h'(t) = -32t + c.$$

From the initial velocity, we have

$$64 = h'(0) = -32(0) + c = c$$

and so,

$$h'(t) = 64 - 32t.$$

Integrating one more time gives us

$$\int h'(t)\,dt = \int (64 - 32t)\,dt$$

TODAY IN MATHEMATICS

Vladimir Arnold (1937–)
A Russian mathematician with important contributions to numerous areas of mathematics, both in research and popular exposition. The esteem in which he is held by his colleagues can be measured by the international conference known as "Arnoldfest" held in Toronto in honor of his 60th birthday. Many of his books are widely used today, including a collection of challenges titled *Arnold's Problems*. A review of this book states that "Arnold did not consider mathematics a game with deductive reasoning and symbols, but a part of natural science (especially of physics), i.e., an experimental science."

or
$$h(t) = 64t - 16t^2 + c.$$

From the initial height we have

$$0 = h(0) = 64(0) - 16(0)^2 + c = c,$$

and so,
$$h(t) = 64t - 16t^2.$$

Since the height function is quadratic, its maximum occurs at the one time when $h'(t) = 0$. [You should also consider the physics of the situation: what happens physically when $h'(t) = 0$?] Solving $64 - 32t = 0$ gives $t = 2$ (the time at the maximum height) and the corresponding height is $h(2) = 64(2) - 16(2)^2 = 64$ feet. Again, the ball lands when $h(t) = 0$. Solving

$$0 = h(t) = 64t - 16t^2 = 16t(4 - t)$$

gives $t = 0$ (launch time) and $t = 4$ (landing time). The time of flight is thus 4 seconds. ■

You can observe an interesting property of projectile motion by graphing the height function from example 5.2 along with the line $y = 48$ (see Figure 5.43). Notice that the graphs intersect at $t = 1$ and $t = 3$. Further, the time interval [1, 3] corresponds to exactly half the time spent in the air. Notice that this says that the ball stays in the top one-fourth of its height for half of its time in the air. You may have marveled at how some athletes jump so high that they seem to "hang in the air." As this calculation suggests, all objects tend to hang in the air.

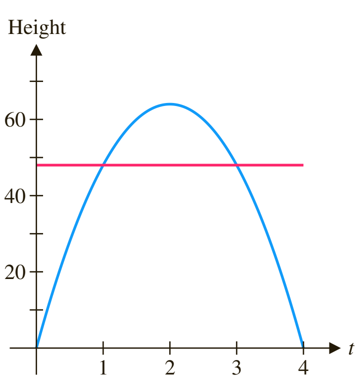


FIGURE 5.43
Height of the ball at time t

EXAMPLE 5.3 Finding the Initial Velocity Required to Reach a Certain Height

It has been reported that basketball star Michael Jordan has a vertical leap of 54″. Ignoring air resistance, what is the initial velocity required to jump this high?

Solution Once again, Newton's second law leads us to the equation $h''(t) = -32$ for the height $h(t)$. We call the initial velocity v_0, so that $h'(0) = v_0$ and look for the value of v_0 that will give a maximum altitude of 54″. As before, we integrate to get

$$h'(t) = -32t + c.$$

Using the initial velocity, we get

$$v_0 = h'(0) = -32(0) + c = c.$$

This gives us the velocity function

$$h'(t) = v_0 - 32t.$$

Integrating once again and using the initial position $h(0) = 0$, we get

$$h(t) = v_0 t - 16t^2.$$

The maximum height occurs when $h'(t) = 0$. (Why?) Setting

$$0 = h'(t) = v_0 - 32t,$$

gives us $t = \frac{v_0}{32}$. The height at this time (i.e., the maximum altitude) is then

$$h\left(\frac{v_0}{32}\right) = v_0\left(\frac{v_0}{32}\right) - 16\left(\frac{v_0}{32}\right)^2 = \frac{v_0^2}{32} - \frac{v_0^2}{64} = \frac{v_0^2}{64}.$$

So, a jump of $54'' = 4.5'$ requires $\frac{v_0^2}{64} = 4.5$ or $v_0^2 = 288$, so that $v_0 = \sqrt{288} \approx 17$ ft/s (equivalent to roughly 11.6 mph). ■

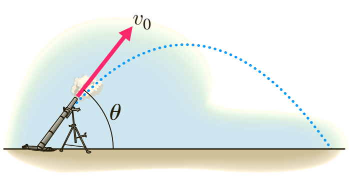


FIGURE 5.44a
Path of projectile

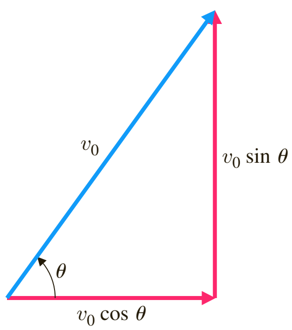


FIGURE 5.44b
Vertical and horizontal components of velocity

So far, we have only considered projectiles moving vertically. In practice, we must also consider movement in the horizontal direction. Ignoring air resistance, these calculations are also relatively straightforward. The idea is to apply Newton's second law separately to the horizontal and vertical components of the motion. If $y(t)$ represents the vertical position, then we have $y''(t) = -g$, as before. Ignoring air resistance, there are no forces acting horizontally on the projectile. So, if $x(t)$ represents the horizontal position, Newton's second law gives us $x''(t) = 0$.

The initial conditions are slightly more complicated here. In general, we want to consider projectiles that are launched with an initial speed v_0 at an angle θ from the horizontal. Figure 5.44a shows a projectile fired with $\theta > 0$. Notice that an initial angle of $\theta < 0$ would mean a downward initial velocity.

As shown in Figure 5.44b, the initial velocity can be separated into horizontal and vertical components. From elementary trigonometry, the horizontal component of the initial velocity is $v_x = v_0 \cos\theta$ and the vertical component is $v_y = v_0 \sin\theta$.

EXAMPLE 5.4 The Motion of a Projectile in Two Dimensions

An object is launched at angle $\theta = \pi/6$ from the horizontal with initial speed $v_0 = 98$ m/s. Determine the time of flight and the (horizontal) range of the projectile.

Solution Starting with the vertical component of the motion (and again ignoring air resistance), we have $y''(t) = -9.8$ (since the initial speed is given in terms of meters per second). Referring to Figure 5.44b, notice that the vertical component of the initial velocity is $y'(0) = 98 \sin \pi/6 = 49$ and the initial altitude is $y(0) = 0$. A pair of simple integrations gives us the velocity function $y'(t) = -9.8t + 49$ and the position function $y(t) = -4.9t^2 + 49t$. The object hits the ground when $y(t) = 0$ (i.e., when its height above the ground is 0). Solving

$$0 = y(t) = -4.9t^2 + 49t = 49t(1 - 0.1t)$$

gives $t = 0$ (launch time) and $t = 10$ (landing time). The time of flight is then 10 seconds. The horizontal component of motion is determined from the equation $x''(t) = 0$ with initial velocity $x'(0) = 98 \cos \pi/6 = 49\sqrt{3}$ and initial position $x(0) = 0$. Integration gives us $x'(t) = 49\sqrt{3}$ and $x(t) = (49\sqrt{3})t$. In Figure 5.45, we plot the path of the ball. [You can do this using the parametric plot mode on your

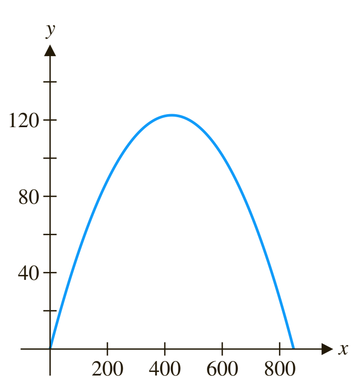


FIGURE 5.45
Path of ball

REMARK 5.1

You should resist the temptation to reduce this section to a few memorized formulas. It is true that if you ignore air resistance, the vertical component of position will always turn out to be $y(t) = -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + y(0)$. However, your understanding of the process and your chances of finding the correct answer will improve dramatically if you start each problem with Newton's second law and work through the integrations (which are not difficult).

graphing calculator or CAS, by entering equations for $x(t)$ and $y(t)$ and setting the range of t-values to be $0 \le t \le 10$. Alternatively, you can easily solve for t, to get $t = \frac{1}{49\sqrt{3}}x$, to see that the curve is simply a parabola.] The horizontal range is then the value of $x(t)$ at $t = 10$ (the landing time),

$$x(10) = (49\sqrt{3})(10) = 490\sqrt{3} \approx 849 \text{ meters.}$$

EXAMPLE 5.5 The Motion of a Tennis Serve

Venus Williams has one of the fastest serves in women's tennis. Suppose that she hits a serve from a height of 10 feet at an initial speed of 120 mph and at an angle of 7° below the horizontal. The serve is "in" if the ball clears a 3′-high net that is 39′ away and hits the ground in front of the service line 60′ away. (We illustrate this situation in Figure 5.46.) Determine whether the serve is in or out.

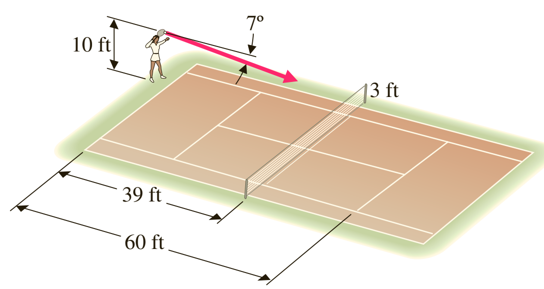


FIGURE 5.46
Height of tennis serve

Solution As in example 5.4, we start with the vertical motion of the ball. Since distance is given in feet, the equation of motion is $y''(t) = -32$. The initial speed must be converted to feet per second: 120 mph = 120 $\frac{5280}{3600}$ ft/s = 176 ft/s. The vertical component of the initial velocity is then $y'(0) = 176\sin(-7°) \approx -21.45$ ft/s. Integration then gives us

$$y'(t) = -32t - 21.45.$$

The initial height is $y(0) = 10$ ft, so another integration gives us

$$y(t) = -16t^2 - 21.45t + 10 \text{ ft.}$$

The horizontal component of motion is determined from $x''(t) = 0$, with initial velocity $x'(0) = 176\cos(-7°) \approx 174.69$ ft/s and initial position $x(0) = 0$. Integrations give us $x'(t) = 174.69$ ft/s and $x(t) = 174.69t$ ft. Summarizing, we have

$$x(t) = 174.69t,$$
$$y(t) = -16t^2 - 21.45t + 10.$$

For the ball to clear the net, y must be at least 3 when $x = 39$. We have $x(t) = 39$ when $174.69t = 39$ or $t \approx 0.2233$. At this time, $y(0.2233) \approx 4.4$, showing that the ball is high enough to clear the net. The second requirement is that we need to have $x \le 60$ when the ball lands ($y = 0$). We have $y(t) = 0$ when $-16t^2 - 21.45t + 10 = 0$. From the quadratic formula, we get $t \approx -1.7$ and $t \approx 0.3662$. Ignoring the negative solution, we compute $x(0.3662) \approx 63.97$, so that the serve lands nearly four feet beyond the service line. The serve is not in.

One reason you should start each problem with Newton's second law is so that you pause to consider the forces that are (and are not) being considered. For example, we have thus far ignored air resistance, as a simplification of reality. Some calculations using such simplified equations are reasonably accurate. Others, such as in example 5.6, are not.

EXAMPLE 5.6 An Example Where Air Resistance Can't Be Ignored

Suppose a raindrop falls from a cloud 3000 feet above the ground. Ignoring air resistance, how fast would the raindrop be falling when it hits the ground?

Solution If the height of the raindrop at time t is given by $y(t)$, Newton's second law of motion tells us that $y''(t) = -32$. Further, we have the initial velocity $y'(0) = 0$ (since the drop falls—as opposed to being thrown down) and the initial altitude $y(0) = 3000$. Integrating and using the initial conditions gives us $y'(t) = -32t$ and $y(t) = 3000 - 16t^2$. The raindrop hits the ground when $y(t) = 0$. Setting

$$0 = y(t) = 3000 - 16t^2$$

gives us $t = \sqrt{3000/16} \approx 13.693$ seconds. The velocity at this time is then

$$y'(\sqrt{3000/16}) = -32\sqrt{3000/16} \approx -438.18 \text{ ft/s}.$$

This corresponds to nearly 300 mph! Fortunately, air resistance does play a significant role in the fall of a raindrop, which has an actual landing speed of about 10 mph. ■

The obvious lesson from example 5.6 is that it is not always reasonable to ignore air resistance. Some of the mathematical tools needed to more fully analyze projectile motion with air resistance are developed in Chapter 7.

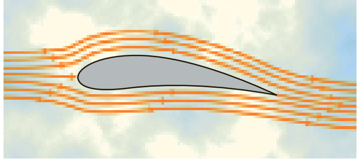


FIGURE 5.47
Cross section of a wing

The air resistance (more precisely, *air drag*) that slows the raindrop down is only one of the ways in which air can affect the motion of an object. The **Magnus force,** produced by the spinning of an object or lack of symmetry in the shape of an object, can cause the object to change directions and curve. Perhaps the most common example of a Magnus force occurs on an airplane. One side of an airplane wing is curved and the other side is comparatively flat (see Figure 5.47). The lack of symmetry causes the air to move over the top of the wing faster than it moves over the bottom. This produces a Magnus force in the upward direction (lift), lifting the airplane into the air.

A more down-to-earth example of a Magnus force occurs in an unusual baseball pitch called the knuckleball. To throw this pitch, the pitcher grips the ball with his fingertips and throws the ball with as little spin as possible. Baseball players claim that the knuckleball "dances around" unpredictably and is exceptionally hard to hit or catch. There still is no complete agreement on why the knuckleball moves so much, but we will present one current theory due to physicists Robert Watts and Terry Bahill.

Regulation baseball, showing stitching

The cover of the baseball is sewn on with stitches that are raised up slightly from the rest of the ball. These curved stitches act much like an airplane wing, creating a Magnus force that affects the ball. The direction of the Magnus force depends on the exact orientation of the ball's stitches. Measurements by Watts and Bahill indicate that the lateral force (left/right from the pitcher's perspective) is approximately $F_m = -0.1 \sin(4\theta)$ lb, where θ is the angle (in radians) of the ball's position rotated from a particular starting position.

Since gravity does not affect the lateral motion of the ball, the only force acting on the ball laterally is the Magnus force. Newton's second law applied to the lateral motion of the

knuckleball gives $mx''(t) = -0.1\sin(4\theta)$. The mass of a baseball is about 0.01 slug. (Slugs are the units of measurement of *mass* in the English system. To get the more familiar *weight* in pounds, simply multiply the mass by $g = 32$.) We now have

$$x''(t) = -10\sin(4\theta).$$

If the ball is spinning at the rate of ω radians per second, then $4\theta = 4\omega t + \theta_0$, where the initial angle θ_0 depends on where the pitcher grips the ball. We then have

$$x''(t) = -10\sin(4\omega t + \theta_0), \tag{5.1}$$

with initial conditions $x'(0) = 0$ and $x(0) = 0$. For a typical knuckleball speed of 60 mph, it takes about 0.68 second for the pitch to reach home plate.

EXAMPLE 5.7 An Equation for the Motion of a Knuckleball

For a spin rate of $\omega = 2$ radians per second and $\theta_0 = 0$, find an equation for the motion of the knuckleball and graph it for $0 \le t \le 0.68$. Repeat this for $\theta_0 = \pi/2$.

Solution For $\theta_0 = 0$, Newton's second law gives us $x''(t) = -10\sin 8t$, from (5.1). Integrating this and using the initial condition $x'(0) = 0$ gives us

$$x'(t) = -\frac{10}{8}[-\cos 8t - (-\cos 0)] = 1.25(\cos 8t - 1).$$

Integrating once again and using the second initial condition $x(0) = 0$, we get

$$x(t) = 1.25\left(\frac{1}{8}\right)(\sin 8t - 0) - 1.25t = 0.15625\sin 8t - 1.25t.$$

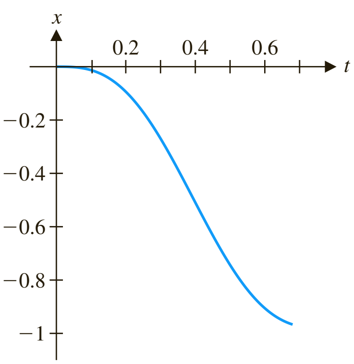


FIGURE 5.48a
Lateral motion of a knuckleball for $\theta_0 = 0$

A graph of this function shows the lateral motion of the ball (see Figure 5.48a). The graph shows the path of the pitch as it would look viewed from above. Notice that after starting out straight, this pitch breaks nearly a foot away from the center of home plate!

For the case where $\theta_0 = \pi/2$, we have from (5.1) that

$$x''(t) = -10\sin\left(8t + \frac{\pi}{2}\right).$$

Integrating this and using the first initial condition gives us

$$x'(t) = -\frac{10}{8}\left\{-\cos\left(8t + \frac{\pi}{2}\right) - \left[-\cos\left(0 + \frac{\pi}{2}\right)\right]\right\} = 1.25\cos\left(8t + \frac{\pi}{2}\right).$$

Integrating a second time yields

$$x(t) = 1.25\left(\frac{1}{8}\right)\left[\sin\left(8t + \frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right] = 0.15625\left[\sin\left(8t + \frac{\pi}{2}\right) - 1\right].$$

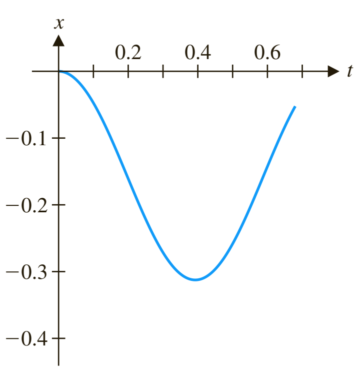


FIGURE 5.48b
Lateral motion of a knuckleball for $\theta_0 = \dfrac{\pi}{2}$

A graph of the lateral motion in this case is shown in Figure 5.48b. Notice that this pitch breaks nearly 4 inches to the pitcher's right and then curves back over the plate for a strike! You can see that, in theory, the knuckleball is very sensitive to spin and initial position and can be very difficult to hit when thrown properly. ■

EXERCISES 5.5

WRITING EXERCISES

1. In example 5.6, the assumption that air resistance can be ignored is obviously invalid. Discuss the validity of this assumption in examples 5.1 and 5.3.

2. In the discussion preceding example 5.3, we showed that Michael Jordan (and any other human) spends half of his air-time in the top one-fourth of the height. Compare his velocities at various points in the jump to explain why relatively more time is spent at the top than at the bottom.

3. In example 5.4, we derived separate equations for the horizontal and vertical components of position. To discover one consequence of this separation, consider the following situation. Two people are standing next to each other with arms raised to the same height. One person fires a bullet horizontally from a gun. At the same time, the other person drops a bullet. Explain why the bullets will hit the ground at the same time.

4. For the falling raindrop in example 5.6, a more accurate model would be $y''(t) = -32 + f(t)$, where $f(t)$ represents the force due to air resistance (divided by the mass). If $v(t)$ is the downward velocity of the raindrop, explain why this equation is equivalent to $v'(t) = 32 - f(t)$. Explain in physical terms why the larger $v(t)$ is, the larger $f(t)$ is. Thus, a model such as $f(t) = v(t)$ or $f(t) = [v(t)]^2$ would be reasonable. (In most situations, it turns out that $[v(t)]^2$ matches the experimental data better.)

In exercises 1–4, identify the initial conditions $y(0)$ and $y'(0)$.

1. An object is dropped from a height of 80 feet.

2. An object is dropped from a height of 100 feet.

3. An object is released from a height of 60 feet with an upward velocity of 10 ft/s.

4. An object is released from a height of 20 feet with a downward velocity of 4 ft/s.

In exercises 5–56, ignore air resistance.

5. A diver drops from 30 feet above the water (about the height of an Olympic platform dive). What is the diver's velocity at impact?

6. A diver drops from 120 feet above the water (about the height of divers at the Acapulco Cliff Diving competition). What is the diver's velocity at impact?

7. Compare the impact velocities of objects falling from 30 feet (exercise 5), 120 feet (exercise 6) and 3000 feet (example 5.6). If height is increased by a factor of h, by what factor does the impact velocity increase?

8. The Washington Monument is 555 feet, $5\frac{1}{8}$ inches high. In a famous experiment, a baseball was dropped from the top of the monument to see if a player could catch it. How fast would the ball be going?

9. A certain not-so-wily coyote discovers that he just stepped off the edge of a cliff. Four seconds later, he hits the ground in a puff of dust. How high was the cliff?

10. A large boulder dislodged by the falling coyote in exercise 9 falls for 3 seconds before landing on the coyote. How far did the boulder fall? What was its velocity when it flattened the coyote?

11. The coyote's next scheme involves launching himself into the air with an Acme catapult. If the coyote is propelled vertically from the ground with initial velocity 64 ft/s, find an equation for the height of the coyote at any time t. Find his maximum height, the amount of time spent in the air and his velocity when he smacks back into the catapult.

12. On the rebound, the coyote in exercise 11 is propelled to a height of 256 feet. What is the initial velocity required to reach this height?

13. One of the authors has a vertical "jump" of 20 inches. What is the initial velocity required to jump this high? How does this compare to Michael Jordan's velocity, found in example 5.3?

14. If the author underwent an exercise program and increased his initial velocity by 10%, by what percentage would he increase his vertical jump?

15. Show that an object dropped from a height of H feet will hit the ground at time $T = \frac{1}{4}\sqrt{H}$ seconds with impact velocity $V = -8\sqrt{H}$ ft/s.

16. Show that an object propelled from the ground with initial velocity v_0 ft/s will reach a maximum height of $v_0^2/64$ ft.

17. You can measure your reaction time using a ruler. Hold your thumb and forefinger on either side of a yardstick. Have a friend drop the yardstick and grab it as fast as you can. Take the distance d that the yardstick falls and compute how long the ruler fell. Show that if d is measured in cm, your reaction time is approximately $t \approx 0.045\sqrt{d}$. For comparison purposes, a top athlete has a reaction time of about 0.15 s.

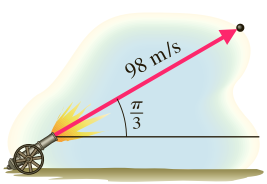
18. The coefficient of restitution of a ball measures how "lively" the bounce is. By definition, the coefficient equals $\dfrac{v_2}{v_1}$, where v_1 is the (downward) speed of the ball when it hits the ground and v_2 is the (upward) launch speed after it hits the ground. If a ball is dropped from a height of H feet and rebounds to a height of cH for some constant c with $0 < c < 1$, compute its coefficient of restitution.

19. According to legend, Galileo dropped two balls from the Leaning Tower of Pisa. When both the heavy lead ball and the light wood ball hit the ground at the same time, Galileo knew that the acceleration due to gravity is the same for all objects. Unfortunately, air resistance would affect such an experiment. Taking into account air resistance, a 6″ wood ball would fall $f(t) = \frac{7225}{8} \ln\left[\cosh\left(\frac{16}{85}t\right)\right]$ feet in t seconds, while a 6″ lead ball would fall $g(t) = 12{,}800 \ln\left[\cosh\left(\frac{1}{20}t\right)\right]$ feet, where $\cosh x = \frac{1}{2}(e^x + e^{-x})$. From a height of 179 feet, find the height of the wood ball when the lead ball hits the ground.

20. If a theatrical production wanted to show the balls of exercise 19 landing at the same time, how much earlier would the wood ball need to be released?

In exercises 21–30, sketch the parametric graphs as in example 5.4 to indicate the flight path.

21. An object is launched at angle $\theta = \pi/3$ radians from the horizontal with an initial speed of 98 m/s. Determine the time of flight and the horizontal range. Compare to example 5.4.

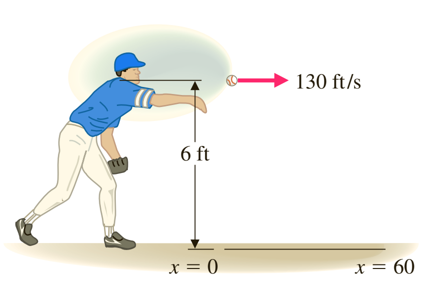


22. Find the time of flight and horizontal range of an object launched at angle 30° with initial speed 120 ft/s. Repeat with an angle of 60°.

23. Repeat example 5.5 with an initial angle of 6°. By trial and error, find the smallest and largest angles for which the serve will be in.

24. Repeat example 5.5 with an initial speed of 170 ft/s. By trial and error, find the smallest and largest initial speeds for which the serve will be in.

25. A baseball pitcher releases the ball horizontally from a height of 6 ft with an initial speed of 130 ft/s. Find the height of the ball when it reaches home plate 60 feet away. (Hint: Determine the time of flight from the x-equation, then use the y-equation to determine the height.)



26. Repeat exercise 25 with an initial speed of 80 ft/s. (Hint: Carefully interpret the negative answer.)

27. A baseball player throws a ball toward first base 120 feet away. The ball is released from a height of 5 feet with an initial speed of 120 ft/s at an angle of 5° above the horizontal. Find the height of the ball when it reaches first base.

28. By trial and error, find the angle at which the ball in exercise 27 will reach first base at the catchable height of 5 feet. At this angle, how far above the first baseman's head would the thrower be aiming?

29. A daredevil plans to jump over 25 cars. If the cars are all compact cars with a width of 5 feet and the ramp angle is 30°, determine the initial velocity required to complete the jump successfully. Repeat with a takeoff angle of 45°. In spite of the reduced initial velocity requirement, why might the daredevil prefer an angle of 30° to 45°?

30. A plane at an altitude of 256 feet wants to drop supplies to a specific location on the ground. If the plane has a horizontal velocity of 100 ft/s, how far away from the target should the plane release the supplies in order to hit the target location? (Hint: Use the y-equation to determine the time of flight, then use the x-equation to determine how far the supplies will drift.)

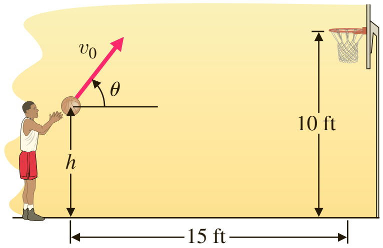
31. Consider a knuckleball (see example 5.7) with lateral motion satisfying the initial value problem $x''(t) = -25\sin(4\omega t + \theta_0)$, $x'(0) = x(0) = 0$. With $\omega = 1$, find an equation for $x(t)$ and graph the solution for $0 \le t \le 0.68$ with (a) $\theta_0 = 0$ and (b) $\theta_0 = \pi/2$.

32. Repeat exercise 31 for $\theta_0 = \pi/4$ and (a) $\omega = 2$ and (b) $\omega = 1$.

33. For the Olympic diver in exercise 5, what would be the average angular velocity (measured in radians per second) necessary to complete $2\frac{1}{2}$ somersaults?

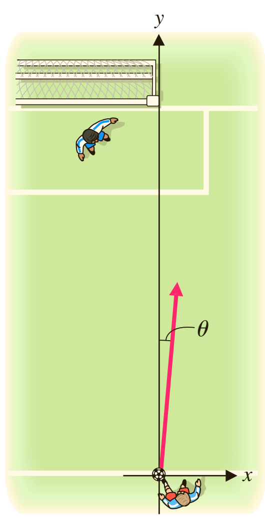
34. In the Flying Zucchini Circus' human cannonball act, a performer is shot out of a cannon from a height of 10 feet at an angle of 45° with an initial speed of 160 ft/s. If the safety net stands 5 feet above the ground, how far should the safety net be placed from the cannon? If the safety net can withstand an impact velocity of only 160 ft/s, will the Flying Zucchini land safely or come down squash?

35. In a basketball free throw, a ball is shot from a height of h feet toward a basket 10 feet above the ground at a horizontal distance of 15 feet. (See the figure on the following page.) If $h = 6$, $\theta = 52°$ and $v_0 = 25$ ft/s, show that the free throw is good. Since the basket is larger than the ball, a free throw has a margin of error of several inches. If any shot that passes through height 10 ft with $14.65 \le x \le 15.35$ is good, show that, for the given initial speed v_0, the margin of error is $48° \le \theta \le 57°$. Sketch parametric graphs to show several of these free throws.



36. For the basketball shot in exercise 35, fix $h = 6$ and $\theta = 52°$. Find the range of speeds v_0 for which the free throw will be good. Given that it requires a force of $F = 0.01v_0^2$ pounds to launch a free throw with speed v_0, how much margin of error is there in the applied force?

37. Soccer player Roberto Carlos of Brazil is known for his curving kicks. Suppose that he has a free kick from 30 yards out. Orienting the x- and y-axes as shown in the figure, suppose the kick has initial speed 100 ft/s at an angle of 5° from the positive y-axis. Assume that the only force on the ball is a Magnus force to the left caused by the spinning of the ball. With $x''(t) = -20$ and $y''(t) = 0$, determine whether the ball goes in the goal at $y = 90$ and $-24 \le x \le 0$.



38. In exercise 37, a wall of players is lined up 10 yards away, extending from $x = -10$ to $x = 1$. Determine whether the kick goes around the wall.

39. To train astronauts to operate in a weightless environment, NASA sends them up in a special plane (nicknamed the *Vomit Comet*). To allow the passengers to experience weightlessness, the vertical acceleration of the plane must exactly match the acceleration due to gravity. If $y''(t)$ is the vertical acceleration of the plane, then $y''(t) = -g$. Show that, for a constant horizontal velocity, the plane follows a parabolic path. NASA's plane flies parabolic paths of approximately 2500 feet in height (2500 feet up and 2500 down). The time to complete such a path is the amount of weightless time for the passengers. Compute this time.

40. In the 1992 Summer Olympics in Barcelona, Spain, an archer lit the Olympic flame by shooting an arrow toward a cauldron at a distance of about 70 meters horizontally and 30 meters vertically. If the arrow reached the cauldron at the peak of its trajectory, determine the initial speed and angle of the arrow. (Hint: Show that $y'(t) = 0$ if $t = (v_0 \sin\theta)/9.8$. For this t, show that $\dfrac{x(t)}{y(t)} = 2\cot\theta = \dfrac{7}{3}$ and solve for θ. Then solve for v_0.)

In exercises 41–46, we explore two aspects of juggling. More information can be found in *The Mathematics of Juggling* by Burkard Polster.

41. Professional jugglers generally agree that 10 is the maximum number of balls that a human being can successfully maintain. To get an idea why, suppose that it takes $\frac{1}{2}$ second to catch and toss a ball. (In other words, using both hands, the juggler can process 4 balls per second.) To juggle 10 balls, each ball would need to be in the air for 2.5 seconds. Neglecting air resistance, how high would the ball have to be tossed to stay in the air this long? How much higher would the balls need to be tossed to juggle 11 balls?

42. Another aspect of juggling balls is accuracy. A ball juggled from the right hand to the left hand must travel the correct horizontal distance to be catchable. Suppose that a ball is tossed with initial horizontal velocity v_{0x} and initial vertical velocity v_{0y}. Assume that the ball is caught at the height from which it is thrown. Show that the horizontal distance traveled is $w = \dfrac{v_{0x}v_{0y}}{16}$ feet. (Hint: This is a basic projectile problem, like example 5.4.)

43. Referring to exercise 42, suppose that a ball is tossed at an angle of α from the vertical. Show that $\tan\alpha = \dfrac{v_{0x}}{v_{0y}}$. Combining this result with exercises 16 and 42, show that $w = 4h\tan\alpha$, where h is the maximum height of the toss.

44. Find a linear approximation for $\tan^{-1} x$ at $x = 0$. Use this approximation and exercise 43 to show that $\alpha \approx \dfrac{w}{4h}$. If an angle of α produces a distance of w and an angle of $\alpha + \Delta\alpha$ produces a distance of $w + \Delta w$, show that $\Delta\alpha \approx \dfrac{\Delta w}{4h}$.

45. Suppose that Δw is the difference between the ideal horizontal distance for a toss and the actual horizontal distance of a toss. For the average juggler, an error of $\Delta w = 1$ foot is manageable. Let $\Delta\alpha$ be the corresponding error in the angle of toss. If h is the height needed to juggle 10 balls (see exercise 41), find the maximum error in tossing angle.

46. Repeat exercise 45 using the height needed to juggle 11 balls. How much more accurate does the juggler need to be to juggle 11 balls?

47. In a typical golf tee shot, the ball is launched at an angle of 9.3° at an initial speed of 220 ft/s. In the absence of air resistance, how far (horizontally) would this shot carry? The actual carry on such a shot is about 240 yards (720 feet). In this case, a backspin of 4000 rpm gives the ball a huge upward Magnus force, which offsets most of the air resistance and gravity.

48. Suppose that a firefighter holds a water hose at slope m and the water exits the hose with speed v ft/s. Show that the water follows the path $y = -16\left(\frac{1+m^2}{v^2}\right)x^2 - mx$. If the firefighter stands 20 feet from a wall, what is the maximum height that the water can reach?

49. Astronaut Alan Shepard modified some of his lunar equipment and became the only person to hit a golf ball on the Moon. Assume that the ball was hit with speed 60 ft/s at an angle of 25° above the horizontal. Assuming no air resistance, find the distance the ball would have traveled on Earth. Then find how far it would travel on the moon, where there really is no air resistance (use $g = 5.2$ ft/s^2).

50. The gravitational force of the Moon is about one-sixth that of Earth. A simple guess might be that a golf ball would travel six times as high and six times as far on the Moon compared to on Earth. Determine whether this is correct.

51. Sprinters have been timed at speeds up to 28 mph. If this could be converted to a vertical velocity, how high would the person jump on Earth? Ignore air resistance.

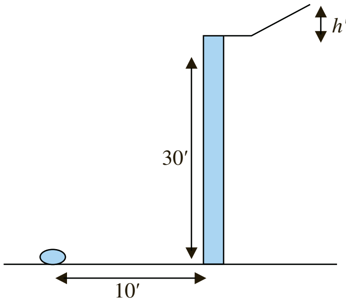
52. One version of Murphy's Law states that a piece of bread with butter on it will always fall off the table and land butter-side down. This is actually more a result of physics than of bad luck. An object knocked off of a table will typically rotate with an angular velocity ω rad/s. At a constant angular velocity ω, the object will rotate ωt radians in t seconds. Let $\theta = 0$ represent a flat piece of bread with butter-side up. If the bread hits the floor with $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, it has landed butter-side down. Assume that the bread falls from a height of 3 feet and with an initial tilt of $\theta = \frac{\pi}{4}$. Find the range of ω-values such that the bread falls butter-side down. For a falling piece of bread, ω is fairly small. Based on your calculation, if ω varies from $\frac{1}{10}$ revolution per second to 1 revolution per second, how likely is the bread to land butter-side down?

53. Suppose a target is dropping vertically at a horizontal distance of 20 feet from you. If you fire a paint ball directly at the target, show that you will hit it (assuming no air resistance and assuming that the paint ball reaches the target before either hits the ground).

54. An object is dropped from a height of 100 feet. Another object directly below the first is launched vertically from the ground with initial velocity 40 ft/s. Determine when and how high up the objects collide.

55. How fast is a vert skateboarder like Tony Hawk going at the bottom of a ramp? Ignoring friction and air resistance, the answer comes from conservation of energy, which states that the kinetic energy $\frac{1}{2}mv^2$ plus the potential energy mgy remains constant. Assume that the energy at the top of a trick at height H is all potential energy and the energy at the bottom of the ramp is all kinetic energy. (a) Find the speed at the bottom as a function of H. (b) Compute the speed if $H = 16$ feet. (c) Find the speed halfway down ($y = 8$). (d) If the ramp has the shape $y = x^2$ for $-4 \le x \le 4$, find the horizontal and vertical components of speed halfway at $y = 8$.

Exercise 55.



Exercise 56.

56. A science class builds a ramp to roll a bowling ball out of a window that is 30 feet above the ground. Their goal is for the ball to land on a watermelon that is 10 feet from the building. Assuming no friction or air resistance, determine how high the ramp should be to smash the watermelon.

EXPLORATORY EXERCISES

1. In the text and exercises 31 and 32, we discussed the differential equation $x''(t) = -25\sin(4\omega t + \theta_0)$ for the lateral motion of a knuckleball. Integrate and apply the initial conditions $x'(0) = 0$ and $x(0) = 0$ to derive the general equation $x(t) = \frac{25}{16\omega^2}\sin(4\omega t + \theta_0) - \left(\frac{25}{4\omega}\cos\theta_0\right)t - \frac{25}{16\omega^2}\sin\theta_0$. If you have access to three-dimensional graphics, graph $x(t, \omega)$ for $\theta_0 = 0$ with $0 \le t \le 0.68$ and $0.01 \le \omega \le 10$. (Note: Some plotters will have trouble with $\omega = 0$.) Repeat with $\theta_0 = \pi/4$, $\theta_0 = \pi/2$ and two choices of your own for θ_0. A pitcher wants the ball to move as much as possible back and forth but end up near home plate ($x = 0$). Based on these criteria, pick the combinations of θ_0 and ω that produce the four best pitches. Graph these pitches in two dimensions with $x = x(t)$ as in Figures 5.48a and 5.48b.

2. Although we have commented on some inadequacies of the gravity-only model of projectile motion, we have not presented any alternatives. Such models tend to be somewhat more mathematically complex. The model explored in this exercise takes into account air resistance in a way that is mathematically tractable but still not very realistic. Assume that the force of

air resistance is proportional to the speed and acts in the opposite direction of velocity. For a horizontal motion (no gravity), we have $a(t) = F(t)/m = -cv(t)$ for some constant c. Explain what the minus sign indicates. Since $a(t) = v'(t)$, the model is $v'(t) = -cv(t)$. Show that the function $v(t) = v_0 e^{-ct}$ satisfies the equation $v'(t) = -cv(t)$ and the initial condition $v(0) = v_0$. If an object starts at $x(0) = a$, integrate $v(t) = v_0 e^{-ct}$ to find its position at any time t. Show that the amount of time needed to reach $x = b$ (for $a < b$) is given by $T = -\frac{1}{c} \ln\left(1 - c\frac{b-a}{v_0}\right)$. For a baseball ($c = 0.15$) thrown at 125 ft/s from $a = 0$, determine how long it takes to reach $b = 60$ and compute its velocity at that point. By what percentage has the velocity decreased? In baseball, two different types of radar guns are used to measure the velocity of a pitch. One measures the speed of the ball right as it leaves the pitcher's hand. The other measures the speed of the ball partway to home plate. If the first gun registers 94 mph and the second registers 89 mph, how far from the plate does the second gun make its measurement?

3. The goal in the old computer game called "Gorillas" is to enter a speed and angle to launch an explosive banana to try to hit a gorilla at some other location. Suppose that you are located at the origin and your opponent at (40, 20). (a) Find two speed/angle combinations that will hit the gorilla. (b) Estimate the smallest speed that can be used to hit the target. (c) Repeat parts (a) and (b) if there is a building in the way that occupies $20 \le x \le 30$ and $0 \le y \le 30$.

5.6 APPLICATIONS OF INTEGRATION TO PHYSICS AND ENGINEERING

In this section, we explore several applications of integration in physics. In each case, we will define a basic concept to help solve a specific problem. We'll then use the definite integral to generalize the concept to solve a much wider range of problems. This use of integration is an excellent example of how the mathematical theory helps you find solutions of practical problems.

Imagine that you are at the bottom of a snow-covered hill with a sled. To get a good ride, you want to push the sled as far up the hill as you can. A physicist would say that the higher up you are, the more *potential energy* you have. Sliding down the hill converts the potential energy into *kinetic energy*. (This is the fun part!) But pushing the sled up the hill requires you to do some work: you must exert a force over a long distance.

Our first task is to quantify work. If you push a sled up a hill, you're doing work, but can you give a measure of how much? Certainly, if you push twice the weight (i.e., exert twice the force), you're doing twice the work. Further, if you push the sled twice as far, you've done twice the work. In view of these observations, for any constant force F applied over a distance d, we define the **work** W done as

$$W = Fd.$$

Unfortunately, forces are generally not constant. We extend this notion of work to the case of a nonconstant force $F(x)$ applied on the interval $[a, b]$ as follows. First, we divide the interval $[a, b]$ into n equal subintervals, each of width $\Delta x = \frac{b-a}{n}$ and consider the work done on each subinterval. If Δx is small, then the force $F(x)$ applied on the subinterval $[x_{i-1}, x_i]$ can be approximated by the constant force $F(c_i)$ for some point $c_i \in [x_{i-1}, x_i]$. The work done moving the object along the subinterval is then approximately $F(c_i)\,\Delta x$.

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The total work done W is then approximately

$$W \approx \sum_{i=1}^{n} F(c_i)\,\Delta x.$$

You should recognize this as a Riemann sum, which, as n gets larger, approaches the actual work,

Work

$$\boxed{W = \lim_{n\to\infty} \sum_{i=1}^{n} F(c_i)\,\Delta x = \int_a^b F(x)\,dx.} \tag{6.1}$$

We take (6.1) as our definition of work.

You've probably noticed that the farther a spring is compressed (or stretched) from its natural resting position, the more force is required to further compress (or stretch) the spring. According to **Hooke's Law,** the force required to maintain a spring in a given position is proportional to the distance it's compressed (or stretched). That is, if x is the distance a spring is compressed (or stretched) from its natural length, the force $F(x)$ exerted by the spring is given by

$$F(x) = kx, \tag{6.2}$$

for some constant k (the **spring constant**).

EXAMPLE 6.1 Computing the Work Done Stretching a Spring

A force of 3 pounds stretches a spring $\frac{1}{4}$ foot from its natural length (see Figure 5.49). Find the work done in stretching the spring 6 inches beyond its natural length.

Solution First, we determine the value of the spring constant. From Hooke's Law (6.2), we have that

$$3 = F\left(\frac{1}{4}\right) = k\left(\frac{1}{4}\right),$$

so that $k = 12$ and $F(x) = 12x$. From (6.1), the work done in stretching the spring 6 inches (1/2 foot) is then

$$W = \int_0^{1/2} F(x)\,dx = \int_0^{1/2} 12x\,dx = \frac{3}{2} \text{ foot-pounds.}$$

In this case, notice that stretching the spring transfers potential energy to the spring. (If the spring is later released, it springs back toward its resting position, converting the potential energy to kinetic energy.) ■

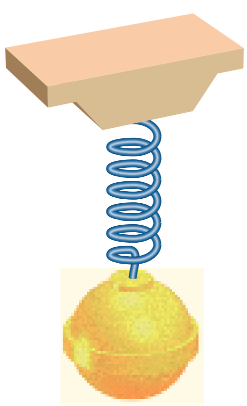


FIGURE 5.49
Stretched spring

EXAMPLE 6.2 Computing the Work Done by a Weightlifter

A weightlifter lifts a 200-pound barbell a distance of 3 feet. How much work was done? Also, determine the work done by the weightlifter if the weight is raised 4 feet above the ground and then lowered back into place.

Solution Since the force (the weight) is constant here, we simply have

$$W = Fd = 200 \times 3 = 600 \text{ foot-pounds.}$$

On the other hand, find the amount of work done if the weightlifter lifts the same weight 4 feet from the ground and then lowers it back into place. It may seem strange, but since the barbell ends up in the same place as it started, the net distance covered is zero and the work done is zero. Of course, it would feel like work to the weightlifter, but this is where the mathematical notion of work differs from the usual use of the word. As we

have defined it, work accounts for the energy change in the object. Since the barbell has the same kinetic and potential energy that it started with, the total work done on it is zero. ■

In example 6.3, both the force and the distance are nonconstant. This presents some unique challenges and we'll need to first approximate the work and then recognize the definite integral that this approximation process generates.

EXAMPLE 6.3 Computing the Work Required to Pump the Water Out of a Tank

A spherical tank of radius 10 feet is filled with water. Find the work done in pumping all of the water out through the top of the tank.

Solution The basic formula $W = Fd$ does not directly apply here, for several reasons. The most obvious of these is that the distance traveled by the water in each part of the tank is different, as the water toward the bottom of the tank must be pumped all the way to the top, while the water near the top of the tank must be pumped only a short distance. Let x represent distance as measured from the bottom of the tank, as in Figure 5.50a. The entire tank corresponds to the interval $0 \le x \le 20$, which we partition into

$$0 = x_0 < x_1 < \cdots < x_n = 20,$$

where $x_i - x_{i-1} = \Delta x = \dfrac{20}{n}$, for each $i = 1, 2, \ldots, n$. This partitions the tank into n thin layers, each corresponding to an interval $[x_{i-1}, x_i]$ (see Figure 5.50b). You can think of the water in the layer corresponding to $[x_{i-1}, x_i]$ as being approximately cylindrical, of height Δx. This layer must be pumped a distance of approximately $20 - c_i$, for some $c_i \in [x_{i-1}, x_i]$. Notice from Figure 5.50b that the radius of the ith layer depends on the value of x. From Figure 5.50c (where we show a cross section of the tank), the radius r_i corresponding to a depth of $x = c_i$ is the base of a right triangle with hypotenuse 10 and height $|10 - c_i|$. From the Pythagorean Theorem, we now have

$$(10 - c_i)^2 + r_i^2 = 10^2.$$

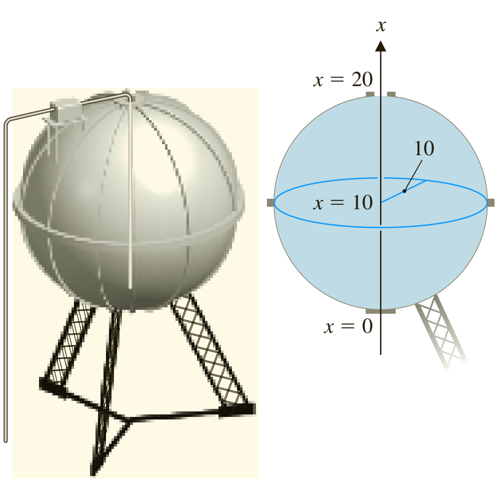


FIGURE 5.50a
Spherical tank

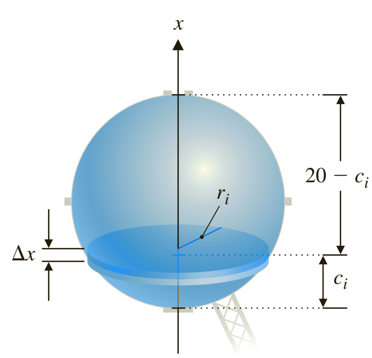


FIGURE 5.50b
The ith slice of water

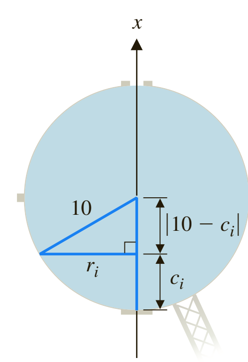


FIGURE 5.50c
Cross section of tank

Solving this for r_i^2, we have

$$\begin{aligned} r_i^2 &= 10^2 - (10 - c_i)^2 = 100 - \left(100 - 20c_i + c_i^2\right) \\ &= 20c_i - c_i^2. \end{aligned}$$

The force F_i required to move the ith layer is then simply the force exerted on the water by gravity (i.e., its weight). For this we will need to know the weight density of water: 62.4 lb/ft^3. We now have

$$\begin{aligned} F_i &\approx \text{(Volume of cylindrical slice) (weight of water per unit volume)} \\ &= \left(\pi r_i^2 h\right)(62.4\ \text{lb/ft}^3) \\ &= 62.4\pi\left(20c_i - c_i^2\right)\Delta x. \end{aligned}$$

The work required to pump out the ith slice is then given approximately by

$$\begin{aligned} W_i &\approx \text{(Force) (distance)} \\ &= 62.4\pi\left(20c_i - c_i^2\right)\Delta x(20 - c_i) \\ &= 62.4\pi c_i(20 - c_i)^2\Delta x. \end{aligned}$$

The work required to pump out all of the water is then the sum of the work required for each of the n slices:

$$W \approx \sum_{i=1}^{n} 62.4\pi c_i(20 - c_i)^2\Delta x.$$

Finally, taking the limit as $n \to \infty$ gives the exact work, which you should recognize as a definite integral:

$$\begin{aligned} W &= \lim_{n\to\infty}\sum_{i=1}^{n} 62.4\pi c_i(20 - c_i)^2\Delta x = \int_0^{20} 62.4\pi x(20 - x)^2\,dx \\ &= 62.4\pi\int_0^{20}(400x - 40x^2 + x^3)\,dx \\ &= 62.4\pi\left[400\frac{x^2}{2} - 40\frac{x^3}{3} + \frac{x^4}{4}\right]_0^{20} \\ &= 62.4\pi\left(\frac{40{,}000}{3}\right) \approx 2.61 \times 10^6 \text{ foot-pounds.} \end{aligned}$$

Impulse is a physical quantity closely related to work. Instead of relating force and distance to account for changes in energy, impulse relates force and time to account for changes in velocity. First, suppose that a constant force F is applied to an object from time $t = 0$ to time $t = T$. If the position of the object at time t is given by $x(t)$, then Newton's second law says that $F = ma = mx''(t)$. Integrating this equation once with respect to t gives us

$$\int_0^T F\,dt = m\int_0^T x''(t)\,dt,$$

or

$$F(T - 0) = m[x'(T) - x'(0)].$$

Recall that $x'(t)$ is the velocity $v(t)$, so that

$$FT = m[v(T) - v(0)]$$

or $FT = m\Delta v$, where $\Delta v = v(T) - v(0)$ is the change in velocity. The quantity FT is called the **impulse,** $mv(t)$ is the **momentum** at time t and the equation relating the impulse to the change in velocity is called the **impulse-momentum equation.**

Since we defined the impulse for a constant force, we must generalize the notion to the case of a nonconstant force. Think about this and try to guess what the definition should be.

We define the **impulse** J of a force $F(t)$ applied over the time interval $[a, b]$ to be

Impulse

$$\boxed{J = \int_a^b F(t)\,dt.}$$

We leave the derivation of the impulse integral for the case of a nonconstant force as an exercise. The impulse-momentum equation also generalizes to the case of a nonconstant force:

Impulse-momentum equation

$$\boxed{J = m[v(b) - v(a)].}$$

EXAMPLE 6.4 Estimating the Impulse for a Baseball

Suppose that a baseball traveling at 130 ft/s (about 90 mph) collides with a bat. The following data (adapted from *The Physics of Baseball* by Robert Adair) shows the force exerted by the bat on the ball at 0.0001-second intervals.

t (s)	0	0.0001	0.0002	0.0003	0.0004	0.0005	0.0006	0.0007
$F(t)$ (lb)	0	1250	4250	7500	9000	5500	1250	0

Estimate the impulse of the bat on the ball and (using $m = 0.01$ slug) the speed of the ball after impact.

Solution In this case, the impulse J is given by $\int_0^{0.0007} F(t)\,dt$. Since we're given only a fixed number of measurements of $F(t)$, the best we can do is approximate the integral numerically (e.g., using Simpson's Rule). Recall that Simpson's Rule requires an even number n of subintervals, which means that you need an odd number $n + 1$ of points in the partition. Using $n = 8$ and adding a 0 function value at $t = 0.0008$ (why is it fair to do this?), Simpson's Rule gives us

$$\begin{aligned} J &\approx [0 + 4(1250) + 2(4250) + 4(7500) + 2(9000) + 4(5500) \\ &\quad + 2(1250) + 4(0) + 0]\frac{0.0001}{3} \\ &\approx 2.867. \end{aligned}$$

In this case, the impulse-momentum equation $J = m\,\Delta v$ becomes $2.867 = 0.01\,\Delta v$ or $\Delta v = 286.7$ ft/s. Since the ball started out with a speed of 130 ft/s in one direction and it ended up moving in the opposite direction, the speed after impact is 156.7 ft/s. ■

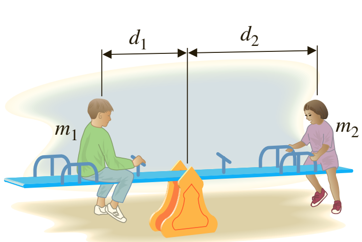


FIGURE 5.51a
Balancing two masses

The concept of the *first moment,* like work, involves force and distance. Moments are used to solve problems of balance and rotation. Consider two children on a playground seesaw (or teeter-totter). Suppose that the child on the left in Figure 5.51a is heavier (i.e., has larger mass) than the child on the right. If the children sit an equal distance from the pivot point, you know what will happen: the left side will be pulled down. However, the

children can balance each other if the heavier child moves closer to the pivot point. That is, the balance is determined both by weight (force) and distance from the pivot point. If the children have masses m_1 and m_2 and are sitting at distances d_1 and d_2, respectively, from the pivot point, then they balance each other if and only if

$$m_1 d_1 = m_2 d_2. \tag{6.3}$$

Let's turn the problem around slightly. Suppose there are two objects, of mass m_1 and m_2, located at x_1 and x_2, respectively, with $x_1 < x_2$. We consider the objects to be **point-masses.** That is, each is treated as a single point, with all of the mass concentrated at that point (see Figure 5.51b).

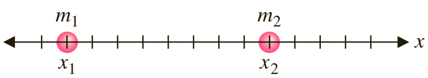


FIGURE 5.51b
Two point-masses

Suppose that you want to find the **center of mass** $\bar{x}$, that is, the location at which you could place the pivot of a seesaw and have the objects balance. From the balance equation (6.3), you'll need $m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$. Solving this equation for $\bar{x}$ gives us

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

Notice that the denominator in this equation is the total mass of the "system" (i.e., the total mass of the two objects). The numerator of this expression is called the **first moment** of the system.

More generally, for a system of n masses $m_1, m_2, \ldots, m_n$, located at $x = x_1$, $x_2, \ldots, x_n$, respectively, the center of mass $\bar{x}$ is given by the first moment divided by the total mass, that is,

Center of mass

$$\boxed{\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}.}$$

Now, suppose that we wish to find the mass and center of mass of an object of variable density that extends from $x = a$ to $x = b$. Here, we assume that the density function $\rho(x)$ (measured in units of mass per unit length) is known. Note that if the density is a constant ρ, the mass of the object is simply given by $m = \rho L$, where $L = b - a$ is the length of the object. On the other hand, if the density varies throughout the object, we can approximate the mass by dividing the interval $[a, b]$ into n pieces of equal width $\Delta x = \dfrac{b - a}{n}$. On each subinterval $[x_{i-1}, x_i]$, the mass is approximately $\rho(c_i)\Delta x$, where c_i is a point in the subinterval. The total mass is then approximately

$$m \approx \sum_{i=1}^{n} \rho(c_i)\, \Delta x.$$

You should recognize this as a Riemann sum, which approaches the total mass as $n \to \infty$,

Mass

$$\boxed{m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho(c_i)\, \Delta x = \int_a^b \rho(x)\, dx.} \tag{6.4}$$

EXAMPLE 6.5 Computing the Mass of a Baseball Bat

A 30-inch baseball bat can be represented approximately by an object extending from $x = 0$ to $x = 30$ inches, with density $\rho(x) = \left(\frac{1}{46} + \frac{x}{690}\right)^2$ slugs per inch. The density takes into account the fact that a baseball bat is similar to an elongated cone. Find the mass of the object.

Solution From (6.4), the mass is given by

$$\begin{aligned} m &= \int_0^{30} \left(\frac{1}{46} + \frac{x}{690}\right)^2 dx \\ &= \frac{690}{3}\left(\frac{1}{46} + \frac{x}{690}\right)^3 \Bigg|_0^{30} = \frac{690}{3}\left[\left(\frac{1}{46} + \frac{30}{690}\right)^3 - \left(\frac{1}{46}\right)^3\right] \\ &\approx 6.144 \times 10^{-2} \text{ slug.} \end{aligned}$$

To compute the weight (in ounces), multiply the mass by $32 \cdot 16$. The bat weighs roughly 31.5 ounces. ■

To compute the first moment for an object of nonconstant density $\rho(x)$ extending from $x = a$ to $x = b$, we again divide the interval into n equal pieces. From our earlier argument, for each $i = 1, 2, \ldots, n$, the mass of the ith slice of the object is approximately $\rho(c_i)\,\Delta x$, for any choice of $c_i \in [x_{i-1}, x_i]$. We then represent the ith slice of the object with a particle of mass $m_i = \rho(c_i)\,\Delta x$ located at $x = c_i$. We can now think of the original object as having been approximated by n distinct point-masses, as indicated in Figure 5.52.

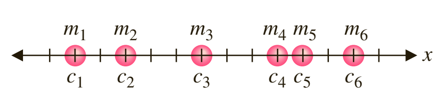


FIGURE 5.52
Six point-masses

Notice that the first moment M_n of this approximate system is

$$\begin{aligned} M_n &= [\rho(c_1)\,\Delta x]c_1 + [\rho(c_2)\,\Delta x]c_2 + \cdots + [\rho(c_n)\,\Delta x]c_n \\ &= [c_1\rho(c_1) + c_2\rho(c_2) + \cdots + c_n\rho(c_n)]\,\Delta x = \sum_{i=1}^{n} c_i\rho(c_i)\Delta x. \end{aligned}$$

Taking the limit as $n \to \infty$, the sum approaches the first moment

First moment

$$\boxed{M = \lim_{n\to\infty} \sum_{i=1}^{n} c_i\rho(c_i)\,\Delta x = \int_a^b x\rho(x)\,dx.} \qquad (6.5)$$

The center of mass of the object is then given by

Center of mass

$$\boxed{\bar{x} = \frac{M}{m} = \frac{\displaystyle\int_a^b x\rho(x)\,dx}{\displaystyle\int_a^b \rho(x)\,dx}.} \qquad (6.6)$$

EXAMPLE 6.6 Finding the Center of Mass (Sweet Spot) of a Baseball Bat

Find the center of mass of the baseball bat from example 6.5.

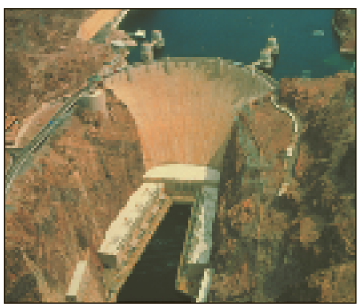
Solution From (6.5), the first moment is given by

$$M = \int_0^{30} x\left(\frac{1}{46} + \frac{x}{690}\right)^2 dx = \left[\frac{x^2}{4232} + \frac{x^3}{47{,}610} + \frac{x^4}{1{,}904{,}400}\right]_0^{30} \approx 1.205.$$

Recall that we had already found the mass to be $m \approx 6.144 \times 10^{-2}$ slug and so, from (6.6), the center of mass of the bat is

$$\bar{x} = \frac{M}{m} \approx \frac{1.205}{6.144 \times 10^{-2}} \approx 19.6 \text{ inches.}$$

Note that for a baseball bat, the center of mass is one candidate for the so-called "sweet spot" of the bat, the best place to hit the ball. ■



Hoover Dam

For our final application of integration in this section, we consider hydrostatic force. Imagine a dam holding back a lake full of water. What force must the dam withstand?

As usual, we solve a simpler problem first. If you have a flat rectangular plate oriented horizontally underwater, notice that the force exerted on the plate by the water (the **hydrostatic force**) is simply the weight of the water lying above the plate. This is the product of the volume of the water lying above the plate and the weight density of water (62.4 lb/ft^3). If the area of the plate is A ft^2 and it lies d ft below the surface (see Figure 5.53), then the force on the plate is

$$F = 62.4Ad.$$

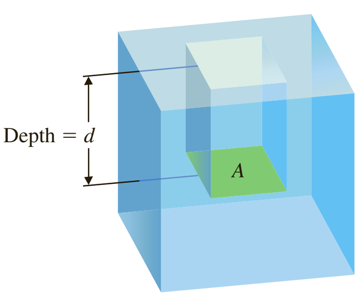


FIGURE 5.53
A plate of area A submerged to depth d

According to Pascal's Principle, the pressure at a given depth d in a fluid is the same in all directions. This says that if a flat plate is submerged in a fluid, then the pressure on one side of the plate at any given point is $\rho \cdot d$, where ρ is the weight density of the fluid and d is the depth. In particular, this says that it's irrelevant whether the plate is submerged vertically, horizontally or otherwise (see Figure 5.54).

Consider now a vertically oriented wall (a dam) holding back a lake. It is convenient to orient the x-axis vertically with $x = 0$ located at the surface of the water and the bottom of the wall at $x = a > 0$ (see Figure 5.55). In this way, x measures the depth of a section of the dam. Suppose $w(x)$ is the width of the wall at depth x (where all distances are measured in feet).

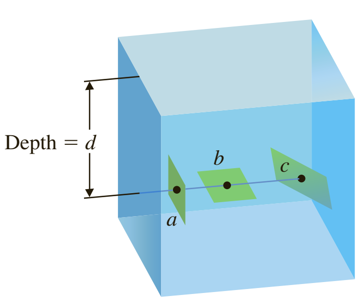


FIGURE 5.54
Pressure at a given depth is the same, regardless of the orientation

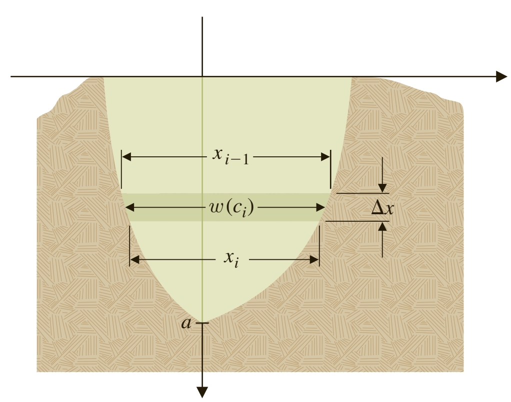


FIGURE 5.55
Force acting on a dam

Partition the interval $[0, a]$ into n subintervals of equal width $\Delta x = \frac{a}{n}$. This has the effect of slicing the dam into n slices, each of width Δx. For each $i = 1, 2, \ldots, n$, observe that the area of the ith slice is approximately $w(c_i)\,\Delta x$, where c_i is some point in the subinterval $[x_{i-1}, x_i]$. Further, the depth at every point on this slice is approximately c_i. We can then approximate the force F_i acting on this slice of the dam by the weight of the water lying above a plate the size of this portion but which is oriented horizontally:

$$F_i \approx \underbrace{62.4}_{\text{weight density}} \underbrace{w(c_i)}_{\text{length}} \underbrace{\Delta x}_{\text{width}} \underbrace{c_i}_{\text{depth}} = 62.4\, c_i\, w(c_i)\,\Delta x.$$

Adding together the forces acting on each slice, the total force F on the dam is approximately

$$F \approx \sum_{i=1}^{n} 62.4 c_i\, w(c_i)\,\Delta x.$$

Recognizing this as a Riemann sum and taking the limit as $n \to \infty$, we obtain the total hydrostatic force on the dam,

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 62.4 c_i\, w(c_i) \Delta x = \int_0^a 62.4 x\, w(x)\, dx. \tag{6.7}$$

EXAMPLE 6.7 Finding the Hydrostatic Force on a Dam

A dam is shaped like a trapezoid with height 60 ft. The width at the top is 100 ft and the width at the bottom is 40 ft (see Figure 5.56). Find the maximum hydrostatic force that the dam will need to withstand. Find the hydrostatic force if a drought lowers the water level by 10 ft.

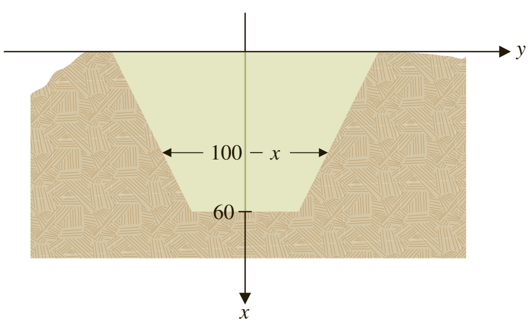


FIGURE 5.56
Trapezoidal dam

Solution Notice that the width function is a linear function of depth with $w(0) = 100$ and $w(60) = 40$. The slope is then $\frac{60}{-60} = -1$ and so, $w(x) = 100 - x$. From (6.7), the hydrostatic force is then

$$F = \int_0^{60} \underbrace{62.4}_{\text{weight density}} \underbrace{x}_{\text{depth}} \underbrace{(100 - x)}_{\text{width}}\, dx$$

$$= 3120x^2 - 62.4\frac{x^3}{3}\bigg|_0^{60} = 6{,}739{,}200 \text{ lb.}$$

If the water level dropped 10 ft, the width of the dam at the water level would be 90 ft. Lowering the origin by 10 ft, the new width function satisfies $w(0) = 90$ and $w(50) = 40$. The slope is still -1 and so, the width is given by $w(x) = 90 - x$. From (6.7), the hydrostatic force is now

$$F = \int_0^{50} \underbrace{62.4}_{\text{weight density}} \underbrace{x}_{\text{depth}} \underbrace{(90-x)}_{\text{width}}\, dx$$

$$= 2808x^2 - 62.4\frac{x^3}{3}\bigg|_0^{50} = 4{,}420{,}000 \text{ lb.}$$

Notice that this represents a reduction in force of over 34%. ■

EXERCISES 5.6

WRITING EXERCISES

1. For each of work, impulse and the first moment: identify the quantities in the definition (e.g., force and distance) and the calculations for which it is used (e.g., change in velocity).

2. The center of mass is not always the location at which half the mass is on one side and half the mass is on the other side. Give an example where more than half the mass is on one side (see examples 6.5 and 6.6) and explain why the object balances at the center of mass.

3. People who play catch have a seemingly instinctive method of pulling their hand back as they catch the ball. To catch a ball, you must apply an impulse equal to the mass times velocity of the ball. By pulling your hand back, you increase the amount of time in which you decelerate the ball. Use the impulse-momentum equation to explain why this reduces the average force on your hand.

4. A tennis ball comes toward you at 100 mph. After you hit the ball, it is moving away from you at 100 mph. Work measures changes in energy. Explain why work has been done by the tennis racket on the ball even though the ball has the same speed before and after the hit.

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1. A force of 5 pounds stretches a spring 4 inches. Find the work done in stretching this spring 6 inches beyond its natural length.

2. A force of 10 pounds stretches a spring 2 inches. Find the work done in stretching this spring 3 inches beyond its natural length.

3. A weightlifter lifts 250 pounds a distance of 20 inches. Find the work done (as measured in foot-pounds).

4. A wrestler lifts his 300-pound opponent overhead, a height of 6 feet. Find the work done (as measured in foot-pounds).

5. A rocket full of fuel weighs 10,000 pounds at launch. After launch, the rocket gains altitude and loses weight as the fuel burns. Assume that the rocket loses 1 pound of fuel for every 15 feet of altitude gained. Explain why the work done raising the rocket to an altitude of 30,000 feet is $\int_0^{30{,}000} (10{,}000 - x/15)\,dx$ and compute the integral.

6. Referring to exercise 5, suppose that a rocket weighs 8000 pounds at launch and loses 1 pound of fuel for every 10 feet of altitude gained. Find the work needed to raise the rocket to a height of 10,000 feet.

7. Suppose that a car engine exerts a force of $800x(1-x)$ pounds when the car is at position x miles, $0 \le x \le 1$. Compute the work done.

8. Horsepower measures the rate of work done as a function of time. For the situation in exercise 7, explain why this is not equal to $800x(1-x)$. If the car in exercise 7 takes 80 seconds to travel the mile, compute the average horsepower (1 hp = 550 ft-lb/s).

9. A water tower is spherical in shape with radius 50 feet, extending from 200 feet to 300 feet above ground. Compute the work done in filling the tank from the ground.

10. Compute the work done in filling the tank of exercise 9 halfway.

11. Compute the work done in pumping half of the water out of the top of the tank in example 6.3.

12. A water tank is in the shape of a right circular cone of altitude 10 feet and base radius 5 feet, with its vertex at the ground. (Think of an ice cream cone with its point facing down.) If the tank is full, find the work done in pumping all of the water out the top of the tank.

13. Two laborers share the job of digging a rectangular hole 10 feet deep. The dirt from the hole is cleared away by other laborers. Assuming a constant density of dirt, how deep should the first worker dig to do half the work? Explain why 5 feet is not the answer.

14. A trough is to be dug 6 feet deep. Cross sections have the shape  and are 2 feet wide at the bottom and 5 feet wide at the top. Find the depth at which half the work has been done.

15. In example 6.4, suppose that the baseball was traveling at 100 ft/s. The force exerted by the bat on the ball would change to the values in the table. Estimate the impulse and the speed of the ball after impact.

t (s)	0	0.0001	0.0002	0.0003	0.0004
F (lb)	0	1000	2100	4000	5000

t (s)	0.0005	0.0006	0.0007	0.0008
F (lb)	5200	2500	1000	0

16. In exercise 15, suppose that the baseball was traveling at 85 ft/s. The force exerted by the bat on the ball would change to the values in the table. Estimate the impulse and the speed of the ball after impact.

t (s)	0	0.0001	0.0002	0.0003	0.0004
F (lb)	0	600	1200	2000	2500

t (s)	0.0005	0.0006	0.0007	0.0008
F (lb)	3000	2500	1100	300

17. A crash test is performed on a vehicle. The force of the wall on the front bumper is shown in the table. Estimate the impulse and the speed of the vehicle (use $m = 200$ slugs).

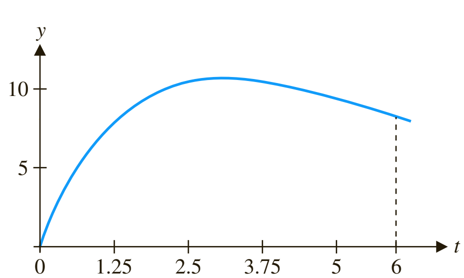
t (s)	0	0.1	0.2	0.3	0.4	0.5	0.6
F (lb)	0	8000	16,000	24,000	15,000	9000	0

18. Two football players collide. The force of the defensive player on the offensive player is given in the table. Estimate the impulse. If the offensive player has mass $m = 7$ and velocity of 29 ft/s before the collision, does the defensive player stop the offensive player?

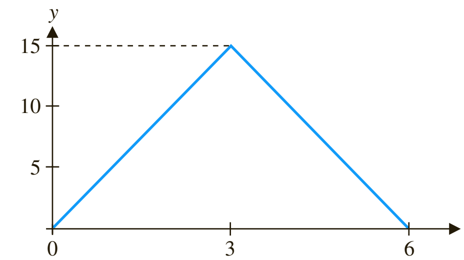
t (s)	0	0.1	0.2	0.3	0.4	0.5
F (lb)	0	300	500	400	250	150

t (s)	0.6	0.7	0.8	0.9	1.0
F (lb)	100	100	80	40	0

19. A **thrust-time curve** $f(t) = 10te^{-t/3}$ for a model rocket is shown. Compute the maximum thrust. Compute the impulse.



20. A thrust-time curve for a model rocket is shown. Compute the impulse. Based on your answers to exercises 19 and 20, which rocket would reach a higher altitude?



21. Compute the mass and center of mass of an object with density $\rho(x) = \dfrac{x}{6} + 2$ kg/m, $0 \le x \le 6$. Briefly explain in terms of the density function why the center of mass is not at $x = 3$.

22. Compute the mass and center of mass of an object with density $\rho(x) = 3 - \dfrac{x}{6}$ kg/m, $0 \le x \le 6$. Briefly explain in terms of the density function why the center of mass is not at $x = 3$.

23. Compute the weight in ounces of an object extending from $x = -3$ to $x = 27$ with density $\rho(x) = \left(\dfrac{1}{46} + \dfrac{x+3}{690}\right)^2$ slugs/in.

24. Compute the weight in ounces of an object extending from $x = 0$ to $x = 32$ with density $\rho(x) = \left(\dfrac{1}{46} + \dfrac{x+3}{690}\right)^2$ slugs/in.

25. Compute the center of mass of the object in exercise 23. This object models the baseball bat of example 6.5 "choked up" (held 3 inches up the handle). Compare the masses and centers of mass of the two bats.

26. Compute the center of mass of the object in exercise 24. This object models a baseball bat that is 2 inches longer than the bat of example 6.5. Compare the masses and centers of mass of the two bats.

27. Compute the mass and weight in ounces and center of mass of an object extending from $x = 0$ to $x = 30$ with density $\rho(x) = 0.00468\left(\dfrac{3}{16} + \dfrac{x}{60}\right)$ slugs/in.

28. The object in exercise 27 models an aluminum baseball bat (hollow and $\frac{1}{4}$ inch thick). Compare the mass and center of mass to the wooden bat of example 6.5. Baseball experts claim that it is easier to hit an inside pitch (small x value) with an aluminum bat. Explain why your calculations indicate that this is true.

29. The accompanying figure shows the outline of a model rocket. Assume that the vertical scale is 3 units high and the horizontal scale is 6 units wide. Use basic geometry to compute the area of each of the three regions of the rocket outline. Assuming a

constant density ρ, locate the x-coordinate of the center of mass of each region. (Hint: The first region can be thought of as extending from $x = 0$ to $x = 1$ with density $\rho(3 - 2x)$. The third region extends from $x = 5$ to $x = 6$ with density $\rho(6 - x)$.)

30. For the model rocket in exercise 29, replace the rocket with 3 particles, one for each region. Assume that the mass of each particle equals the area of the region and the location of the particle on the x-axis equals the center of mass of the region. Find the center of mass of the 3-particle system. [Rockets are designed with bottom fins large enough that the center of mass is shifted near the bottom (or, in the figure here, left) of the rocket. This improves the flight stability of the rocket.]

In exercises 31–34, find the centroid of each region. The centroid is the center of mass of a region with constant density. (Hint: Modify (6.6) to find the y-coordinate $\bar{y}$.)

31. The triangle with vertices (0, 0), (4, 0) and (4, 6)

32. The rhombus with vertices (0, 0), (3, 4), (8, 4) and (5, 0)

33. The region bounded by $y = 4 - x^2$ and $y = 0$

34. The region bounded by $y = x$, $y = -x$ and $x = 4$

35. A dam is in the shape of a trapezoid with height 60 feet. The width at the top is 40 feet and the width at the bottom is 100 feet. Find the maximum hydrostatic force the wall would need to withstand. Explain why the force is so much greater than the force in example 6.7.

36. Find the maximum hydrostatic force in exercise 35 if a drought lowers the water level by 10 feet.

37. An underwater viewing window is installed at an aquarium. The window is circular with radius 5 feet. The center of the window is 40 feet below the surface of the water. Find the hydrostatic force on the window.

38. An underwater viewing window is rectangular with width 40 feet. The window extends from the surface of the water to a depth of 10 feet. Find the hydrostatic force on the window.

39. The camera's window on a robotic submarine is circular with radius 3 inches. How much hydrostatic force would the window need to withstand to descend to a depth of 1000 feet?

40. A diver wears a watch to a depth of 60 feet. The face of the watch is circular with a radius of 1 inch. How much hydrostatic force will the face need to withstand if the watch is to keep on ticking?

41. In the text, we mentioned that work measures changes in energy. For example, a 200-pound pole vaulter is propelled by a pole to a height of 20 feet. The work done by the pole, equal to 4000 ft-lb, gives the vaulter a large potential energy. To see what this means, compute the speed v of the vaulter when he reaches the ground. Show that the kinetic energy at impact, given by $\frac{1}{2}mv^2$ ($m = 200/32$), also equals 4000 ft-lb. This illustrates the concept of conservation of energy.

42. Compute the speed of the falling vaulter in exercise 41 at the 10-foot mark. Show that the sum of the potential energy ($32mh$) and the kinetic energy ($\frac{1}{2}mv^2$) equals 4000 ft-lb.

43. Given that power is the product of force and velocity, compute the horsepower needed to lift a 100-ton object such as a blue whale at 20 mph (1 hp = 550 ft-lb/s). (Note that blue whales swim so efficiently that they can maintain this speed with an output of 60–70 hp.)

44. For a constant force F exerted over a length of time t, impulse is defined by $F \cdot t$. For a variable force $F(t)$, derive the impulse formula $J = \int_a^b F(t)\,dt$.

45. The first moment of a solid of density $\rho(x)$ is $\int_a^b x\rho(x)\,dx$. The second moment about the y-axis, defined by $\int_a^b x^2\rho(x)\,dx$, is also important in applications. The larger this number is, the more difficult it is to rotate the solid about the y-axis. Compute the second moments of the baseball bats in example 6.5 and exercise 23. Choking up on a bat makes it easier to swing (and control). Compute the percentage by which the second moment is reduced by choking up 3 inches.

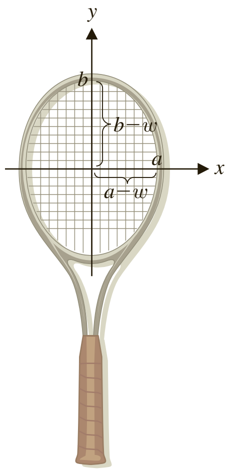
46. Occasionally, baseball players illegally "cork" their bats by drilling out a portion of wood from the end of the bats and filling the hole with a light substance such as cork. The advantage of this procedure is that the second moment is significantly reduced. To model this, take the bat of example 6.5 and change the density to

$$\rho(x) = \begin{cases} \left(\dfrac{1}{46} + \dfrac{x}{690}\right)^2 & \text{if } 0 \le x \le 28 \\ \left(\dfrac{1}{92} + \dfrac{x}{690}\right)^2 & \text{if } 28 < x \le 30, \end{cases}$$

representing a hole of radius $\frac{1}{4}''$ and length $2''$. Compute the mass and second moment of the corked bat and compare to the original bat.

47. The second moment (see exercise 45) of a disk of density ρ in the shape of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by $\int_{-a}^{a} 2\rho b x^2 \sqrt{1 - \frac{x^2}{a^2}}\,dx$. Use your CAS to evaluate this integral.

48. Use the result from exercise 47 to show that the second moment of the tennis racket head in the diagram (on the following page) is $M = \rho\frac{\pi}{4}[ba^3 - (b - w)(a - w)^3]$.



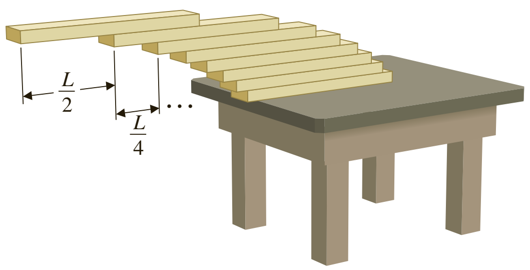
49. For tennis rackets, a large second moment (see exercises 47 and 48) means less twisting of the racket on off-center shots. Compare the second moment of a wooden racket ($a = 9$, $b = 12$, $w = 0.5$), a midsize racket ($a = 10$, $b = 13$, $w = 0.5$) and an oversized racket ($a = 11$, $b = 14$, $w = 0.5$).

50. Let M be the second moment found in exercise 48. Show that $\frac{dM}{da} > 0$ and conclude that larger rackets have larger second moments. Also, show that $\frac{dM}{dw} > 0$ and interpret this result.

EXPLORATORY EXERCISES

1. As equipment has improved, heights cleared in the pole vault have increased. A crude estimate of the maximum pole vault possible can be derived from conservation of energy principles. Assume that the maximum speed a pole-vaulter could run carrying a long pole is 25 mph. Convert this speed to ft/s. The kinetic energy of this vaulter would be $\frac{1}{2}mv^2$. (Leave m as an unknown for the time being.) This initial kinetic energy would equal the potential energy at the top of the vault minus whatever energy is absorbed by the pole (which we will ignore). Set the potential energy, $32mh$, equal to the kinetic energy and solve for h. This represents the maximum amount the vaulter's center of mass could be raised. Add 3 feet for the height of the vaulter's center of mass and you have an estimate of the maximum vault possible. Compare this to Sergei Bubka's 1994 world record vault of $20'1\frac{3}{4}''$.

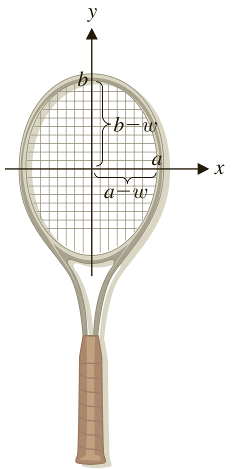
2. An object will remain on a table as long as the center of mass of the object lies over the table. For example, a board of length 1 will balance if half the board hangs over the edge of the table. Show that two homogeneous boards of length 1 will balance if $\frac{1}{4}$ of the first board hangs over the edge of the table and $\frac{1}{2}$ of the second board hangs over the edge of the first board. Show that three boards of length 1 will balance if $\frac{1}{6}$ of the first board hangs over the edge of the table, $\frac{1}{4}$ of the second board hangs over the edge of the first board and $\frac{1}{2}$ of the third board hangs over the edge of the second board. Generalize this to a procedure for balancing n boards. How many boards are needed so that the last board hangs completely over the edge of the table?



5.7 PROBABILITY

The mathematical fields of probability and statistics focus on the analysis of random processes. In this section, we give a brief introduction to the use of calculus in probability theory. It may surprise you to learn that calculus provides insight into random processes, but this is in fact, a very important application of integration.

We begin with a simple example involving coin-tossing. Suppose that you toss two coins, each of which has a 50% chance of coming up heads. Because of the randomness involved, you cannot calculate exactly how many heads you will get on a given number of tosses. But you *can* calculate the *likelihood* of each of the possible outcomes. If we denote heads by H and tails by T, then the four possible outcomes from tossing two coins are HH, HT, TH and TT. Each of these four outcomes is equally likely, so we can say that each has probability $\frac{1}{4}$. This means that, *on average,* each of these events will occur in one-fourth of your tries. Said a different way, the **relative frequency** with which each event occurs in a large number of trials will be approximately $\frac{1}{4}$.



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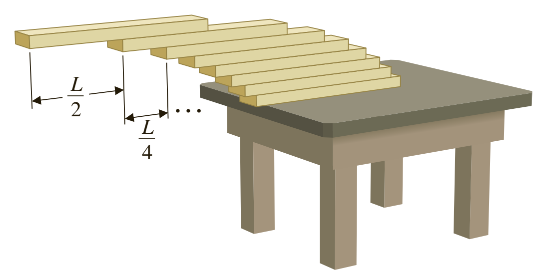

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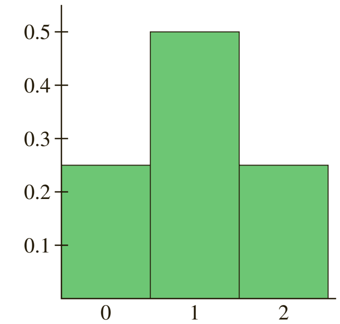


FIGURE 5.57
Histogram for two-coin toss

Suppose that we record the number of heads. Based on our calculations above, the probability of getting two heads is $\frac{1}{4}$, the probability of getting one head is $\frac{2}{4}$ (since there are two ways for this to happen: HT and TH) and the probability of getting zero heads is $\frac{1}{4}$. We often summarize such information by displaying it in a **histogram,** a bar graph where the outcomes are listed on the horizontal axis (see Figure 5.57).

Suppose that we instead toss eight coins. The probabilities for getting a given number of heads are given in the accompanying table and the corresponding histogram is shown in Figure 5.58. You should notice that the sum of all the probabilities is 1 (or 100%, since it's certain that *one* of the possible outcomes will occur on a given try). This is one of the defining properties of probability theory. Another basic property is called the **addition principle:** to compute the probability of getting 6, 7 or 8 heads (or any other mutually exclusive outcomes), simply add together the individual probabilities:

$$P(6, 7 \text{ or } 8 \text{ heads}) = \frac{28}{256} + \frac{8}{256} + \frac{1}{256} = \frac{37}{256} \approx 0.145.$$

Number of Heads	*Probability*
0	1/256
1	8/256
2	28/256
3	56/256
4	70/256
5	56/256
6	28/256
7	8/256
8	1/256

A graphical interpretation of this calculation is very revealing. In the histogram in Figure 5.58, notice that each bar is a rectangle of width 1. Then the probability associated with each bar equals the area of the rectangle. In graphical terms,

- The total area in such a histogram is 1.
- The probability of getting between 6 and 8 heads (inclusive) equals the sum of the areas of the rectangles located between 6 and 8 (inclusive).

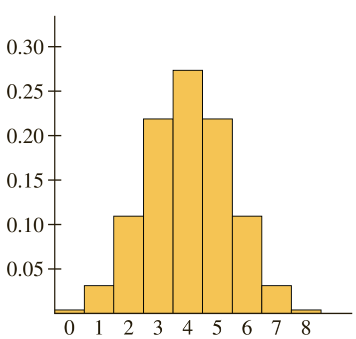


FIGURE 5.58
Histogram for eight-coin toss

Not all probability events have the nice theoretical structure of coin-tossing. For instance, suppose that we want to find the probability that a randomly chosen person will have a height of 5′9″ or 5′10″. There is no easy theory we can use here to compute the probabilities (since not all heights are equally likely). In this case, we use the correspondence between probability and relative frequency. If we collect information about the heights of a large number of adults, we might find the following.

Height	<64″	64″	65″	66″	67″	68″	69″	70″	71″	72″	73″	>73″
Number of people	23	32	61	94	133	153	155	134	96	62	31	26

Since the total number of people in the survey is 1000, the relative frequency of the height 5′9″ (69″) is $\frac{155}{1000} = 0.155$ and the relative frequency of the height 5′10″ (70″) is $\frac{134}{1000} = 0.134$. An estimate of the probability of being 5′9″ or 5′10″ is then $0.155 + 0.134 = 0.289$. A histogram is shown in Figure 5.59.

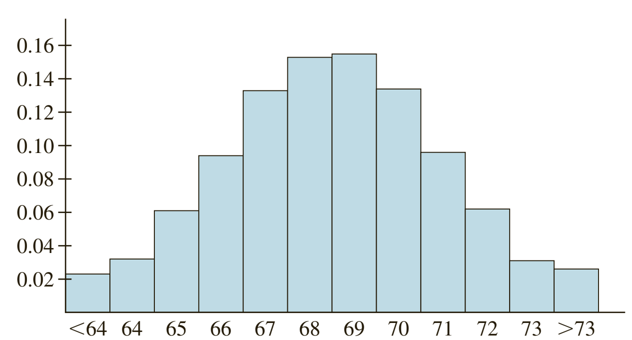


FIGURE 5.59
Histogram for relative frequency of heights

Suppose that we want to be more specific: for example, what is the probability that a randomly chosen person is $5'8\frac{1}{2}''$ or $5'9''$? To answer this question, we would need to have our data broken down further, as in the following partial table.

$66\frac{1}{2}''$	$67''$	$67\frac{1}{2}''$	$68''$	$68\frac{1}{2}''$	$69''$	$69\frac{1}{2}''$	$70''$	$70\frac{1}{2}''$	$71''$
52	61	72	71	82	81	74	69	65	58

The probability that a person is $5'9''$ can be estimated by the relative frequency of $5'9''$ people in our survey, which is $\frac{81}{1000} = 0.081$. Similarly, the probability that a person is $5'8\frac{1}{2}''$ is approximately $\frac{82}{1000} = 0.082$. The probability of being $5'8\frac{1}{2}''$ or $5'9''$ is then approximately $0.081 + 0.082 = 0.163$. A histogram for this portion of the data is shown in Figure 5.60a.

Notice that since each bar of the histogram now represents a half-inch range of height, we can no longer interpret area in the histogram as the probability. We will modify the histogram to make the area connection clearer. In Figure 5.60b, we have labeled the horizontal axis with the height in inches, while the vertical axis shows twice the relative frequency. The bar at $69''$ has height 0.162 and width $\frac{1}{2}$. Its area, $\frac{1}{2}(0.162) = 0.081$, corresponds to the relative frequency (or probability) of the height $5'9''$.

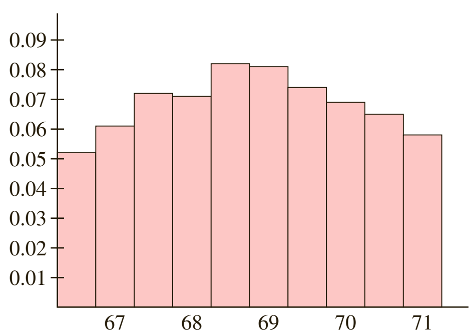


FIGURE 5.60a
Histogram for relative frequency of heights

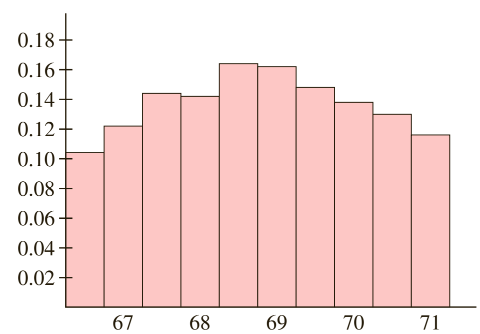


FIGURE 5.60b
Histogram showing double the relative frequency

Of course, we could continue subdividing the height intervals into smaller and smaller pieces. Think of doing this while modifying the scale on the vertical axis so that the area of each rectangle (length times width of interval) always gives the relative frequency (probability) of that height interval. For example, suppose that there are n height intervals between $5'8''$ and $5'9''$. Let x represent height in inches and $f(x)$ equal the height of the histogram bar for the interval containing x. Let $x_1 = 68 + \frac{1}{n}$, $x_2 = 68 + \frac{2}{n}$ and so on, so that $x_i = 68 + \frac{i}{n}$, for $1 \le i \le n$ and let $\Delta x = \frac{1}{n}$. For a randomly selected person, the probability that their height is between $5'8''$ and $5'9''$ is estimated by the sum of the areas of the corresponding histogram rectangles, given by

$$P(68 \le x \le 69) \approx f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x = \sum_{i=1}^{n} f(x_i)\,\Delta x. \qquad (7.1)$$

Observe that as n increases, the histogram of Figure 5.61 will "smooth out," approaching a curve like the one shown in Figure 5.62.

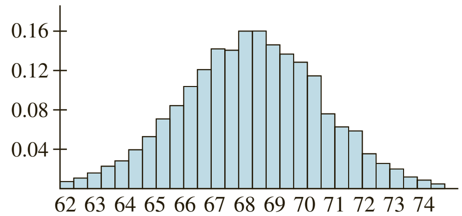


FIGURE 5.61
Histogram for heights

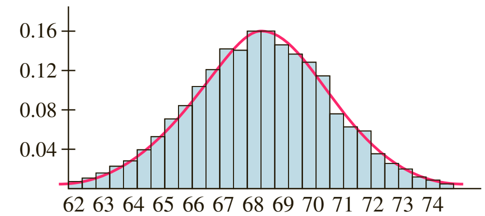


FIGURE 5.62
Probability density function and histogram for heights

We call this limiting function $f(x)$, the **probability density function (pdf)** for heights. Notice that for any given $i = 1, 2, \ldots, n$, $f(x_i)$ does not give the probability that a person's height equals x_i. Instead, for small values of Δx, the quantity $f(x_i)\,\Delta x$ is an approximation of the probability that a randomly selected height is in the range $[x_{i-1}, x_i]$.

Look carefully at (7.1) and observe that as $n \to \infty$, the Riemann sum on the right should approach an integral $\int_a^b f(x)\,dx$. Here, the limits of integration are 68 (5′8″) and 69 (5′9″). We have

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_{68}^{69} f(x)\,dx.$$

Notice that by adjusting the function values so that probability corresponds to area, we have found a familiar and direct technique for computing probabilities. We now summarize our dicussion with some definitions. The preceding examples are of **discrete probability distributions** (*discrete* since the quantity being measured can only assume values from a certain finite set). For instance, in coin-tossing, the number of heads must be an integer. By contrast, many distributions are **continuous.** That is, the quantity of interest (the random variable) assumes values from a continuous range of numbers (an interval). For instance, although height is normally rounded off to the nearest integer number of inches, a person's actual height can be any number.

For continuous distributions, the graph corresponding to a histogram is the graph of a probability density function (pdf). We now give a precise definition of a pdf.

HISTORICAL NOTES

Blaise Pascal (1623–1662) A French mathematician and physicist who teamed with Pierre Fermat to begin the systematic study of probability. Pascal is credited with numerous inventions, including a wrist watch, barometer, hydraulic press, syringe and a variety of calculating machines. He also discovered what is now known as Pascal's Principle in hydrostatics (see section 5.6). Pascal may well have become one of the founders of calculus, but poor health and large periods of time devoted to religious and philosophical contemplation reduced his mathematical output.

DEFINITION 7.1

Suppose that X is a random variable that may assume any value x with $a \le x \le b$. A probability density function for X is a function $f(x)$ satisfying

(i) $f(x) \ge 0$ for $a \le x \le b$. — Probability density functions are never negative.

and

(ii) $\int_a^b f(x)\,dx = 1.$ — The total probability is 1.

The probability that the (observed) value of X falls between c and d is given by the area under the graph of the pdf on that interval. That is,

$$P(c \le X \le d) = \int_c^d f(x)\,dx.$$ Probability corresponds to area under the curve.

To verify that a function defines a pdf for *some* (unknown) random variable, we must show that it satisfies properties (i) and (ii) of Definition 7.1.

EXAMPLE 7.1 Verifying That a Function Is a pdf on an Interval

Show that $f(x) = 3x^2$ defines a pdf on the interval [0, 1] by verifying properties (i) and (ii) of Definition 7.1.

Solution Clearly, $f(x) \geq 0$. For property (ii), we integrate the pdf over its domain. We have

$$\int_0^1 3x^2 dx = x^3 \Big|_0^1 = 1.$$

EXAMPLE 7.2 Using a pdf to Estimate Probabilities

Suppose that $f(x) = \frac{0.4}{\sqrt{2\pi}} e^{-0.08(x-68)^2}$ is the probability density function for the heights in inches of adult American males. Find the probability that a randomly selected adult American male will be between 5′8″ and 5′9″. Also, find the probability that a randomly selected adult American male will be between 6′2″ and 6′4″.

Solution To compute the probabilities, you first need to convert the specified heights into inches. The probability of being between 68 and 69 inches tall is

$$P(68 \leq X \leq 69) = \int_{68}^{69} \frac{0.4}{\sqrt{2\pi}} e^{-0.08(x-68)^2} dx \approx 0.15542.$$

Here, we approximated the value of the integral numerically. (You can use Simpson's Rule or the numerical integration method built into your calculator or CAS.) Similarly, the probability of being between 74 and 76 inches is

$$P(74 \leq X \leq 76) = \int_{74}^{76} \frac{0.4}{\sqrt{2\pi}} e^{-0.08(x-68)^2} dx \approx 0.00751,$$

where we have again approximated the value of the integral numerically.

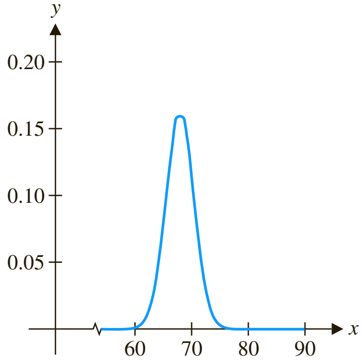


FIGURE 5.63
Heights of adult males

According to data in Gyles Brandreth's *Your Vital Statistics,* the pdf for the heights of adult males in the United States looks like the graph of $f(x) = \frac{0.4}{\sqrt{2\pi}} e^{-0.08(x-68)^2}$ shown in Figure 5.63 and used in example 7.2. You probably have seen *bell-shaped curves* like this before. This distribution is referred to as a **normal distribution.** Besides the normal distribution, there are many other probability distributions that are important in applications.

EXAMPLE 7.3 Computing Probability with an Exponential pdf

Suppose that the lifetime in years of a certain brand of lightbulb is exponentially distributed with pdf $f(x) = 4e^{-4x}$. Find the probability that a given lightbulb lasts 3 months or less.

Solution First, since the random variable measures lifetime in years, convert 3 months to $\frac{1}{4}$ year. The probability is then

$$P\left(0 \leq X \leq \frac{1}{4}\right) = \int_0^{1/4} 4e^{-4x} dx = 4\left(-\frac{1}{4}\right) e^{-4x}\Big|_0^{1/4}$$
$$= -e^{-1} + e^0 = 1 - e^{-1} \approx 0.63212.$$

In some cases, there may be theoretical reasons for assuming that a pdf has a certain form. In this event, the first task is to determine the values of any constants to achieve the properties of a pdf.

EXAMPLE 7.4 Determining the Coefficient of a pdf

Suppose that the pdf for a random variable has the form $f(x) = ce^{-3x}$ for some constant c, with $0 \le x \le 1$. Find the value of c that makes this a pdf.

Solution To be a pdf, we first need that $f(x) = ce^{-3x} \ge 0$, for all $x \in [0, 1]$. (This will be the case as long as $c \ge 0$.) Also, the integral over the domain must equal 1. So, we set

$$1 = \int_0^1 c\,e^{-3x}\,dx = c\left(-\frac{1}{3}\right)e^{-3x}\Big|_0^1 = -\frac{c}{3}e^{-3} + \frac{c}{3} = \frac{c}{3}(1 - e^{-3}).$$

It now follows that $c = \dfrac{3}{1 - e^{-3}} \approx 3.1572$. ■

TODAY IN MATHEMATICS

Persi Diaconis (1945–) An American statistician who was one of the first recipients of a lucrative MacArthur Foundation Fellowship, often called a "genius grant." Diaconis trained on the violin at Juilliard until age 14, when he left home to become a professional magician for 10 years. His varied interests find expression in his work, where he uses all areas of mathematics and statistics to solve problems from throughout science and engineering. He says, "What makes somebody a good applied mathematician is a balance between finding an interesting real-world problem and finding an interesting real-world problem which relates to beautiful mathematics."

Given a pdf, it is possible to compute various statistics to summarize the properties of the random variable. The most common statistic is the **mean,** the best-known measure of average value. If you wanted to average test scores of 85, 89, 93 and 93, you would probably compute the mean, given by $\frac{85+89+93+93}{4} = 90$.

Notice here that there were three different test scores recorded: 85, which has a relative frequency of $\frac{1}{4}$, 89, also with a relative frequency of $\frac{1}{4}$ and 93, with a relative frequency of $\frac{2}{4}$. We can also compute the mean by multiplying each value by its relative frequency and then summing: $(85)\frac{1}{4} + (89)\frac{1}{4} + (93)\frac{2}{4} = 90$.

Now, suppose we wanted to compute the mean height of the people in the following table.

Height	63″	64″	65″	66″	67″	68″	69″	70″	71″	72″	73″	74″
Number	23	32	61	94	133	153	155	134	96	62	31	26

It would be silly to write out the heights of all 1000 people, add and divide by 1000. It is much easier to multiply each height by its relative frequency and add the results. Following this route, the mean m is given by

$$\begin{aligned} m &= (63)\frac{23}{1000} + (64)\frac{32}{1000} + (65)\frac{61}{1000} + (66)\frac{94}{1000} + (67)\frac{133}{1000} + \cdots + (74)\frac{26}{1000} \\ &= 68.523. \end{aligned}$$

If we denote the heights by $x_1, x_2, \ldots, x_n$ and let $f(x_i)$ be the relative frequency or probability corresponding to $x = x_i$, the mean then has the form

$$m = x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3) + \cdots + x_{12} f(x_{12}).$$

If the heights in our data set were given for every half-inch or tenth-of-an-inch, we would compute the mean by multiplying each x_i by the corresponding probability $f(x_i)\Delta x$, where Δx is the fraction of an inch between data points. The mean now has the form

$$m = [x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3) + \cdots + x_n f(x_n)]\,\Delta x = \sum_{i=1}^{n} x_i\,f(x_i)\,\Delta x,$$

where n is the number of data points. Notice that, as n increases and Δx approaches 0, the Riemann sum approaches the integral $\int_a^b x f(x)\,dx$. This gives us the following definition.

DEFINITION 7.2

The **mean** μ of a random variable with pdf $f(x)$ on the interval $[a, b]$ is given by

$$\mu = \int_a^b x f(x)\, dx. \tag{7.2}$$

Although the mean is commonly used to report the average value of a random variable, it is important to realize that it is not the only measure of *average* used by statisticians. An alternative measurement of average is the **median,** the x-value that divides the probability in half. (That is, half of all values of the random variable lie at or below the median and half lie at or above the median.) In example 7.5 and in the exercises, you will explore situations in which each measure provides a different indication about the average of a random variable.

EXAMPLE 7.5 Finding the Mean Age and Median Age of a Group of Cells

Suppose that the age in days of a type of single-celled organism has pdf $f(x) = (\ln 2)e^{-kx}$, where $k = \frac{1}{2}\ln 2$. The domain is $0 \le x \le 2$. (The assumption here is that upon reaching an age of 2 days, each cell divides into two daughter cells.) Find (a) the mean age of the cells, (b) the proportion of cells that are younger than the mean and (c) the median age of the cells.

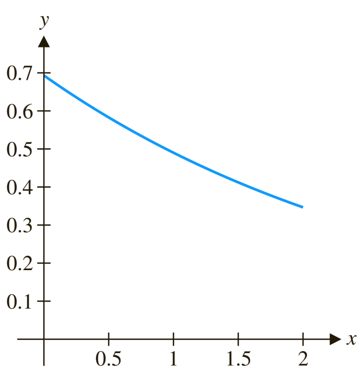


FIGURE 5.64
$y = (\ln 2)e^{-(\ln 2)x/2}$

Solution For part (a), we have from (7.2) that the mean is given by

$$\mu = \int_0^2 x(\ln 2)e^{-(\ln 2)x/2} dx \approx 0.88539 \text{ day},$$

where we have approximated the value of the integral numerically. Notice that even though the cells range in age from 0 to 2 days, the mean is not 1. The graph of the pdf in Figure 5.64 shows that younger ages are more likely than older ages, and this causes the mean to be less than 1.

For part (b), notice that the proportion of cells younger than the mean is the same as the probability that a randomly selected cell is younger than the mean. This probability is given by

$$P(0 \le X \le \mu) = \int_0^{0.88539} (\ln 2)e^{-(\ln 2)x/2} dx \approx 0.52848,$$

where we have again approximated the value of the integral numerically. Therefore, the proportion of cells younger than the mean is about 53%. Notice that in this case the mean does not represent the 50% mark for probabilities. In other words, the mean is not the same as the median.

To find the median in part (c), we must solve for the constant c such that

$$0.5 = \int_0^c (\ln 2)e^{-(\ln 2)x/2} dx.$$

Since an antiderivative of $e^{-(\ln 2)x/2}$ is $-\frac{2}{\ln 2}e^{-(\ln 2)x/2}$, we have

$$\begin{aligned}0.5 &= \int_0^c \ln 2 e^{-(\ln 2)x/2}\,dx \\ &= \ln 2\left[-\frac{2}{\ln 2}e^{-(\ln 2)x/2}\right]_0^c \\ &= -2e^{-(\ln 2)c/2} + 2.\end{aligned}$$

Subtracting 2 from both sides, we have

$$-1.5 = -2e^{-(\ln 2)c/2},$$

so that dividing by -2 yields $$0.75 = e^{-(\ln 2)c/2}.$$

Taking the natural log of both sides gives us

$$\ln 0.75 = -(\ln 2)c/2.$$

Finally, solving for c gives us $$c = \frac{-2\ln 0.75}{\ln 2},$$

so that the median is $-2\ln 0.75/\ln 2 \approx 0.83$. We can now conclude that half of the cells are younger than 0.83 day and half the cells are older than 0.83 day. ■

EXERCISES 5.7

WRITING EXERCISES

1. In the text, we stated that the probability of tossing two fair coins and getting two heads is $\frac{1}{4}$. If you try this experiment four times, explain why you will not always get two heads exactly one out of four times. If probability doesn't give precise predictions, what is its usefulness? To answer this question, discuss the information conveyed by knowing that in the above experiment the probability of getting one head and one tail is $\frac{1}{2}$ (twice as big as $\frac{1}{4}$).

2. Suppose you toss two coins numerous times (or simulate this on your calculator or computer). Theoretically, the probability of getting two heads is $\frac{1}{4}$. In the long run (as the coins are tossed more and more often), what proportion of the time should two heads occur? Try this and discuss how your results compare to the theoretical calculation.

3. Based on Figures 5.57 and 5.58, describe what you expect the histogram to look like for larger numbers of coins. Compare to Figure 5.63.

4. The height of a person is determined by numerous factors, both hereditary and environmental (e.g., diet). Explain why this might produce a histogram similar to that produced by tossing a large number of coins.

In exercises 1–6, show that the given function is a pdf on the indicated interval.

1. $f(x) = 4x^3$, $[0, 1]$

2. $f(x) = \frac{3}{8}x^2$, $[0, 2]$

3. $f(x) = x + 2x^3$, $[0, 1]$

4. $f(x) = \cos x$, $[0, \pi/2]$

5. $f(x) = \frac{1}{2}\sin x$, $[0, \pi]$

6. $f(x) = e^{-x/2}$, $[0, \ln 4]$

In exercises 7–12, find a value of c for which $f(x)$ is a pdf on the indicated interval.

7. $f(x) = cx^3$, $[0, 1]$

8. $f(x) = cx + x^2$, $[0, 1]$

9. $f(x) = ce^{-4x}$, $[0, 1]$

10. $f(x) = ce^{-x/2}$, $[0, 2]$

11. $f(x) = 2ce^{-cx}$, $[0, 2]$

12. $f(x) = 2ce^{-cx}$, $[0, 4]$

In exercises 13–16, use the pdf in example 7.2 to find the probability that a randomly selected American male has height in the indicated range.

13. Between 5′10″ and 6′.

14. Between 6′6″ and 6′10″.

15. Between 7′ and 10′.

16. Between 2′ and 5′.

In exercises 17–20, find the indicated probabilities, given that the lifetime of a lightbulb is exponentially distributed with pdf $f(x) = 6e^{-6x}$ (with x measured in years).

17. The lightbulb lasts less than 3 months.

18. The lightbulb lasts less than 6 months.

19. The lightbulb lasts between 1 and 2 years.

20. The lightbulb lasts between 3 and 10 years.

In exercises 21–24, suppose the lifetime of an organism has pdf $f(x) = 4xe^{-2x}$ (with x measured in years).

21. Find the probability that the organism lives less than 1 year.

22. Find the probability that the organism lives between 1 and 2 years.

23. Find the mean lifetime ($0 \le x \le 10$).

24. Graph the pdf and compare the maximum value of the pdf to the mean.

In exercises 25–30, find (a) the mean and (b) the median of the random variable with the given pdf.

25. $f(x) = 3x^2, 0 \le x \le 1$

26. $f(x) = 4x^3, 0 \le x \le 1$

27. $f(x) = \frac{1}{2}\sin x, 0 \le x \le \pi$

28. $f(x) = \cos x, 0 \le x \le \pi/2$

29. $f(x) = \frac{1}{2}(\ln 3)e^{-kx}, k = \frac{1}{3}\ln 3, 0 \le x \le 3$

30. $f(x) = \dfrac{4}{1 - e^{-4}}e^{-4x}, 0 \le x \le 1$

31. For $f(x) = ce^{-4x}$, find c so that $f(x)$ is a pdf on the interval $[0, b]$ for $b > 0$. What happens to c as $b \to \infty$?

32. For the pdf of exercise 31, find the mean exactly (use a CAS for the antiderivative). As b increases, what happens to the mean?

33. Repeat exercises 31 and 32 for $f(x) = ce^{-6x}$.

34. Based on the results of exercises 31–33, conjecture the values for c and the mean as $b \to \infty$, for $f(x) = ce^{-ax}, a > 0$.

35. For eight coins being tossed, the probabilities of getting various numbers of heads are shown in the table. Use the addition principle to find the probability of each event indicated below.

Number of heads	0	1	2	3	4
Probability	$\frac{1}{256}$	$\frac{8}{256}$	$\frac{28}{256}$	$\frac{56}{256}$	$\frac{70}{256}$

Number of heads	5	6	7	8
Probability	$\frac{56}{256}$	$\frac{28}{256}$	$\frac{8}{256}$	$\frac{1}{256}$

(a) three or fewer heads (b) more heads than tails
(c) all heads or all tails (d) an odd number of heads

36. In one version of the game of keno, you choose 10 numbers between 1 and 80. A random drawing selects 20 numbers between 1 and 80. Your payoff depends on how many of your numbers are selected. Use the given probabilities (rounded to 4 digits) to find the probability of each event indicated below. (To win, at least 5 of your numbers must be selected. On a \$2 bet, you win \$40 or more if 6 or more of your numbers are selected.)

Number selected	0	1	2	3	4
Probability	0.0458	0.1796	0.2953	0.2674	0.1473

Number selected	5	6	7	8	9	10
Probability	0.0514	0.0115	0.0016	0.0001	0.0	0.0

(a) winning (at least 5 selected)
(b) losing (4 or fewer selected)
(c) winning big (6 or more)
(d) 3 or 4 numbers selected

37. In the baseball World Series, two teams play games until one team or the other wins four times. Suppose team A should win each game with probability 0.6. The probabilities for team A's record (given as wins/losses) in the World Series are shown. Find the probability of each event indicated below.

Wins/losses	0/4	1/4	2/4	3/4
Probability	0.0256	0.0615	0.0922	0.1106

Wins/losses	4/3	4/2	4/1	4/0
Probability	0.1659	0.2073	0.2073	0.1296

(a) Team A wins the World Series
(b) Team B wins the World Series
(c) One team wins all four games
(d) The teams play six or seven games

38. Suppose a basketball player makes 70% of her free throws. If she shoots three free throws and the probability of making each one is 0.7, the probabilities for the total number made are as shown. Find the probability of each event indicated below.

Number made	0	1	2	3
Probability	0.027	0.189	0.441	0.343

(a) She makes 2 or 3 (b) She makes at least 1

39. In any given time period, some species become extinct. Mass extinctions (such as that of the dinosaurs) are relatively rare. Fossil evidence indicates that the probability that p percent ($1 < p < 100$) of the species become extinct in a 4-million-year period is approximately $e(p) = cp^{-2}$ for some constant c. Find c to make $e(p)$ a pdf and estimate the probability that in a 4-million-year period 60–70% of the species will become extinct.

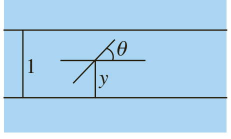
40. In example 7.5, we found the median (also called the second quartile). Now find the first and third quartiles, the ages such that the probability of being younger are 0.25 and 0.75, respectively.

41. The pdf in example 7.2 is the pdf for a normally distributed random variable. The mean is easily read off from $f(x)$; in example 7.2, the mean is 68. The mean and a number called the **standard deviation** characterize normal distributions. As Figure 5.63 indicates, the graph of the pdf has a maximum at the mean and has two inflection points located on opposite sides of the mean. The standard deviation equals the distance from the mean to an inflection point. Find the standard deviation in example 7.2.

42. In exercise 41, you found the standard deviation for the pdf in example 7.2. Denoting the mean as μ and the standard deviation as σ, find the probability that a given height is between $\mu - \sigma$ and $\mu + \sigma$ (that is, within one standard deviation of the mean). Find the probability that a given height is within two standard deviations of the mean ($\mu - 2\sigma$ to $\mu + 2\sigma$) and within three standard deviations of the mean. These probabilities are the same for any normal distribution. So, if you know the mean and standard deviation of a normally distributed random variable, you automatically know these probabilities.

43. If the probability of an event is p, the probability that it will happen m times in n tries is $f(p) = \dfrac{n!}{m!(n-m)!} p^m (1-p)^{n-m}$. Find the value of p that maximizes $f(p)$. This is called the maximum likelihood estimator of p. Briefly explain why your answer makes sense.

44. The **Buffon needle problem** is one of the oldest and most famous of probability problems. Suppose that a series of horizontal lines are spaced one unit apart and a needle of length one is placed randomly. What is the probability that the needle intersects one of the horizontal lines?



In the figure, y is the distance from the center of the needle to the nearest line and θ is the positive angle that the needle makes with the horizontal. Show that the needle intersects the line if and only if $0 \le y \le \frac{1}{2}\sin\theta$. Since $0 \le \theta \le \pi$ and $0 \le y \le \frac{1}{2}$, the desired probability is $\dfrac{\int_0^{\pi} \frac{1}{2}\sin\theta\, d\theta}{\int_0^{\pi} \frac{1}{2} d\theta}$. Compute this.

45. Suppose that a game player has won m games out of n, with a winning percentage of $100\frac{m}{n} < 75$. The player then wins several games in a row, so that the winning percentage exceeds 75%. Show that at some point in this process the player's winning percentage is *exactly* 75%.

46. Generalize exercise 45 to any winning percentage that can be written as $100\dfrac{k}{k+1}$, for some integer k.

47. Suppose that a soccer team has a probability p of scoring the next goal in a game. The probability of a 2-goal game ending in a 1-1 tie is $2p(1-p)$, the probability of a 4-goal game ending in a 2-2 tie is $\dfrac{4\cdot 3}{2\cdot 1}p^2(1-p)^2$, the probability of a 6-goal game ending in a 3-3 tie is $\dfrac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1}p^3(1-p)^3$ and so on. Assume that an even number of goals is scored. Show that the probability of a tie is a decreasing function of the number of goals scored.

48. The pdf for inter-spike intervals of neurons firing in the cochlear nucleus of a cat is $f(t) = kt^{-3/2}e^{bt-a/t}$, where $a = 100$, $b = 0.38$ and t is measured in microseconds (see Mackey and Glass, *From Clocks to Chaos*). Use your CAS to find the value of k that makes f a pdf on the interval [0, 40]. Then find the probability that neurons fire between 20 and 30 microseconds apart.

49. The Maxwell-Boltzmann pdf for molecular speeds in a gas at equilibrium is $f(x) = ax^2e^{-b^2x^2}$, for positive parameters a and b. Find the most common speed [i.e., find x to maximize $f(x)$].

EXPLORATORY EXERCISES

1. The mathematical theory of chaos indicates that numbers generated by very simple algorithms can look random. Chaos researchers look at a variety of graphs to try to distinguish randomness from deterministic chaos. For example, iterate the function $f(x) = 4x(1-x)$ starting at $x = 0.1$. That is, compute $f(0.1) = 0.36$, $f(0.36) = 0.9216$, $f(0.9216) \approx 0.289$ and so on. Iterate 50 times and record how many times each first digit occurs (so far, we've got a 1, a 3, a 9 and a 2). If the process were truly random, the digits would occur about the same number of times. Does this seem to be happening? To unmask this process as nonrandom, you can draw a **phase portrait.** To do this, take consecutive iterates as coordinates of a point (x, y) and plot the points. The first three points are (0.1, 0.36), (0.36, 0.9216) and (0.9216, 0.289). Describe the (nonrandom) pattern that appears, identifying it as precisely as possible.

2. Suppose that a spring is oscillating up and down with vertical position given by $u(t) = \sin t$. If you pick a random time and look at the position of the spring, would you be more likely to find the spring near an extreme ($u = 1$ or $u = -1$) or near the middle ($u = 0$)? The pdf is inversely proportional to speed. (Why is this reasonable?) Show that speed is given by $|\cos t| = \sqrt{1-u^2}$, so the pdf is $f(u) = c/\sqrt{1-u^2}$, $-1 \le u \le 1$, for some constant c. Show that $c = 1/\pi$, then graph $f(x)$ and describe the positions in which the spring is likely to be found. Use this result to explain the following. If you are driving in a residential neighborhood, you are more likely to meet a car coming the other way at an intersection than in the middle of a block.