Ch07-2 Jacobi & Gauss-Seidel Iterative Techniques II

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Outline

1 The Gauss-Seidel Method

2 The Gauss-Seidel Algorithm

3 Convergence Results for General Iteration Methods

4 Application to the Jacobi & Gauss-Seidel Methods

The Gauss-Seidel Method

Looking at the Jacobi Method

- A possible improvement to the Jacobi Algorithm can be seen by re-considering

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(-a_{ij} x_j^{(k-1)} \right) + b_i \right], \qquad \text{for } i = 1, 2, \ldots, n$$

- The components of $\mathbf{x}^{(k-1)}$ are used to compute all the components $x_i^{(k)}$ of $\mathbf{x}^{(k)}$.
- But, for $i > 1$, the components $x_1^{(k)}, \ldots, x_{i-1}^{(k)}$ of $\mathbf{x}^{(k)}$ have already been computed and are expected to be better approximations to the actual solutions $x_1, \ldots, x_{i-1}$ than are $x_1^{(k-1)}, \ldots, x_{i-1}^{(k-1)}$.

The Gauss-Seidel Method

Instead of using

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1\\ j\neq i}}^{n}\left(-a_{ij}x_j^{(k-1)}\right) + b_i\right], \qquad \text{for } i = 1, 2, \ldots, n$$

it seems reasonable, then, to compute $x_i^{(k)}$ using these most recently calculated values.

The Gauss-Seidel Iterative Technique

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1}(a_{ij}x_j^{(k)}) - \sum_{j=i+1}^{n}(a_{ij}x_j^{(k-1)}) + b_i\right]$$

for each $i = 1, 2, \ldots, n$.

The Gauss-Seidel Method

Example

Use the Gauss-Seidel iterative technique to find approximate solutions to

$$\begin{aligned} 10x_1 - x_2 + 2x_3 \qquad &= 6 \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11, \\ 3x_2 - x_3 + 8x_4 &= 15 \end{aligned}$$

starting with $\mathbf{x} = (0, 0, 0, 0)^t$ and iterating until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}$$

Note: The solution $\mathbf{x} = (1,\ 2,\ -1,\ 1)^t$ was approximated by Jacobi's method in an earlier example.

The Gauss-Seidel Method

Solution (1/3)

For the Gauss-Seidel method we write the system, for each $k = 1, 2, \ldots$ as

$$
\begin{aligned}
x_1^{(k)} &= \phantom{-\frac{1}{5}x_1^{(k)} +} \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} + \frac{3}{5} \\
x_2^{(k)} &= \frac{1}{11}x_1^{(k)} + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11} \\
x_3^{(k)} &= -\frac{1}{5}x_1^{(k)} + \frac{1}{10}x_2^{(k)} + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10} \\
x_4^{(k)} &= - \frac{3}{8}x_2^{(k)} + \frac{1}{8}x_3^{(k)} + \frac{15}{8}
\end{aligned}
$$

The Gauss-Seidel Method

Solution (2/3)

When $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$, we have $\mathbf{x}^{(1)} = (0.6000, 2.3272, -0.9873, 0.8789)^t$. Subsequent iterations give the values in the following table:

k	0	1	2	3	4	5
$x_1^{(k)}$	0.0000	0.6000	1.030	1.0065	1.0009	1.0001
$x_2^{(k)}$	0.0000	2.3272	2.037	2.0036	2.0003	2.0000
$x_3^{(k)}$	0.0000	−0.9873	−1.014	−1.0025	−1.0003	−1.0000
$x_4^{(k)}$	0.0000	0.8789	0.984	0.9983	0.9999	1.0000

The Gauss-Seidel Method

Solution (3/3)

Because

$$\frac{\|\mathbf{x}^{(5)} - \mathbf{x}^{(4)}\|_\infty}{\|\mathbf{x}^{(5)}\|_\infty} = \frac{0.0008}{2.000} = 4 \times 10^{-4}$$

$\mathbf{x}^{(5)}$ is accepted as a reasonable approximation to the solution.

Note that, in an earlier example, Jacobi's method required twice as many iterations for the same accuracy.

The Gauss-Seidel Method: Matrix Form

Re-Writing the Equations

To write the Gauss-Seidel method in matrix form, multiply both sides of

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1}(a_{ij}x_j^{(k)}) - \sum_{j=i+1}^{n}(a_{ij}x_j^{(k-1)}) + b_i\right]$$

by a_{ii} and collect all kth iterate terms, to give

$$a_{i1}x_1^{(k)} + a_{i2}x_2^{(k)} + \cdots + a_{ii}x_i^{(k)} = -a_{i,i+1}x_{i+1}^{(k-1)} - \cdots - a_{in}x_n^{(k-1)} + b_i$$

for each $i = 1, 2, \ldots, n$.

The Gauss-Seidel Method: Matrix Form

Re-Writing the Equations (Cont'd)

Writing all *n* equations gives

$$
\begin{aligned}
a_{11}x_1^{(k)} \qquad\qquad\qquad\qquad &= -a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{1n}x_n^{(k-1)} + b_1 \\
a_{21}x_1^{(k)} + a_{22}x_2^{(k)} \qquad\qquad &= \qquad\qquad -a_{23}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2 \\
\vdots \qquad\qquad\qquad\qquad & \\
a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \cdots + a_{nn}x_n^{(k)} &= b_n
\end{aligned}
$$

With the definitions of *D*, *L*, and *U* given previously, we have the Gauss-Seidel method represented by

$$(D - L)\mathbf{x}^{(k)} = U\mathbf{x}^{(k-1)} + \mathbf{b}$$

The Gauss-Seidel Method: Matrix Form

$$(D - L)\mathbf{x}^{(k)} = U\mathbf{x}^{(k-1)} + \mathbf{b}$$

Re-Writing the Equations (Cont'd)

Solving for $\mathbf{x}^{(k)}$ finally gives

$$\mathbf{x}^{(k)} = (D - L)^{-1}U\mathbf{x}^{(k-1)} + (D - L)^{-1}\mathbf{b}, \quad \text{for each } k = 1, 2, \ldots$$

Letting $T_g = (D - L)^{-1}U$ and $\mathbf{c}_g = (D - L)^{-1}\mathbf{b}$, gives the Gauss-Seidel technique the form

$$\mathbf{x}^{(k)} = T_g\mathbf{x}^{(k-1)} + \mathbf{c}_g$$

For the lower-triangular matrix $D - L$ to be nonsingular, it is necessary and sufficient that $a_{ii} \neq 0$, for each $i = 1, 2, \ldots, n$.