Ch06-3 Linear Systems of Equations, Matrix Factorization

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Outline

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2 Constructing the Matrix Factorization

3 Example: *LU Factorization of a 4 × 4 Matrix*

4 The *LU Factorization Algorithm*

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Matrix Factorization

Background

- Gaussian elimination is the principal tool in the direct solution of linear systems of equations.
- We will now see that the steps used to solve a system of the form $A\mathbf{x} = \mathbf{b}$ can be used to factor a matrix.
- The factorization is particularly useful when it has the form $A = LU$, where L is lower triangular and U is upper triangular.
- Although not all matrices have this type of representation, many do that occur frequently in the application of numerical techniques.

Matrix Factorization

Computational Cost Considerations

- Gaussian elimination applied to an arbitrary linear system $A\mathbf{x} = \mathbf{b}$ requires $O(n^3/3)$ arithmetic operations to determine $\mathbf{x}$.
- However, to solve a linear system that involves an upper-triangular system requires only backward substitution, which takes $O(n^2)$ operations.
- The number of operations required to solve a lower-triangular systems is similar.

Matrix Factorization

Solution Strategy

Suppose that A has been factored into the triangular form $A = LU$, where L is lower triangular and U is upper triangular. Then we can solve for $\mathbf{x}$ more easily by using a two-step process:

- First we let $\mathbf{y} = U\mathbf{x}$ and solve the lower triangular system $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$. Since L is triangular, determining $\mathbf{y}$ from this equation requires only $O(n^2)$ operations.
- Once $\mathbf{y}$ is known, the upper triangular system $U\mathbf{x} = \mathbf{y}$ requires only an additional $O(n^2)$ operations to determine the solution $\mathbf{x}$.

Solving a linear system $A\mathbf{x} = \mathbf{b}$ in factored form means that the number of operations needed to solve the system $A\mathbf{x} = \mathbf{b}$ is reduced from $O(n^3/3)$ to $O(2n^2)$.

Matrix Factorization

Constructing L & U

- First, suppose that Gaussian elimination can be performed on the system $A\mathbf{x} = \mathbf{b}$ without row interchanges.
- With the notation used earlier, this is equivalent to having nonzero pivot elements $a_{ii}^{(i)}$, for each $i = 1, 2, \ldots, n$.
- The first step in the Gaussian elimination process consists of performing, for each $j = 2, 3, \ldots, n$, the operations

$$(E_j - m_{j,1}E_1) \to (E_j), \quad \text{where} \quad m_{j,1} = \frac{a_{j1}^{(1)}}{a_{11}^{(1)}}$$

- These operations transform the system into one in which all the entries in the first column below the diagonal are zero.

Matrix Factorization: Constructing L & U (Cont'd)

The system of operations in

$$(E_j - m_{j,1}E_1) \to (E_j), \quad \text{where} \quad m_{j,1} = \frac{a_{j1}^{(1)}}{a_{11}^{(1)}}$$

can be viewed in another way. It is simultaneously accomplished by multiplying the original matrix A on the left by the matrix

$$M^{(1)} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -m_{21} & 1 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -m_{n1} & 0 & \cdots & 0 & 1 \end{bmatrix}$$

This is called the first Gaussian transformation matrix.

Matrix Factorization

Constructing L & U (Cont'd)

- We denote the product of this matrix with $A^{(1)} \equiv A$ by $A^{(2)}$ and with $\mathbf{b}$ by $\mathbf{b}^{(2)}$, so

$$A^{(2)}\mathbf{x} = M^{(1)}A\mathbf{x} = M^{(1)}\mathbf{b} = \mathbf{b}^{(2)}$$

- In a similar manner we construct $M^{(2)}$, the identity matrix with the entries below the diagonal in the second column replaced by the negatives of the multipliers

$$m_{j,2} = \frac{a_{j2}^{(2)}}{a_{22}^{(2)}}.$$

Matrix Factorization

Constructing L & U (Cont'd)

- The product of $M^{(2)}$ with $A^{(2)}$ has zeros below the diagonal in the first two columns, and we let

$$A^{(3)}\mathbf{x} = M^{(2)}A^{(2)}\mathbf{x} = M^{(2)}M^{(1)}A\mathbf{x} = M^{(2)}M^{(1)}\mathbf{b} = \mathbf{b}^{(3)}$$

Matrix Factorization

Constructing L & U (Cont'd)

In general, with $A^{(k)}\mathbf{x} = \mathbf{b}^{(k)}$ already formed, multiply by the ***k*th Gaussian transformation matrix**

$$M^{(k)} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & & 0 & \ddots & \ddots & & & \vdots \\ \vdots & & \vdots & -m_{k+1,k} & \ddots & \ddots & & \vdots \\ \vdots & & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & 0 & -m_{n,k} & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Matrix Factorization

Constructing L & U (Cont'd)

to obtain

$$\begin{aligned} A^{(k+1)}\mathbf{x} &= M^{(k)}A^{(k)}\mathbf{x} \\ &= M^{(k)}\ldots M^{(1)}A\mathbf{x} \\ &= M^{(k)}\mathbf{b}^{(k)} \\ &= \mathbf{b}^{(k+1)} \\ &= M^{(k)}\ldots M^{(1)}\mathbf{b} \end{aligned}$$

Matrix Factorization

Constructing L & U (Cont'd)

The process ends with the formation of $A^{(n)}\mathbf{x} = \mathbf{b}^{(n)}$, where $A^{(n)}$ is the upper triangular matrix

$$A^{(n)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_{n-1,n}^{(n-1)} \\ 0 & \cdots & \cdots & 0 & a_{n,n}^{(n)} \end{bmatrix}$$

given by

$$A^{(n)} = M^{(n-1)}M^{(n-2)}\ldots M^{(1)}A$$

Matrix Factorization

Constructing L & U (Cont'd)

- This process forms the $U = A^{(n)}$ portion of the matrix factorization $A = LU$.
- To determine the complementary lower triangular matrix L, first recall the multiplication of $A^{(k)}\mathbf{x} = \mathbf{b}^{(k)}$ by the Gaussian transformation of $M^{(k)}$ used to obtain:

$$A^{(k+1)}\mathbf{x} = M^{(k)}A^{(k)}\mathbf{x} = M^{(k)}\mathbf{b}^{(k)} = \mathbf{b}^{(k+1)},$$

where $M^{(k)}$ generates the row operations

$$(E_j - m_{j,k}E_k) \rightarrow (E_j), \quad \text{for } j = k+1, \ldots, n.$$

Matrix Factorization

Constructing L & U (Cont'd)

To reverse the effects of this transformation and return to $A^{(k)}$ requires that the operations $(E_j + m_{j,k}E_k) \rightarrow (E_j)$ be performed for each $j = k+1, \ldots, n$. This is equivalent to multiplying by $\left[M^{(k)}\right]^{-1}$:

$$L^{(k)} = \left[M^{(k)}\right]^{-1} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & & 0 & \ddots & \ddots & & & \vdots \\ \vdots & & \vdots & m_{k+1,k} & \ddots & \ddots & & \vdots \\ \vdots & & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & 0 & m_{n,k} & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Matrix Factorization

Constructing L & U (Cont'd)

The lower-triangular matrix L in the factorization of A, then, is the product of the matrices $L^{(k)}$:

$$L = L^{(1)}L^{(2)}\cdots L^{(n-1)} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ m_{21} & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ m_{n1} & \cdots & \cdots & m_{n,n-1} & 1 \end{bmatrix}$$

since the product of L with the upper-triangular matrix $U = M^{(n-1)}\cdots M^{(2)}M^{(1)}A$ gives

Matrix Factorization

Constructing L & U (Cont'd)

$$
\begin{aligned}
LU &= L^{(1)}L^{(2)}\ldots L^{(n-3)}L^{(n-2)}L^{(n-1)} \\
&\qquad\qquad \cdot M^{(n-1)}M^{(n-2)}M^{(n-3)}\ldots M^{(2)}M^{(1)}A \\
&= [M^{(1)}]^{-1}[M^{(2)}]^{-1}\ldots[M^{(n-2)}]^{-1}[M^{(n-1)}]^{-1} \\
&\qquad\qquad \cdot M^{(n-1)}M^{(n-2)}\ldots M^{(2)}M^{(1)}A \\
&= A
\end{aligned}
$$

We now state a theorem which follows from these observations.

Matrix Factorization

Theorem

If Gaussian elimination can be performed on the linear system $A\mathbf{x} = \mathbf{b}$ without row interchanges, then the matrix A can be factored into the product of a lower-triangular matrix L and an upper-triangular matrix U, that is, $A = LU$, where $m_{ji} = a_{ji}^{(i)}/a_{ii}^{(i)}$,

$$U = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_{n-1,n}^{(n-1)} \\ 0 & \cdots & \cdots & 0 & a_{n,n}^{(n)} \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ m_{21} & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ m_{n1} & \cdots & \cdots & m_{n,n-1} & 1 \end{bmatrix}$$

Matrix Factorization

Example

(a) Determine the LU factorization for matrix A in the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 4 \end{bmatrix}$$

(b) Then use the factorization to solve the system

$$\begin{aligned} x_1 + x_2 + 3x_4 &= 8 \\ 2x_1 + x_2 - x_3 + x_4 &= 7 \\ 3x_1 - x_2 - x_3 + 2x_4 &= 14 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= -7 \end{aligned}$$

Matrix Factorization: 4×4 Example

Part (a) Solution (1/2)

The original system was considered under Gaussian Elimination where we saw that the sequence of operations

$$
\begin{array}{ll}
(E_2 - 2E_1) \to (E_2) & (E_3 - 3E_1) \to (E_3) \\
(E_4 - (-1)E_1) \to (E_4) & (E_3 - 4E_2) \to (E_3) \\
(E_4 - (-3)E_2) \to (E_4) &
\end{array}
$$

converts the system to the triangular system

$$
\begin{array}{rcrcrcrcr}
x_1 & + & x_2 & & & + & 3x_4 & = & 4 \\
 & - & x_2 & - & x_3 & - & 5x_4 & = & -7 \\
 & & & & 3x_3 & + & 13x_4 & = & 13 \\
 & & & & & - & 13x_4 & = & -13
\end{array}
$$

Matrix Factorization: 4×4 Example

Part (a) Solution (2/2)

The multipliers m_{ij} and the upper triangular matrix produce the factorization

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \\
&= LU
\end{aligned}
$$

Matrix Factorization: 4 × 4 Example

Part (b) Solution (1/3)

To solve

$$Ax = LU\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix}$$

we first introduce the substitution $\mathbf{y} = U\mathbf{x}$. Then $\mathbf{b} = L(U\mathbf{x}) = L\mathbf{y}$.

Matrix Factorization: 4×4 Example

Part (b) Solution (2/3)

First, solve $L\mathbf{y} = \mathbf{b}$ (where $\mathbf{y} = U\mathbf{x}$:

$$L\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix}.$$

This system is solved for $\mathbf{y}$ by a simple forward-substitution process:

$$\begin{aligned} y_1 &= 8 \\ 2y_1 + y_2 &= 7 \quad \Rightarrow \quad y_2 = 7 - 2y_1 = -9 \\ 3y_1 + 4y_2 + y_3 &= 14 \quad \Rightarrow \quad y_3 = 14 - 3y_1 - 4y_2 = 26 \\ -y_1 - 3y_2 + y_4 &= -7 \quad \Rightarrow \quad y_4 = -7 + y_1 + 3y_2 = -26 \end{aligned}$$

Matrix Factorization: 4×4 Example

Part (b) Solution (3/3)

We then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, the solution of the original system; that is,

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ 26 \\ -26 \end{bmatrix}$$

Using backward substitution we obtain $x_4 = 2$, $x_3 = 0$, $x_2 = -1$, $x_1 = 3$.

Discuss algorithm

Matrix Factorization

Using the LU Factorization to solve $A\mathbf{x} = \mathbf{b}$

Once the matrix factorization is complete, the solution to a linear system of the form

$$A\mathbf{x} = LU\mathbf{x} = \mathbf{b}$$

is found by first letting

$$\mathbf{y} = U\mathbf{x}$$

and solving

$$L\mathbf{y} = \mathbf{b}$$

for $\mathbf{y}$.

Matrix Factorization

Using the LU Factorization (Cont'd)

- Since L is lower triangular, we have $y_1 = \dfrac{b_1}{l_{11}}$ and, for each $i = 2, 3, \ldots, n$,

$$y_i = \frac{1}{l_{ii}} \left[b_i - \sum_{j=1}^{i-1} l_{ij} y_j \right]$$

- After $\mathbf{y}$ is found by this forward-substitution process, the upper-triangular system $U\mathbf{x} = \mathbf{y}$ is solved for $\mathbf{x}$ by backward substitution using the equations

$$x_n = \frac{y_n}{u_{nn}} \quad \text{and} \quad x_i = \frac{1}{u_{ii}} \left[y_i - \sum_{j=i+1}^{n} u_{ij} x_j \right]$$

Matrix Factorization: Permutation Matrices

Limitations of the LU Factorization Algorithm

- We assumed that $A\mathbf{x} = \mathbf{b}$ can be solved using Gaussian elimination without row interchanges.
- From a practical standpoint, this factorization is useful only when row interchanges are not required to control round-off error.
- We will now consider the modifications that must be made when row interchanges are required.

Matrix Factorization: Permutation Matrices

We begin with the introduction of a class of matrices that are used to rearrange, or permute, rows of a given matrix.

Permutation Matrix

An $n \times n$ permutation matrix $P = [p_{ij}]$ is a matrix obtained by rearranging the rows of I_n, the identity matrix. This gives a matrix with precisely one nonzero entry in each row and in each column, and each nonzero entry is a 1.

Matrix Factorization: Permutation Matrices

Example

The matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

is a 3×3 permutation matrix. For any 3×3 matrix A, multiplying on the left by P has the effect of interchanging the second and third rows of A:

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Similarly, multiplying A on the right by P interchanges the second and third columns of A.

Matrix Factorization: Permutation Matrices

Two useful properties of permutation matrices (1/2)

Suppose $k_1, \ldots, k_n$ is a permutation of the integers $1, \ldots, n$ and the permutation matrix $P = (p_{ij})$ is defined by

$$p_{ij} = \begin{cases} 1, & \text{if } j = k_i \\ 0, & \text{otherwise} \end{cases}$$

Matrix Factorization: Permutation Matrices

Two useful properties of permutation matrices (2/2)

Then

- PA permutes the rows of A; that is,

$$PA = \begin{bmatrix} a_{k_1 1} & a_{k_1 2} & \cdots & a_{k_1 n} \\ a_{k_2 1} & a_{k_2 2} & \cdots & a_{k_2 n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k_n 1} & a_{k_n 2} & \cdots & a_{k_n n} \end{bmatrix}$$

- P^{-1} exists and $P^{-1} = P^t$.

Matrix Factorization: Permutation Matrices

Permutation Matrices & Gaussian Elimination

- Earlier, we saw that for any nonsingular matrix A, the linear system $A\mathbf{x} = \mathbf{b}$ can be solved by Gaussian elimination, with the possibility of row interchanges.
- If we knew the row interchanges that were required to solve the system by Gaussian elimination, we could arrange the original equations in an order that would ensure that no row interchanges are needed.
- Hence there is a rearrangement of the equations in the system that permits Gaussian elimination to proceed *without* row interchanges.

Matrix Factorization: Permutation Matrices

Permutation Matrices & Gaussian Elimination (Cont'd)

- This implies that for any nonsingular matrix A, a permutation matrix P exists for which the system

$$PA\mathbf{x} = P\mathbf{b}$$

  can be solved without row interchanges. As a consequence, this matrix PA can be factored into $PA = LU$, where L is lower triangular and U is upper triangular.
- Because $P^{-1} = P^t$, this produces the factorization

$$A = P^{-1}LU = (P^tL)U.$$

- The matrix U is still upper triangular, but P^tL is not lower triangular unless $P = I$.

Matrix Factorization: Permutation Matrices

Example

Determine a factorization in the form $A = (P^t L)U$ for the matrix

$$A = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix}$$

Note

The matrix A cannot have an LU factorization because $a_{11} = 0$.

Matrix Factorization: Permutation Matrices

Solution (1/4)

However, using the row interchange $(E_1) \leftrightarrow (E_2)$, followed by $(E_3 + E_1) \rightarrow (E_3)$ and $(E_4 - E_1) \rightarrow (E_4)$, produces

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Then, the row interchange $(E_2) \leftrightarrow (E_4)$, followed by $(E_4 + E_3) \rightarrow (E_4)$, gives the matrix

$$U = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Matrix Factorization: Permutation Matrices

Solution (2/4)

The permutation matrix associated with the row interchanges $(E_1) \leftrightarrow (E_2)$ and $(E_2) \leftrightarrow (E_4)$ is

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and

$$PA = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & 2 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Matrix Factorization: Permutation Matrices

Solution (3/4)

- Gaussian elimination is performed on PA using the same operations as on A, except without the row interchanges.
- That is, $(E_2 - E_1) \to (E_2)$, $(E_3 + E_1) \to (E_3)$, followed by $(E_4 + E_3) \to (E_4)$.
- The nonzero multipliers for PA are consequently,

$$m_{21} = 1, \quad m_{31} = -1, \quad \text{and} \quad m_{43} = -1,$$

and the LU factorization of PA is

$$PA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} = LU$$

Matrix Factorization: Permutation Matrices

Solution (4/4)

Multiplying by $P^{-1} = P^t$ produces the factorization

$$
\begin{aligned}
A = P^{-1}(LU) = P^t(LU) &= (P^tL)U \\
&= \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}
\end{aligned}
$$