Ch06-1 Linear Systems of Equations Gaussian Elimination

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Outline

1 Notation & Basic Terminology

2 Three Operations to Simplify a Linear System of Equations

3 Gaussian Elimination Procedure

4 The Gaussian Elimination with Backward Substitution Algorithm

Introduction

Linear Systems of Equations

We will consider direct methods for solving a linear system of n equations in n variables. Such a system has the form:

$$
\begin{aligned}
E_1: &\quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
E_2: &\quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\qquad\qquad\qquad\qquad \vdots \\
E_n: &\quad a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n
\end{aligned}
$$

In this system we are given the constants a_{ij}, for each $i, j = 1, 2, \ldots, n$, and b_i, for each $i = 1, 2, \ldots, n$, and we need to determine the unknowns $x_1, \ldots, x_n$.

Introduction

Direct Methods & Round-off Error

- Direct techniques are methods that theoretically give the exact solution to the system in a finite number of steps.
- In practice, of course, the solution obtained will be contaminated by the round-off error that is involved with the arithmetic being used.
- Analyzing the effect of this round-off error and determining ways to keep it under control will be a major component of this presentation.

We begin, however, by introducing some important terminology and notation.

Matrices & Vectors

Definition of a Matrix

An $\boldsymbol{n} \times \boldsymbol{m}$ ($\boldsymbol{n}$ by $\boldsymbol{m}$) **matrix** is a rectangular array of elements with n rows and m columns in which not only is the value of an element important, but also its position in the array.

Notation

The notation for an $n \times m$ matrix will be a capital letter such as A for the matrix and lowercase letters with double subscripts, such as a_{ij}, to refer to the entry at the intersection of the ith row and jth column; that is:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Matrices & Vectors

A Vector is a special case

The $1 \times n$ matrix

$$A = [a_{11} \;\; a_{12} \;\; \cdots \;\; a_{1n}]$$

is called an ***n*-dimensional row vector**, and an $n \times 1$ matrix

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}$$

is called an ***n*-dimensional column vector**.

Matrices & Vectors

A Vector is a special case (Cont'd)

Usually the unnecessary subscripts are omitted for vectors, and a boldface lowercase letter is used for notation. Thus

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

denotes a column vector, and

$$\mathbf{y} = [y_1 \;\; y_2 \;\; \dots \;\; y_n]$$

a row vector.

Matrices & Vectors: Augmented Matrix

The Augmented Matrix (1/2)

An $n \times (n+1)$ matrix can be used to represent the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n, \end{aligned}$$

by first constructing

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Matrices & Vectors: Augmented Matrix

The Augmented Matrix (2/2)

and then forming the new array $[A, \mathbf{b}]$:

$$[A, \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

where the vertical line is used to separate the coefficients of the unknowns from the values on the right-hand side of the equations.

The array $[A, \mathbf{b}]$ is called an augmented matrix.

Matrices & Vectors: Augmented Matrix

Representing the Linear System

In what follows, the $n \times (n+1)$ matrix

$$[A, \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

will used to represent the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Simplifying a Linear Systems of Equations

The Linear System

Returning to the linear system of n equations in n variables:

$$\begin{aligned} E_1: &\quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ E_2: &\quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\qquad\qquad\qquad \vdots \\ E_n: &\quad a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{aligned}$$

where we are given the constants a_{ij}, for each $i, j = 1, 2, \ldots, n$, and b_i, for each $i = 1, 2, \ldots, n$, we need to determine the unknowns $x_1, \ldots, x_n$.

Simplifying a Linear Systems of Equations

Permissible Operations

We will use 3 operations to simplify the linear system:

1. Equation E_i can be multiplied by any nonzero constant λ with the resulting equation used in place of E_i. This operation is denoted $(\lambda E_i) \rightarrow (E_i)$.
2. Equation E_j can be multiplied by any constant λ and added to equation E_i with the resulting equation used in place of E_i. This operation is denoted $(E_i + \lambda E_j) \rightarrow (E_i)$.
3. Equations E_i and E_j can be transposed in order. This operation is denoted $(E_i) \leftrightarrow (E_j)$.

By a sequence of these operations, a linear system will be systematically transformed into to a new linear system that is more easily solved and has the same solutions.

Simplifying a Linear Systems of Equations

Illustration

The four equations

$$\begin{aligned}
E_1: \quad x_1 + x_2 + 3x_4 &= 4 \\
E_2: \quad 2x_1 + x_2 - x_3 + x_4 &= 1 \\
E_3: \quad 3x_1 - x_2 - x_3 + 2x_4 &= -3 \\
E_4: \quad -x_1 + 2x_2 + 3x_3 - x_4 &= 4
\end{aligned}$$

will be solved for x_1, x_2, x_3, and x_4.

We first use equation E_1 to eliminate the unknown x_1 from equations E_2, E_3, and E_4 by performing:

$$\begin{aligned}
(E_2 - 2E_1) &\rightarrow (E_2) \\
(E_3 - 3E_1) &\rightarrow (E_3) \\
\text{and} \qquad (E_4 + E_1) &\rightarrow (E_4)
\end{aligned}$$

Simplifying a Linear Systems of Equations

$$\begin{aligned} E_1: &\quad x_1 + x_2 + 3x_4 = 4 \\ E_2: &\quad 2x_1 + x_2 - x_3 + x_4 = 1 \end{aligned}$$

Illustration Cont'd (2/5)

For example, in the second equation

$$(E_2 - 2E_1) \rightarrow (E_2)$$

produces

$$(2x_1 + x_2 - x_3 + x_4) - 2(x_1 + x_2 + 3x_4) = 1 - 2(4)$$

which simplifies to the result shown as E_2 in

$$\begin{aligned} E_1: &\quad x_1 + x_2 + 3x_4 = 4 \\ E_2: &\quad - x_2 - x_3 - 5x_4 = -7 \end{aligned}$$

Simplifying a Linear Systems of Equations

Illustration Cont'd (3/5)

Similarly for equations E_3 and E_4 so that we obtain the new system:

$$\begin{aligned}
E_1 &: \quad x_1 + x_2 + 3x_4 = 4 \\
E_2 &: \quad -x_2 - x_3 - 5x_4 = -7 \\
E_3 &: \quad -4x_2 - x_3 - 7x_4 = -15 \\
E_4 &: \quad 3x_2 + 3x_3 + 2x_4 = 8
\end{aligned}$$

For simplicity, the new equations are again labeled E_1, E_2, E_3, and E_4.

Simplifying a Linear Systems of Equations

Illustration Cont'd (4/5)

In the new system, E_2 is used to eliminate the unknown x_2 from E_3 and E_4 by performing $(E_3 - 4E_2) \rightarrow (E_3)$ and $(E_4 + 3E_2) \rightarrow (E_4)$. This results in

$$\begin{aligned}
E_1: &\quad x_1 + x_2 \qquad\quad + \; 3x_4 = 4, \\
E_2: &\quad\;\; - x_2 - x_3 - 5x_4 = -7, \\
E_3: &\qquad\qquad\; 3x_3 + 13x_4 = 13, \\
E_4: &\qquad\qquad\qquad - 13x_4 = -13.
\end{aligned}$$

This latter system of equations is now in triangular (or reduced) form and can be solved for the unknowns by a backward-substitution process.

Simplifying a Linear Systems of Equations

Illustration Cont'd (5/5)

Since E_4 implies $x_4 = 1$, we can solve E_3 for x_3 to give

$$x_3 = \frac{1}{3}(13 - 13x_4) = \frac{1}{3}(13 - 13) = 0.$$

Continuing, E_2 gives

$$x_2 = -(-7 + 5x_4 + x_3) = -(-7 + 5 + 0) = 2,$$

and E_1 gives

$$x_1 = 4 - 3x_4 - x_2 = 4 - 3 - 2 = -1.$$

The solution is therefore, $x_1 = -1$, $x_2 = 2$, $x_3 = 0$, and $x_4 = 1$.

Constructing an Algorithm to Solve the Linear System

$$
\begin{aligned}
E_1 &: \quad x_1 + x_2 \phantom{{}- x_3} + 3x_4 = 4 \\
E_2 &: \quad 2x_1 + x_2 - x_3 + x_4 = 1 \\
E_3 &: \quad 3x_1 - x_2 - x_3 + 2x_4 = -3 \\
E_4 &: \quad -x_1 + 2x_2 + 3x_3 - x_4 = 4
\end{aligned}
$$

Converting to Augmented Form

Repeating the operations involved in the previous illustration with the matrix notation results in first considering the augmented matrix:

$$
\left[\begin{array}{rrrr|r}
1 & 1 & 0 & 3 & 4 \\
2 & 1 & -1 & 1 & 1 \\
3 & -1 & -1 & 2 & -3 \\
-1 & 2 & 3 & -1 & 4
\end{array}\right]
$$

Constructing an Algorithm to Solve the Linear System

Reducing to Triangular Form

Performing the operations as described in the earlier example produces the augmented matrices:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & -4 & -1 & -7 & -15 \\ 0 & 3 & 3 & 2 & 8 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & 0 & 3 & 13 & 13 \\ 0 & 0 & 0 & -13 & -13 \end{array}\right]$$

The final matrix can now be transformed into its corresponding linear system, and solutions for x_1, x_2, x_3, and x_4, can be obtained. The procedure is called Gaussian elimination with backward substitution.

Gaussian Elimination with Backward Substitution

Basic Steps in the Procedure

The general Gaussian elimination procedure applied to the linear system

$$
\begin{aligned}
E_1: &\quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
E_2: &\quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\quad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
E_n: &\quad a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n
\end{aligned}
$$

will be handled in a similar manner.

Gaussian Elimination with Backward Substitution

Basic Steps in the Procedure (Cont'd)

- First form the augmented matrix $\tilde{A}$:

$$\tilde{A} = [A, \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ a_{21} & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & a_{n,n+1} \end{array}\right]$$

  where A denotes the matrix formed by the coefficients.
- The entries in the $(n+1)$st column are the values of $\mathbf{b}$; that is, $a_{i,n+1} = b_i$ for each $i = 1, 2, \ldots, n$.

Gaussian Elimination with Backward Substitution

Basic Steps in the Procedure (Cont'd)

- Provided $a_{11} \neq 0$, we perform the operations corresponding to

$$(E_j - (a_{j1}/a_{11})E_1) \to (E_j) \quad \text{for each } j = 2, 3, \ldots, n$$

  to eliminate the coefficient of x_1 in each of these rows.
- Although the entries in rows $2, 3, \ldots, n$ are expected to change, for ease of notation we again denote the entry in the ith row and the jth column by a_{ij}.
- With this in mind, we follow a sequential procedure for $i = 2, 3, \ldots, n-1$ and perform the operation

$$(E_j - (a_{ji}/a_{ii})E_i) \to (E_j) \quad \text{for each } j = i+1, i+2, \ldots, n,$$

  provided $a_{ii} \neq 0$.

Gaussian Elimination with Backward Substitution

Basic Steps in the Procedure (Cont'd)

- This eliminates (changes the coefficient to zero) x_i in each row below the ith for all values of $i = 1, 2, \ldots, n-1$.
- The resulting matrix has the form:

$$\tilde{\tilde{A}} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ 0 & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{nn} & a_{n,n+1} \end{array}\right]$$

  where, except in the first row, the values of a_{ij} are not expected to agree with those in the original matrix $\tilde{A}$.
- The matrix $\tilde{\tilde{A}}$ represents a linear system with the same solution set as the original system.

Gaussian Elimination with Backward Substitution

Basic Steps in the Procedure (Cont'd)

The new linear system is triangular,

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & a_{1,n+1} \\ & & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & a_{2,n+1} \\ & & & \ddots & & & \vdots & & \vdots \\ & & & & \ddots & & \vdots & & \vdots \\ & & & & & \ddots & \vdots & & \vdots \\ & & & & & & a_{nn}x_n & = & a_{n,n+1} \end{array}$$

so backward substitution can be performed. Solving the nth equation for x_n gives

$$x_n = \frac{a_{n,n+1}}{a_{nn}}$$

Gaussian Elimination with Backward Substitution

Basic Steps in the Procedure (Cont'd)

Solving the $(n-1)$st equation for x_{n-1} and using the known value for x_n yields

$$x_{n-1} = \frac{a_{n-1,n+1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

Continuing this process, we obtain

$$\begin{aligned} x_i &= \frac{a_{i,n+1} - a_{i,n}x_n - a_{i,n-1}x_{n-1} - \cdots - a_{i,i+1}x_{i+1}}{a_{ii}} \\ &= \frac{a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}} \end{aligned}$$

for each $i = n-1, n-2, \ldots, 2, 1$.

Gaussian Elimination with Backward Substitution

A More Precise Description

Gaussian elimination procedure is described more precisely, although more intricately, by forming a sequence of augmented matrices $\tilde{A}^{(1)}$, $\tilde{A}^{(2)}, \ldots, \tilde{A}^{(n)}$, where $\tilde{A}^{(1)}$ is the matrix $\tilde{A}$ given earlier and $\tilde{A}^{(k)}$, for each $k = 2, 3, \ldots, n$, has entries $a_{ij}^{(k)}$, where:

$$a_{ij}^{(k)} = \begin{cases} a_{ij}^{(k-1)} & \text{when } i = 1, 2, \ldots, k-1 \text{ and } j = 1, 2, \ldots, n+1 \\ 0 & \text{when } i = k, k+1, \ldots, n \text{ and } j = 1, 2, \ldots, k-1 \\ a_{ij}^{(k-1)} - \dfrac{a_{i,k-1}^{(k-1)}}{a_{k-1,k-1}^{(k-1)}} a_{k-1,j}^{(k-1)} & \text{when } i = k, k+1, \ldots, n \text{ and } j = k, k+1, \ldots, n+1 \end{cases}$$

Gaussian Elimination with Backward Substitution

A More Precise Description (Cont'd)

Thus

$$\tilde{A}^{(k)} = \left[\begin{array}{cccccccc|c}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1,k-1}^{(1)} & a_{1k}^{(1)} & \cdots & a_{1n}^{(1)} & a_{1,n+1}^{(1)} \\
0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2,k-1}^{(2)} & a_{2k}^{(2)} & \cdots & a_{2n}^{(2)} & a_{2,n+1}^{(2)} \\
\vdots & \ddots & \ddots & & \vdots & \vdots & & \vdots & \vdots \\
\vdots & & \ddots & \ddots & \vdots & \vdots & & \vdots & \vdots \\
\vdots & & & \ddots & a_{k-1,k-1}^{(k-1)} & a_{k-1,k}^{(k-1)} & \cdots & a_{k-1,n}^{(k-1)} & a_{k-1,n+1}^{(k-1)} \\
\vdots & & & & 0 & a_{kk}^{(k)} & \cdots & a_{kn}^{(k)} & a_{k,n+1}^{(k)} \\
\vdots & & & & \vdots & \vdots & & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & a_{nk}^{(k)} & \cdots & a_{nn}^{(k)} & a_{n,n+1}^{(k)}
\end{array}\right]$$

represents the equivalent linear system for which the variable x_{k-1} has just been eliminated from equations $E_k, E_{k+1}, \ldots, E_n$.

Gaussian Elimination with Backward Substitution

A More Precise Description (Cont'd)

- The procedure will fail if one of the elements $a_{11}^{(1)}, a_{22}^{(2)}, a_{33}^{(3)}, \ldots,$ $a_{n-1,n-1}^{(n-1)}$, $a_{nn}^{(n)}$ is zero because the step

$$\left(E_i - \frac{a_{i,k}^{(k)}}{a_{kk}^{(k)}}(E_k)\right) \to E_i$$

  either cannot be performed (this occurs if one of $a_{11}^{(1)}, \ldots, a_{n-1,n-1}^{(n-1)}$ is zero), or the backward substitution cannot be accomplished (in the case $a_{nn}^{(n)} = 0$).
- The system may still have a solution, but the technique for finding it must be altered.

Illustration of the Gaussian Elimination Procedure

Example

Represent the linear system

$$
\begin{aligned}
E_1: &\quad x_1 - x_2 + 2x_3 - x_4 = -8 \\
E_2: &\quad 2x_1 - 2x_2 + 3x_3 - 3x_4 = -20 \\
E_3: &\quad x_1 + x_2 + x_3 = -2 \\
E_4: &\quad x_1 - x_2 + 4x_3 + 3x_4 = 4
\end{aligned}
$$

as an augmented matrix and use Gaussian Elimination to find its solution.

Illustration of the Gaussian Elimination Procedure

Solution (1/6)

The augmented matrix is

$$\tilde{A} = \tilde{A}^{(1)} = \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 2 & -2 & 3 & -3 & -20 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & -1 & 4 & 3 & 4 \end{array}\right]$$

Performing the operations

$$(E_2 - 2E_1) \to (E_2),\ (E_3 - E_1) \to (E_3) \quad \text{and} \quad (E_4 - E_1) \to (E_4)$$

gives

$$\tilde{A}^{(2)} = \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right]$$

Illustration of the Gaussian Elimination Procedure

$$\tilde{A}^{(2)} = \left[\begin{array}{rrrr|r} 1 & -1 & 2 & -1 & -8 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right]$$

Solution (2/6)

- The diagonal entry $a_{22}^{(2)}$, called the pivot element, is 0, so the procedure cannot continue in its present form.
- But operations $(E_i) \leftrightarrow (E_j)$ are permitted, so a search is made of the elements $a_{32}^{(2)}$ and $a_{42}^{(2)}$ for the first nonzero element.
- Since $a_{32}^{(2)} \neq 0$, the operation $(E_2) \leftrightarrow (E_3)$ can be performed to obtain a new matrix.

Illustration of the Gaussian Elimination Procedure

$$\tilde{A}^{(2)} = \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right]$$

Solution (3/6)

Perform the operation $(E_2) \leftrightarrow (E_3)$ to obtain a new matrix:

$$\tilde{A}^{(2)'} = \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right]$$

Illustration of the Gaussian Elimination Procedure

$$\tilde{A}^{(2)'} = \left[\begin{array}{rrrr|r} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right]$$

Solution (4/6)

Since x_2 is already eliminated from E_3 and E_4, $\tilde{A}^{(3)}$ will be $\tilde{A}^{(2)'}$, and the computations continue with the operation $(E_4 + 2E_3) \to (E_4)$, giving

$$\tilde{A}^{(4)} = \left[\begin{array}{rrrr|r} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right]$$

Illustration of the Gaussian Elimination Procedure

$$\tilde{A}^{(4)} = \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right]$$

Solution (5/6)

The solution may now be found through backward substitution:

$$\begin{aligned} x_4 &= \frac{4}{2} = 2 \\ x_3 &= \frac{[-4 - (-1)x_4]}{-1} = 2 \\ x_2 &= \frac{[6 - x_4 - (-1)x_3]}{2} = 3 \\ x_1 &= \frac{[-8 - (-1)x_4 - 2x_3 - (-1)x_2]}{1} = -7 \end{aligned}$$

Illustration of the Gaussian Elimination Procedure

Solution (6/6): Some Observations

- The example illustrates what is done if $a_{kk}^{(k)} = 0$ for some $k = 1, 2, \ldots, n-1$.
- The kth column of $\tilde{A}^{(k-1)}$ from the kth row to the nth row is searched for the first nonzero entry.
- If $a_{pk}^{(k)} \neq 0$ for some p,with $k+1 \leq p \leq n$, then the operation $(E_k) \leftrightarrow (E_p)$ is performed to obtain $\tilde{A}^{(k-1)'}$.
- The procedure can then be continued to form $\tilde{A}^{(k)}$, and so on.
- If $a_{pk}^{(k)} = 0$ for each p, it can be shown that the linear system does not have a unique solution and the procedure stops.
- Finally, if $a_{nn}^{(n)} = 0$, the linear system does not have a unique solution, and again the procedure stops.

Gaussian Elimination with Backward Substitution Algorithm (1/3)

To solve the $n \times n$ linear system

$$
\begin{array}{lccccc}
E_1: & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{1,n+1} \\
E_2: & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = a_{2,n+1} \\
\vdots & \vdots \quad \vdots \quad \vdots \quad \vdots \\
E_n: & a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = a_{n,n+1}
\end{array}
$$

INPUT number of unknowns and equations n; augmented matrix $A = [a_{ij}]$, where $1 \le i \le n$ and $1 \le j \le n+1$.

OUTPUT solution $x_1, x_2, \ldots, x_n$ or message that the linear system has no unique solution.

Gaussian Elimination with Backward Substitution Algorithm (2/3)

Step 1 For $i = 1, \ldots, n-1$ do Steps 2–4: (*Elimination process*)

Step 2 Let p be the smallest integer with $i \leq p \leq n$ and $a_{pi} \neq 0$
If no integer p can be found
then OUTPUT ('no unique solution exists')
STOP

Step 3 If $p \neq i$ then perform $(E_p) \leftrightarrow (E_i)$

Step 4 For $j = i+1, \ldots, n$ do Steps 5 and 6:

Step 5 Set $m_{ji} = a_{ji}/a_{ii}$
Step 6 Perform $(E_j - m_{ji}E_i) \rightarrow (E_j)$

Gaussian Elimination with Backward Substitution Algorithm (3/3)

Step 7 If $a_{nn} = 0$
then OUTPUT ('no unique solution exists')

Step 8 Set $x_n = a_{n,n+1}/a_{nn}$ (*Start backward substitution*)

Step 9 For $i = n-1, \ldots, 1$ set $x_i = \left[a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j\right] \Big/ a_{ii}$

Step 10 OUTPUT $(x_1, \ldots, x_n)$ (*Procedure completed successfully*)
STOP