Chapter 1 Vector Analysis

1.1 Vector Algebra

1.2 Differential Calculus

1.3 Integral Calculus

1.4 Curvilinear Coordinate

1.5 The Dirac Delta Function

1.6 The Theory of Vector Fields

1.1 Vector Algebra

1.1.1 Scalar , Vector , Tensor

1.1.2 Vector Operation

1.1.3 Triple Products

1.1.4 Vector Transform

1.1.1 Scalar , Vector , Tensor

Scalar:	**3**	$\longrightarrow$	**magnitude , 0 direction**	
Vector:	$\vec{A}$	$\longrightarrow$	**magnitude , 1 direction**	**Tensor**
Function :	$\overleftrightarrow{A}$	$\longrightarrow$	**magnitude , 2 direction**	

all quantities are tensor .

1.1.1

Scalar : $a = \sum_{i=1}^{n} a_i i$ $\quad i \Rightarrow$ **base of number system**

Vector : $\vec{A} = \sum_{i=1}^{n} A_i \hat{e}_i$ $\quad \hat{e}_i \Rightarrow$ **base of coordinates**

function : $f = \sum_{i=1}^{n} a_i f_i$ $\quad f_i \Rightarrow$ **base of functions**

Any component of the base is independent to rest of the base

⇨ **orthogonal**

that is ,

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} \begin{cases} \mathbf{1} & i = j \\ \mathbf{0} & i \neq j \end{cases}$$

1.1.1

Tensor **:**

magnitude , 0 direction , 0 rank tensor → scale

[no dimension]

magnitude , 1 direction , 1^{st} rank tensor → vector

[in N dimention space : N components]

magnitude , 2 direction , 2^{nd} rank tensor → 2^{nd} rank tensor

[in N dimention space : N^2 component]

magnitude , 3 direction , 3^{rd} rank tensor → 3^{rd} rank tensor

[in N dimention space : N^3 component]

. . .

. . .

. . .

1.1.2 Vector operation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots\cdots$$

$$c_1 = a_1 + b_1 = 2 + 4 = 6$$

- Addition

$$\begin{cases} \vec{C} = \vec{A} + \vec{B} = \sum_{i=1}^{n} a_i \hat{e}_i + \sum_{i=1}^{n} b_i \hat{e}_i = \sum_{i=1}^{n} (a_i + b_i)\hat{e}_i = \sum_{i=1}^{n} c_i \hat{e}_i \\ c_i = a_i + b_i \end{cases}$$

$\Rightarrow$ Vector addition=sum of components

Ex.

$$\vec{A} = 2\hat{i} + 3\hat{j} \ ; \quad \vec{B} = 4\hat{i} + 5\hat{j} \ ; \quad \vec{C} = c_1\hat{i} + c_2\hat{j} = \vec{A} + \vec{B}$$

$$\Rightarrow \quad c_1 = a_1 + b_1 = 2 + 4 = 6 \ ; \quad c_2 = a_2 + b_2 = 3 + 5 = 8$$

- Inner product

$$c = \vec{A}\cdot\vec{B} = \left(\sum_{i=1}^{n}\vec{a}_i\hat{e}_i\right)\cdot\left(\sum_{j=1}^{n}\vec{b}_j\hat{e}_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}a_i b_j\left(\hat{e}_i\cdot\hat{e}_j\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}a_i b_j\delta_{ij} = \sum_{i=1}^{n}a_i b_i = |A||B|\cos\theta$$

Ex.

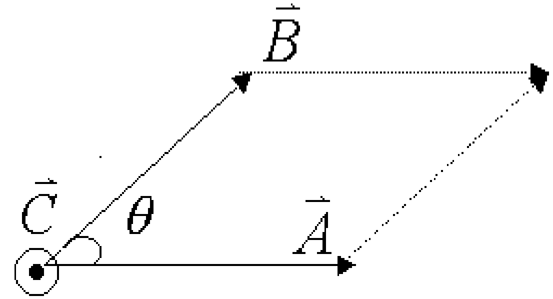
$$\vec{A} = 2\hat{i} + 3\hat{j} \;;\quad \vec{B} = 4\hat{i} + 5\hat{j}$$

$$C = \vec{A}\cdot\vec{B} = \left(2\hat{i} + 3\hat{j}\right)\cdot\left(4\hat{i} + 5\hat{j}\right)$$

$$= 2\hat{i}\cdot 4\hat{i} + 2\hat{i}\cdot 5\hat{j} + 3\hat{j}\cdot 4\hat{i} + 3\hat{j}\cdot 5\hat{j}$$

$$= 2\cdot 4 + 3\cdot 5 = 8 + 15 = 23$$

1.1.2



- Cross product

$$\vec{C} = \vec{A}\times\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta\,\hat{\odot}$$

$$\left|\vec{C}\right| = \left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta =$$ Area of 

$$\vec{C} = \vec{A}\times\vec{B} = \begin{vmatrix} \hat{e}_i & \hat{e}_j & \hat{e}_k \\ a_i & a_j & a_k \\ b_i & b_j & b_k \end{vmatrix} = \hat{e}_i\left(a_j b_k - a_k b_j\right) + \hat{e}_j\left(a_k b_j - a_i b_k\right) + \hat{e}_k\left(a_i b_i - a_j b_i\right)$$

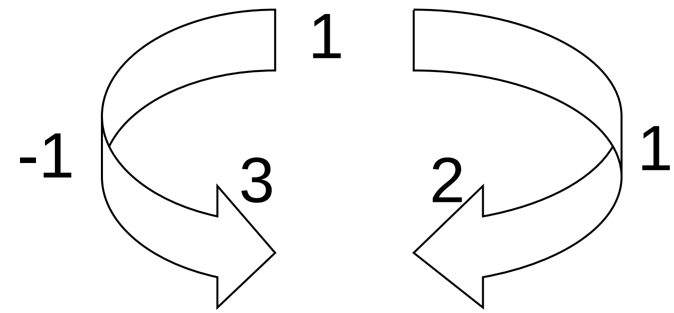
$$= \sum_{\substack{i=1\\ j=1\\ k=1}}^{n} \varepsilon_{ijk}\hat{e}_i a_j b_k = \varepsilon_{ijk}\hat{e}_i a_j b_k$$

1.1.2

$$\vec{A}\times\vec{B}=\varepsilon_{ijk}\hat{e}_i a_j b_k$$

Define :

$$\varepsilon_{ijk}\begin{cases} = 1 & \text{clockwise} \quad i\neq j\neq k \\ = -1 & \text{counterclockwise} \quad i\neq j\neq k \\ = 0 & i=j \ \text{ or } \ j=k \ \text{ or } \ k=i \end{cases}$$



1.1.3 triple products

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=a_i\left(\vec{B}\times\vec{C}\right)_i=a_i\left[\varepsilon_{ijk}b_jc_k\right]=\varepsilon_{ijk}a_ib_jc_k$$

$$=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=ABC$$

(volume enclosed by vectors $\vec{A}$, $\vec{B}$,and $\vec{C}$)

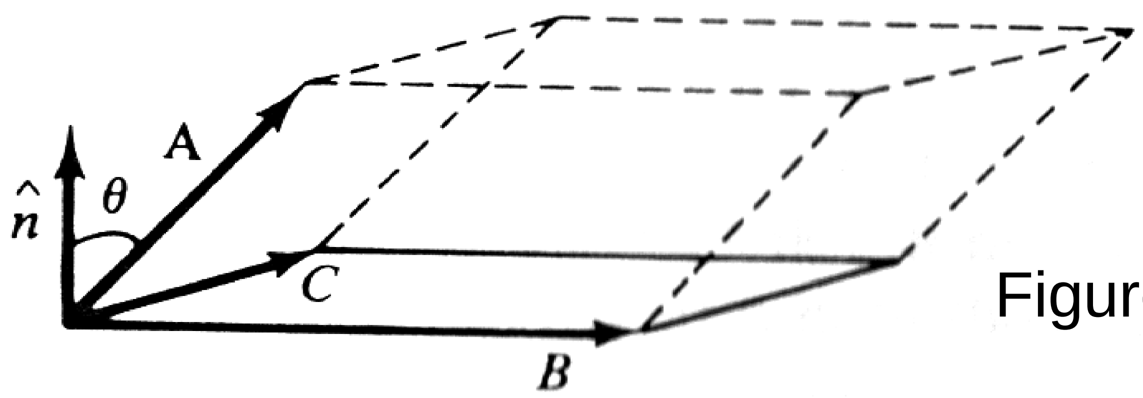
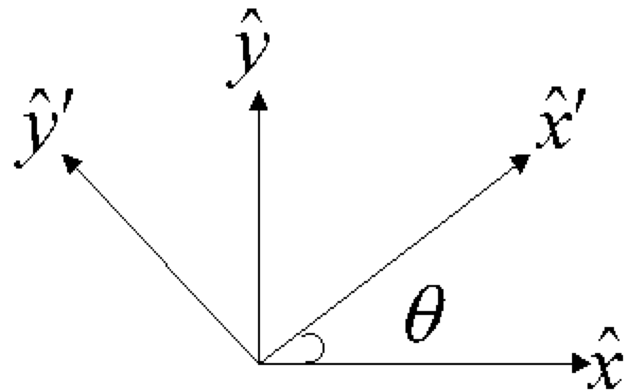


Figure 1.12

1.1.4 Vector transform

$$\vec{A} = \left(A_x \hat{x} + A_y \hat{y} \right)$$

$$\vec{A}' = \left(A'_x \hat{x}' + A'_y \hat{y}' \right)$$



$$\begin{pmatrix} A'_1 \\ A'_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

$$\vec{A}' = \overleftrightarrow{R} \cdot \vec{A} \qquad \vec{B}' = \overleftrightarrow{R} \cdot \vec{B}$$

$$\begin{pmatrix} A'_x \\ A'_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} \qquad \begin{pmatrix} B'_x \\ B'_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} B_x \\ B_y \end{pmatrix}$$

Vector transform $A'_i = R_{ij} A_j$ $\qquad \overleftrightarrow{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

1.2 Differential calculus

1.2.1 Differential Calculus for Rotation

1.2.2 Ordinary Derivative

1.2.3 Gradient

1.2.4 Divergence

1.2.5 The Curl

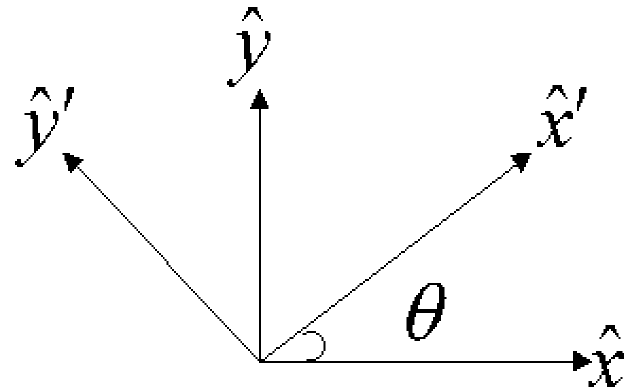
1.2.6 Product Rules

1.2.7 Second Derivatives

1.2.1 Differential Calculus for Rotation

$$\vec{A}\cdot\vec{B} = |A||B|\cos\Phi = A_x B_x + A_y B_y$$

$$\vec{A}'\cdot\vec{B}' = |A'||B'|\cos\Phi' = A'_x B'_x + A'_y B'_y$$



$$\begin{cases} \hat{x} = \cos\theta\,\hat{x}' - \sin\theta\,\hat{y}' \\ \hat{y} = \sin\theta\,\hat{x}' + \cos\theta\,\hat{y}' \end{cases}$$

$$\hat{x} = \hat{x}'\frac{\partial\hat{x}}{\partial\hat{x}'} + \hat{y}'\frac{\partial\hat{x}}{\partial\hat{y}'} = \cos\theta\,\hat{x}' - \sin\theta\,\hat{y}'$$

$$\hat{y} = \hat{x}'\frac{\partial\hat{y}}{\partial\hat{x}'} + \hat{y}'\frac{\partial\hat{y}}{\partial\hat{y}'} = \sin\theta\,\hat{x}' + \cos\theta\,\hat{y}'$$

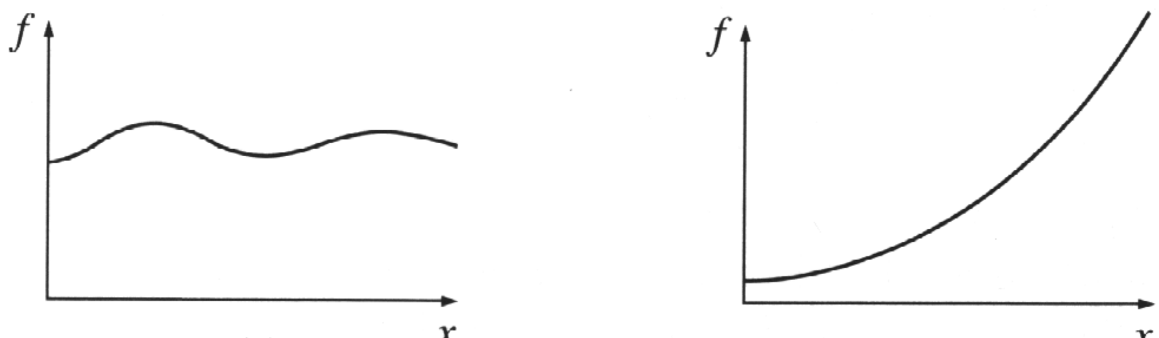
$$\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} = \begin{pmatrix}\frac{\partial\hat{x}}{\partial\hat{x}'} & \frac{\partial\hat{x}}{\partial\hat{y}'}\\ \frac{\partial\hat{y}}{\partial\hat{x}'} & \frac{\partial\hat{y}}{\partial\hat{y}'}\end{pmatrix}\begin{pmatrix}\hat{x}'\\ \hat{y}'\end{pmatrix}$$

1.2.2 Ordinary Derivative

$$df = (\frac{df}{dx})dx$$

Geometrical Interpretation:

The derivative df/dx is the slope of the graph of f versus x



(a) (b)

1.2.3 Gradient

For a function $f(x, y)$

$$df = \frac{df}{dx}\cdot dx + \frac{df}{dy}\cdot dy = \frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy = \left(\frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y\right)\cdot\underbrace{(dx\hat{e}_x + dy\hat{e}_y)}_{=d\vec{l}}$$

$$= \left(\hat{e}_x\frac{\partial}{\partial x} + \hat{e}_y\frac{\partial}{\partial y}\right) f \cdot d\vec{l} = \left(\vec{\nabla} f\right)\cdot d\vec{l}$$

$$\vec{\nabla} = \hat{e}_x\frac{\partial}{\partial x} + \hat{e}_y\frac{\partial}{\partial y}$$

Ex. f =xy $\Longrightarrow \dfrac{\partial f}{\partial x} = y$; $\dfrac{\partial f}{\partial y} = x$

$f=xy^2$ $\Longrightarrow \dfrac{\partial f}{\partial x} = y^2$; $\dfrac{\partial f}{\partial y} = 2xy$

Define $\vec{\nabla} = \hat{e}_x\dfrac{\partial}{\partial x} + \hat{e}_y\dfrac{\partial}{\partial y} = \hat{e}_i\dfrac{\partial}{\partial i}$ $\vec{\nabla} f$: gradient of f

$$df = \vec{\nabla} f \cdot d\vec{l} = \left|\vec{\nabla} f\right|\left|d\vec{l}\right|\cos\theta \Rightarrow \vec{\nabla} f \; \frac{df}{d\vec{l}}$$

1.2.4 Divergence

$$\nabla \cdot \vec{v} = (\hat{e}_i \tfrac{\partial}{\partial i}) \cdot \left(\hat{e}_j v_j\right) = \tfrac{\partial}{\partial i} v_j \left(\hat{e}_i \cdot \hat{e}_j\right) = \tfrac{\partial}{\partial i} v_j \delta_{ij} = \tfrac{\partial}{\partial i} v_i$$

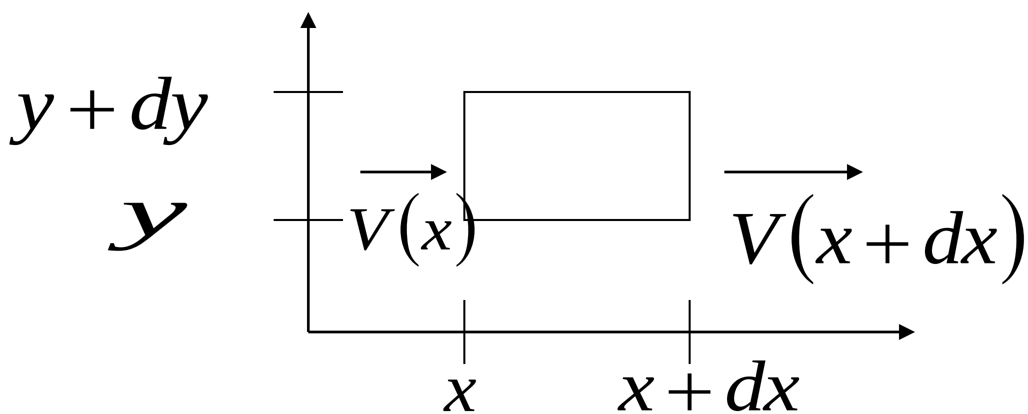
[a scalars]

$$\nabla \cdot \vec{v} = \frac{\partial}{\partial i} v_i = \partial_i v_i = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}$$

$$\frac{\partial V_x}{\partial x} = \frac{V(x+dx) - V(x)}{(x+dx) - (x)} = \frac{V(x+dx) - V(x)}{dx}$$

>0 : blow out

<0 : blow in

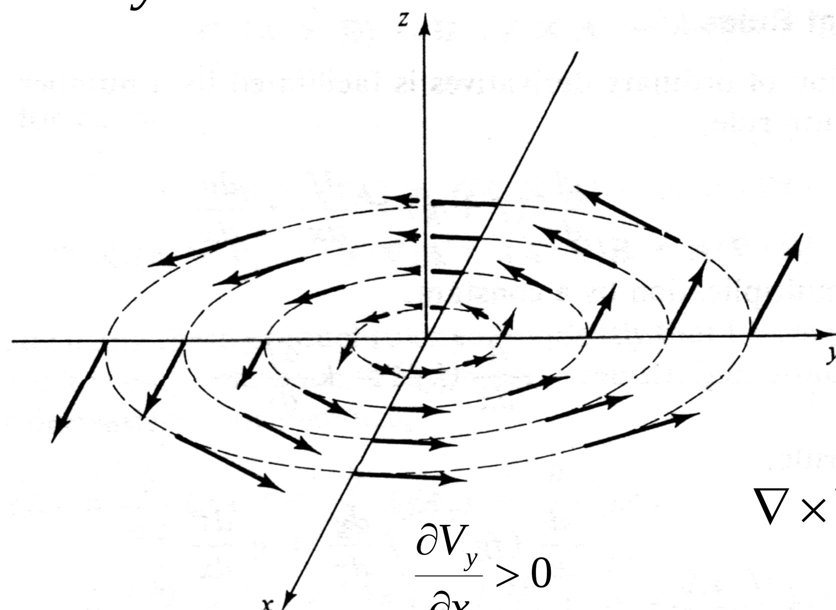


1.2.5 The Curl

$$\nabla \times \vec{v} = \varepsilon_{ijk}\partial_i v_j \hat{e}_k = \varepsilon_{ijk}\hat{e}_i \partial_j v_k$$

$$= \hat{z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) + \cdots$$

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \hat{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{j}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$



$$\frac{\partial V_x}{\partial y} < 0$$

$$\frac{\partial V_y}{\partial x} > 0$$

$$\nabla \times \vec{V} = \hat{z}\left(\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y}\right) > 0$$

Figure 1.19

1.2.6 Product Rules

The similar relations between
calculation of derivatives and vector derivatives

(1)Sum rules

$$\frac{d}{dx}(f+g)=\frac{df}{dx}+\frac{dg}{dx}$$

$$\nabla(f+g)=\nabla f+\nabla g$$

$$\nabla\cdot(\vec{A}+\vec{B})=(\nabla\cdot\vec{A})+(\nabla\cdot\vec{B})$$

$$\nabla\times(\vec{A}+\vec{B})=(\nabla\times\vec{A})+(\nabla\times\vec{B})$$

(2)The rule for multiplication by a constant

$$\frac{d}{dx}(kf)=k\frac{df}{dx}$$

$$\nabla(kf)=k\nabla f$$

$$\nabla\cdot(k\vec{A})=k(\nabla\cdot\vec{A})$$

$$\nabla\times(k\vec{A})=k(\nabla\times\vec{A})$$

1.2.6 (2)

(3) Product rules

there are six product rules:two each for gradient, divergence and curl.

Product rule for divergence:

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

two product rules for gradient

$f \quad g$ (produce of two scalar functions)

$\vec{A} \cdot \vec{B}$ (dot product of two vectors)

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A})$$

$$+(\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A}$$

1.2.6 (3)

two product rules for divergences

$f \ \vec{A}$ (scalar times vector)

$(\vec{A} \times \vec{B})$ (cross product of two vector functions)

$$\nabla \cdot (fA) = f(\nabla \cdot A) + A \cdot (\nabla f)$$

$$\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$$

two product rules for curls

$$\nabla \times (f\vec{A}) = f(\nabla \times \vec{A}) - \vec{A} \times (\nabla f)$$

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B} + \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A})$$

1.2.6 (4)

(4)The quotient rule

The quotient rule for derivative:

$$\frac{d}{dx}\left(\frac{f}{g}\right)=\frac{g\frac{df}{dx}-f\frac{dg}{dx}}{g^2}$$

The quotient rules for gradient,divergence ,and curl

$$\nabla\left(\frac{f}{g}\right)=\frac{g\nabla f-f\nabla g}{g^2}$$

$$\nabla\cdot\left(\frac{\vec{A}}{g}\right)=\frac{g(\nabla\cdot\vec{A})-\vec{A}\cdot(\nabla g)}{g^2}$$

$$\nabla\times\left(\frac{\vec{A}}{g}\right)=\frac{g(\nabla\times\vec{A})+\vec{A}\times(\nabla g)}{g^2}$$

1.2.7 Second Derivatives

$\nabla = \hat{e}_i \partial_i$ a derivative vector

1. Divergence of gradient:

$$\nabla \cdot (\nabla T) = \nabla \cdot (\hat{e}_j \partial_j T) = (\partial_i \hat{e}_i) \cdot (\hat{e}_j \partial_j T)$$

$$= (\hat{e}_i \cdot \hat{e}_j) \partial_i \partial_j T = \delta_{ij} \partial_i \partial_j T = \partial_i{}^2 T$$

(inner product of same vector; $\vec{A} \cdot \vec{A} = A^2$)

so

$$\nabla^2 T = \partial_x{}^2 T + \partial_y{}^2 T + \partial_z{}^2 T$$

$$\nabla^2 \vec{V} = \nabla^2 (\hat{e}_i V_i) = (\nabla^2 V_j) \hat{e}_j$$

∇^2 is called Laplacian,
T is a scalar; $\vec{V}$ is a vector

1.2.7 (2)

2. Curl of gradient :

$$\nabla \times (\nabla T) = (\partial_i \hat{e}_i) \times (\partial_j T \hat{e}_j)$$

$$= \in_{kij} \hat{e}_k \partial_i \partial_j T$$

$$= \hat{e}_1 (\in_{123} \partial_2 \partial_3 T + \in_{132} \partial_3 \partial_2 T) + \hat{e}_2 (\in_{213} \partial_1 \partial_3 T + \in_{231} \partial_3 \partial_1 T)$$

$$+ \hat{e}_3 (\in_{312} \partial_1 \partial_2 T + \in_{321}{}^{a} \partial_2 \partial_1 T)$$

$$= \hat{e}_1 (\partial_2 \partial_3 T - \partial_3 \partial_2 T) + \hat{e}_2 (\partial_3 \partial_1 T - \partial_1 \partial_3 T) + \hat{e}_3 (\partial_1 \partial_2 T - \partial_2 \partial_1 T)$$

$$= 0$$

(cross product of same vector, $\vec{A} \times \vec{A} = 0$)

1.2.7 (3)

3. gradient of divergence: $\nabla(\nabla \cdot \vec{V})$

$$\nabla^2\vec{V} = (\nabla \cdot \nabla)\vec{V} \neq \nabla(\nabla \cdot \vec{V})$$

4. divergence of curl

$$\nabla \cdot (\nabla \times \vec{V}) = (\partial_l \hat{e}_l) \cdot (\in_{kij} \hat{e}_k \partial_i V_j) = \in_{kij} \delta_{lk} \partial_l \partial_i V_j$$

$$= \in_{ijk} \partial_k \partial_i V_j = \sum_{\substack{j=1,cw \\ i \neq j \neq k}}^{3} (\partial_k \partial_i - \partial_i \partial_k) V_j = 0$$

(similar to $\vec{A} \cdot (\vec{A} \times \vec{V}) = 0$)

5.Curl of curl

$$\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V}$$

Poof: $\nabla \times (\nabla \times \vec{V}) = \nabla \times (\in_{ijk} \partial_i V_j \hat{e}_k)$

$$= \hat{e}_l \in_{lmk} \partial_m \in_{ijk} \partial_i V_j$$

$$= \hat{e}_l (\delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi}) \partial_m \partial_i V_j$$

$$= \hat{e}_i \partial_j \partial_i V_j - \hat{e}_j \partial_i^{\,2} V_j$$

$$= \hat{e}_i \partial_i (\partial_j V_j) - (\partial_i^2 V_j) \hat{e}_j$$

$$= \hat{e}_i \partial_i (\nabla \cdot V) - (\nabla^2 V_j) \hat{e}_j$$

$$= \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V}$$

(similar to $\vec{A} \times (\vec{A} \times \vec{V})$)

1.3 Integral Calculus

1.3.1 Line, Surface, and Volume Integrals

1.3.2 The Fundamental Theorem of Calculus

1.3.3 The Fundamental Theorem of Gradients

1.3.4 The Fundamental Theorem of Divergences

1.3.5 The Fundamental Theorem for Curls

1.3.6 Relations Among the Fundamental Theorems

1.3.7 Integration by parts

1.3.1 Line,Surface,and Volume Integrals

(a) Line integrals

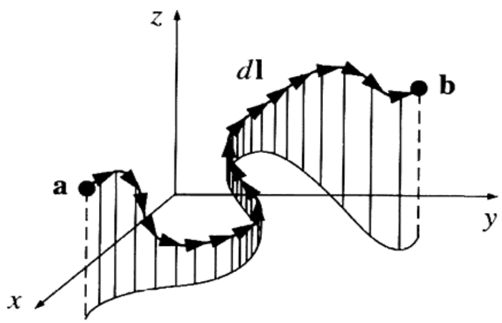


Figure 1.20

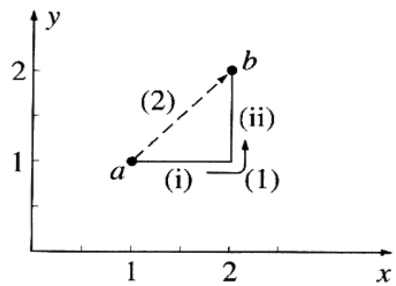


Figure 1.21

$$\int_{aP}^{b} \vec{V} \cdot d\vec{l}$$

P:the path(e.g.(1) or (2))

(b)Surface Integrals

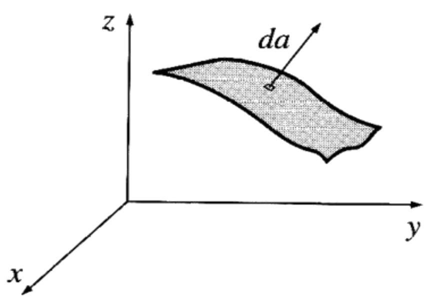


Figure 1.22

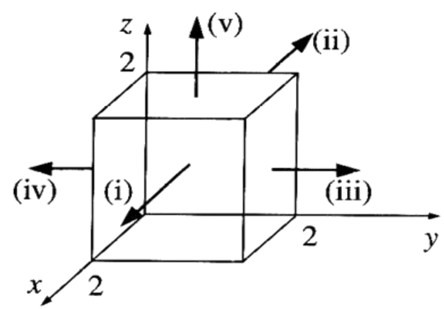


Figure 1.23

$$\int_{s} \vec{V} \cdot d\vec{a}$$

S:the surface of integral

1.3.1 (2)

(c)Volume Integral

$$\int_V T d\tau \qquad d\tau = dx\,dy\,dz$$

$$\int \vec{V} d\tau = \int (V_x\hat{x} + V_y\hat{y} + V_z\hat{z}) d\tau = \hat{x}\int V_x d\tau + \hat{y}\int V_y d\tau + \hat{z}\int V_z d\tau$$

Example1.8

$$T = x\,y\,z^2 \qquad \int_v T d\tau = ?$$

Solution:

$$\int T\ d\tau = \int_0^3 z^2\{\int_0^1 y[\int_0^{1-y} x\,dx]dy\}dz =$$

$$\frac{1}{2}\int_0^3 z^2\ dz \int_0^1 (1-y)^2 y\,dy = \frac{1}{2}(9)(\frac{1}{12}) = \frac{3}{8}$$

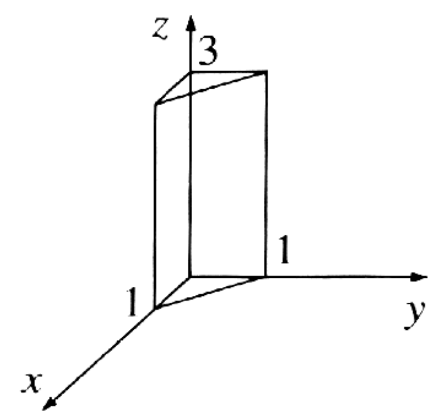


Figure 1.24

1.3.2 The Fundamental Theorem of Calculus

Suppose f(x) is a function of one variable.
The fundamental theorem of calculus:

$$\int_a^b \left(\frac{df}{dx}\right)dx = f(b) - f(a)$$

$$\int_a^b F(x)dx = f(b) - f(a) \qquad F(x) = \frac{df}{dx}$$

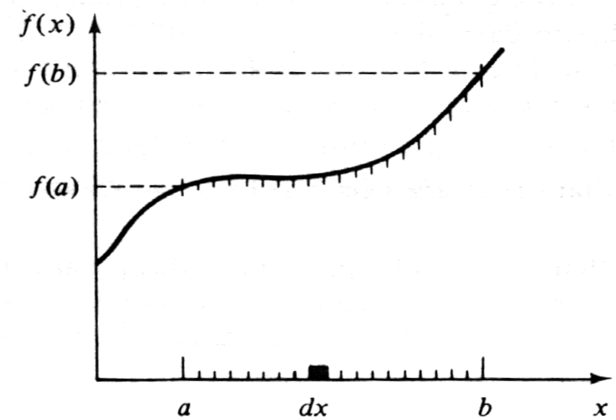


Figure 1.25

1.3.3 The Fundamental Theorem of Gradients

Suppose we have a scalar function of three variables T(x,y,z)
We start at point (a_x, a_y, a_z),and make the journey to point (b_x, b_y, b_z)

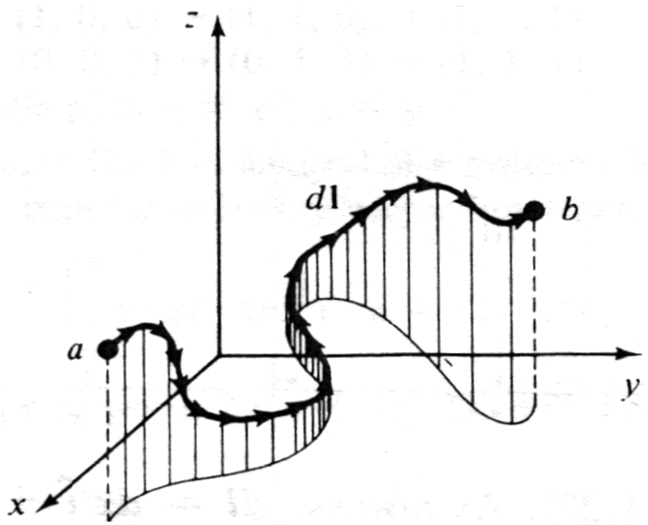


Figure 1.26

$$dT = (\nabla T) \cdot dl$$

$$\int_a^b (\nabla T) \cdot dl = T(b) - T(a)$$

(1)path independent
(2)$\oint (\nabla T) \cdot dl = 0$,a closed loop(a=b)

A line is bounded by two points

1.3.4 The Fundamental Theorem of Divergences

$$\int_{volumn} (\nabla \cdot \vec{V}) d\tau = \oint_{surface} \vec{V} \cdot d\vec{a}$$

Proof:

$$\int_{volumn} (\nabla \cdot \vec{V}) d\tau = \int_{v} \partial_i V_i d_i d_j d_k = \int_{s} V_i d_j d_k = \int_{s} (\vec{V} \cdot \hat{i}) \, da \, \hat{i} = \int_{s} \vec{V} \cdot d\vec{a}$$

$\oint_{surface} \vec{V} \cdot d\vec{a}$ is called the flux of $\vec{V}$ through the surface.

1.3.4 (2)

Example1.10

$$\vec{V} = y^2\hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}$$

$$\int_v \nabla\cdot\vec{V}\, d\tau = \oint_s \vec{V} d\vec{a}$$

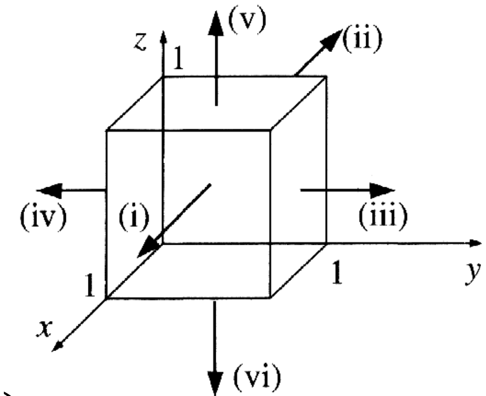


Figure 1.29

Solution:

(1)

$$\int \vec{V}\cdot d\vec{a} = \int_0^1\int_0^1 y^2\ dy\ dz = \frac{1}{3}$$

$$\int \vec{V}\cdot d\vec{a} = -\int_0^1\int_0^1 y^2\ dy\ dz = -\frac{1}{3}$$

$$\int \vec{V}\cdot d\vec{a} = \int_0^1\int_0^1 (2x + z^2)dxdz = \frac{4}{3}$$

$$\int \vec{V}\cdot d\vec{a} = -\int_0^1\int_0^1 z^2 dxdz = -\frac{1}{3}$$

$$\int \vec{V}\cdot d\vec{a} = \int_0^1\int_0^1 2ydxdy = 1$$

$$\int \vec{V}\cdot d\vec{a} = -\int_0^1\int_0^1 0dxdy = 0$$

$$\oint_s \vec{V}\cdot d\vec{a} = \frac{1}{3} - \frac{1}{3} + \frac{4}{3} - \frac{1}{3} + 1 + 0 = 2$$

(2)

$$\nabla\cdot\vec{V} = 2(x + y)$$

$$\int_v 2(x+y)d\tau = 2\int_0^1\int_0^1\int_0^1 (x+y)\,dx\,dy\,dz$$

$$\int_0^1 (x+y)\,dx = \frac{1}{2} + y \quad , \quad \int_0^1 (\frac{1}{2} + y)\,dy = 1$$

$$\int_0^1 1\,dz = 1$$

$$\int_v \nabla\cdot\vec{V} d\tau = 2$$

1.3.5 The Fundamental Theorem for Curls

$$\int_{surface} (\nabla \times \vec{V}) \cdot d\vec{a} = \oint_{\substack{boundary\\ line}} \vec{V} \cdot d\vec{l}$$

Proof:

$$\int_{surface} (\nabla \times \vec{V}) \cdot d\vec{a} = \int_{s} (\in_{ijk} \hat{e}_i \partial_j V_k) \cdot (\hat{e}_i da) = \int_{s} \partial_j V_k d_j d_k = \int_{line} V_k \cdot d_k = \int_{line} \vec{V} \cdot d\vec{l}$$

A surface is enclosed by a closed line

(1) $\int_{surface} (\nabla \times \vec{V}) \cdot d\vec{a}$ dependents only on the boundary line, not on the particular surface used.

(2) $\oint_{surface} (\nabla \times \vec{V}) \cdot d\vec{a} = 0$ for any closed surface.

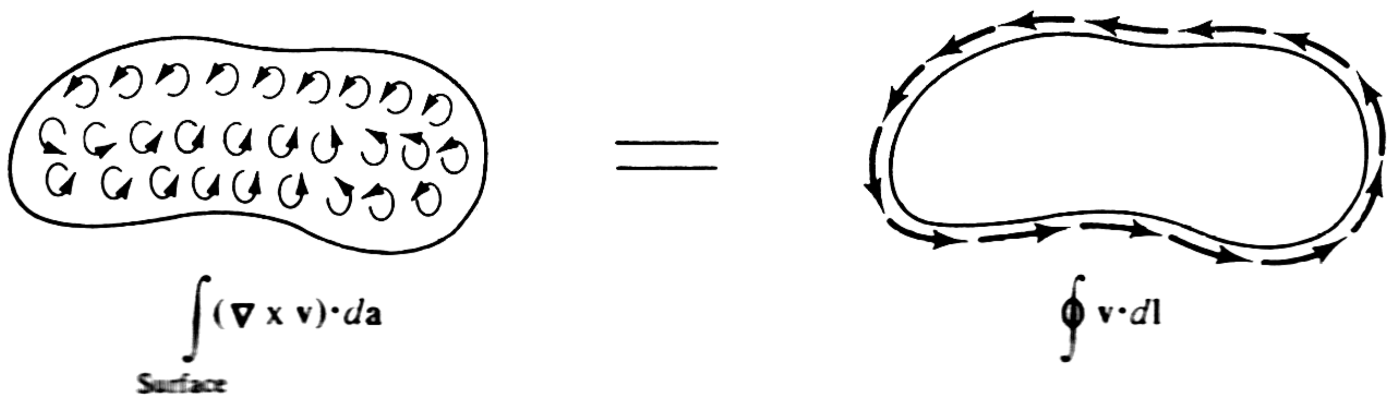


Figure 1.31

1.3.5 (2)

Examples 1.11

$$\vec{V} = (2xz + 3y^2)\hat{y} + (4yz^2)\hat{z} \qquad \int_s (\nabla \times \vec{V}) \cdot d\vec{a} = \oint_p \vec{V} \cdot d\vec{l}$$

Solution:

(1)

$$\nabla \times V = (4z^2 - 2x)\hat{x} + 2z\hat{z} \quad and \quad d\vec{a} = dy\,dz\,\hat{x}$$

$$\int_s (\nabla \times \vec{V}) \cdot d\vec{a} = \int_0^1 \int_0^1 4z^2\ dy\ dz = \frac{4}{3}$$

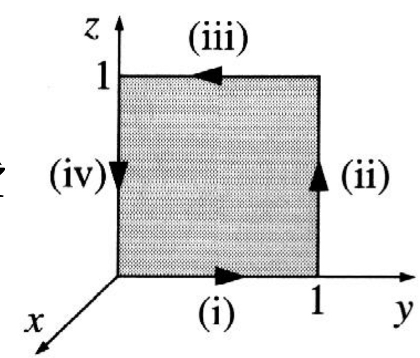


Figure 1.33

(2)

$$x = 0 \quad z = 0, \quad \vec{V} \cdot d\vec{l} = 3y^2 dy, \quad \int \vec{V} \cdot d\vec{l} = \int_0^1 3y^2 dy = 1,$$

$$x = 0 \quad y = 1, \quad \vec{V} \cdot d\vec{l} = 4z^2 dz, \quad \int \vec{V} \cdot d\vec{l} = \int_0^1 4z^2 dz = \frac{4}{3},$$

$$x = 0 \quad z = 1, \quad \vec{V} \cdot d\vec{l} = 3y^2 dy, \quad \int \vec{V} \cdot d\vec{l} = \int_1^0 3y^2 dy = -1,$$

$$x = 0 \quad y = 0, \quad \vec{V} \cdot d\vec{l} = 0, \quad \int \vec{V} \cdot d\vec{l} = \int_1^0 0 dz = 0,$$

$$\oint V \cdot dl = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3}$$

1.3.6 Relations Among the Fundamental Theorems

(1)Gradient : $$\int_a^b (\nabla T)\cdot dl = T(b) - T(a)$$

(2)Divergence : $$\int_{volumn} (\nabla \cdot \vec{V})\, d\tau = \oint_{surface} \vec{V} \cdot d\vec{a}$$

(3)Curl : $$\int_{surface} (\nabla \times \vec{V}) \cdot d\vec{a} = \oint_{\substack{boundary\\line}} \vec{V} \cdot d\vec{l}$$

- combine (1)and (3)

$$\oint_{line} (\nabla T)\cdot d\vec{l} = 0$$

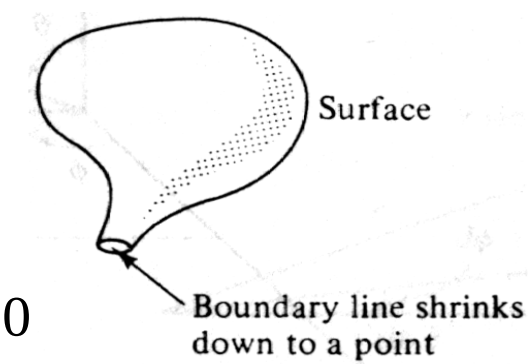
$$\Rightarrow \int_{surface} [\nabla \times (\nabla T)]\cdot d\vec{a} = 0 \quad \Rightarrow \nabla \times (\nabla T) = 0$$

Figure 1.37

- combine (3)and(2)

$$\oint_{surface} (\nabla \times \vec{V})\cdot d\vec{a} = 0$$

$$\Rightarrow \int_{volumn} [\nabla \cdot (\nabla \times \vec{V})]\, d\tau = 0 \quad \Rightarrow \quad \nabla \cdot (\nabla \times \vec{V}) = 0$$



1.3.7 Integration by parts

$$\frac{d}{dx}(fg) = f(\frac{dg}{dx}) + g(\frac{df}{dx})$$

$$\int_a^b \frac{d}{dx}(fg)dx = fg\Big|_a^b = \int_a^b f(\frac{dg}{dx})dx + \int_a^b g(\frac{df}{dx})dx,$$

$$or \quad \int_a^b f(\frac{dg}{dx})dx = -\int_a^b g(\frac{df}{dx})dx + fg\Big|_a^b$$

$$\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f)$$

$$\int \nabla\cdot(f\vec{A})\,d\tau = \int f(\nabla\cdot\vec{A})\,d\tau + \int \vec{A}\cdot(\nabla f)\,d\tau = \oint f\vec{A}\cdot d\vec{a},$$

$$or \quad \int_v f(\nabla\cdot\vec{A})\,d\tau = -\int_v \vec{A}\cdot(\nabla f)\,d\tau + \oint_s f\vec{A}\cdot d\vec{a}$$

1.4 Curvilinear Coordinates

1.4.1 General Coordinates

1.4.2 Gradient

1.4.3 Divergence

1.4.4 Curl

1.4 Curvilinear Coordinates

Spherical Polar Coordinate and Cylindrical Coordinate

(r, θ, ϕ)

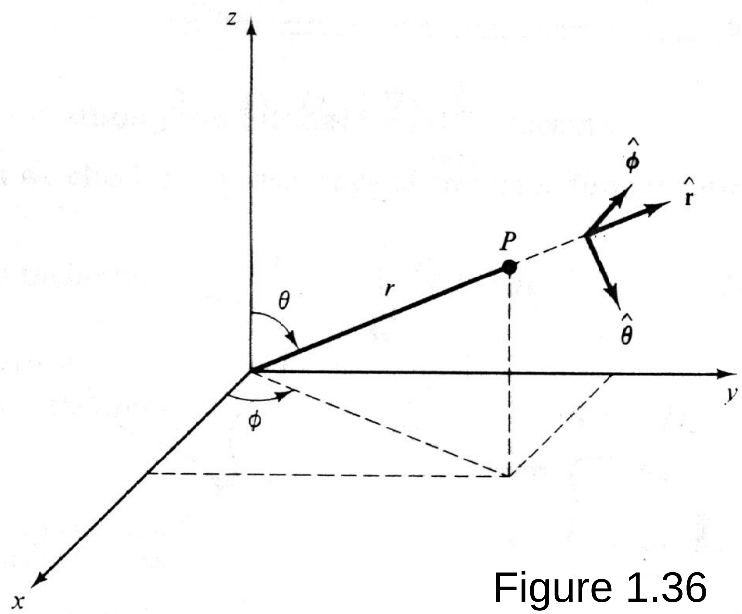


Figure 1.36

(r, ϕ, z)

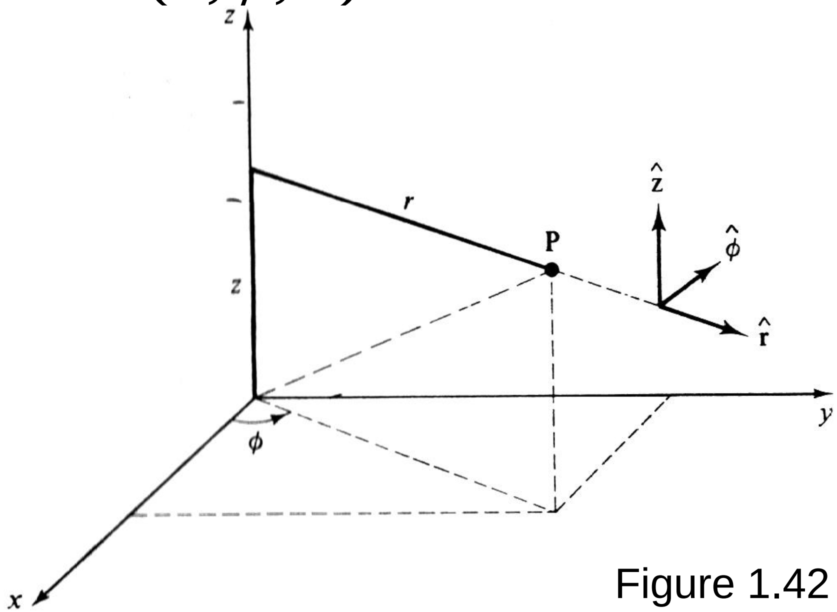


Figure 1.42

$$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

$$d\vec{l} = dx\hat{i} + dy\hat{j} + dz\hat{k} = dl_r\hat{r} + dl_\theta\hat{\theta} + dl_\phi\hat{\phi}$$

$$= dr\hat{r} + r d\theta\hat{\theta} + r\sin\theta\, d\phi\hat{\phi}$$

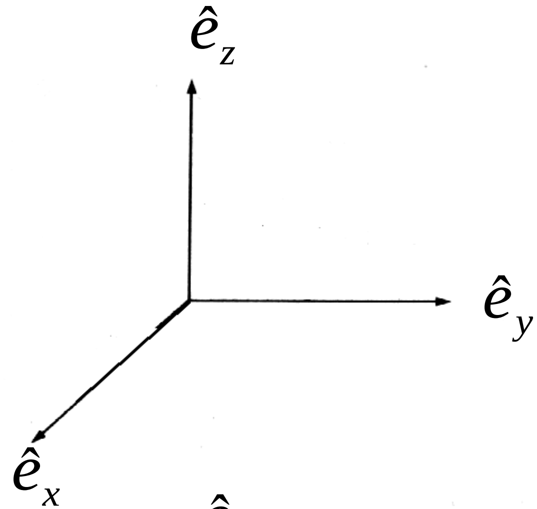
$$x = r\cos\phi \quad y = r\sin\phi \quad z = z$$

$$\vec{A} = A_r\hat{r} + A_\phi\hat{\phi} + A_z\hat{z}$$

$$dl = dr\hat{r} + r d\phi\hat{\phi} + dz\hat{z}$$

1.4.1 General Coordinates

Cartesian coordinate

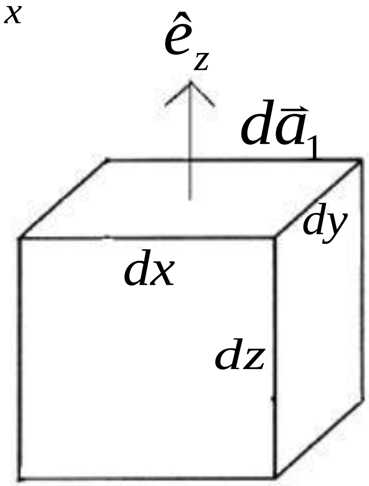


$$q_i = x, y, z \quad h_i = 1,1,1$$

$$d\vec{l} = 1 \cdot dx\,\hat{e}_x + 1 \cdot dy\,\hat{e}_y + 1 \cdot dz\,\hat{e}_z$$

$$= h_1\, dq_1\, \hat{e}_x + h_2\, dq_2\, \hat{e}_y + h_3\, dq_3\, \hat{e}_z$$

$$= h_i\, dq_i\, \hat{e}_i$$



$$d\vec{a} = dx\, dy\, \hat{e}_z = h_1\, dq_1\, h_2\, dq_2\, \hat{e}_3$$

$$d\vec{a} = \epsilon_{ijk}\, h_i\, dq_i\, h_j\, dq_j\, \hat{e}_k$$

$$d\tau = dx\, dy\, dz = h_1\, dq_1\, h_2\, dq_2\, h_3\, dq_3$$

$$= h_1\, h_2\, h_3\, dq_1\, dq_2\, dq_3$$

1.4.1 (2)

Cylindrical coordinate

$$q_i = r, \phi, z \qquad h_i = 1, r, 1$$

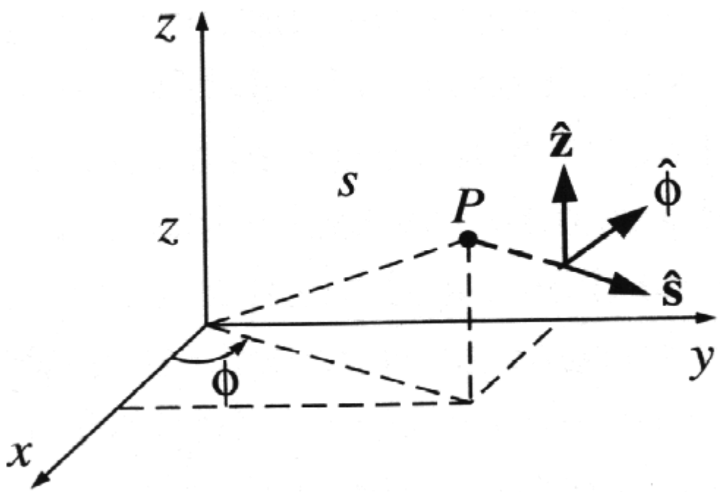


Figure 1.42

$(s \rightarrow r)$

$$dl_s = ds \qquad dl_\varphi = s d\varphi \qquad dl_z = dz$$

$$d\vec{l} = dr\,\hat{r} + r\,d\varphi\hat{\varphi} + dz\,\hat{z} = h_i\,dq_i\,\hat{e}_i$$

$$d\vec{a} = r d\varphi dz\,\hat{r}$$

$$\begin{aligned} d\tau &= dl_r dl_\phi dl_z \\ &= dr\ r d\phi\ dz \\ &= h_r dr\, h_\phi d\phi\ \ h_z dz \\ &= h_1 h_2 h_3 dq_1 dq_2 dq_3 \end{aligned}$$

1.4.1 (3)

Spherical coordinate

$$q_i = r, \theta, \phi \quad h_i = 1, r, r\sin\theta$$

$$dl_r = dr \qquad dl_\theta = r d\theta \qquad dl_\varphi = r\sin\theta\, d\varphi$$

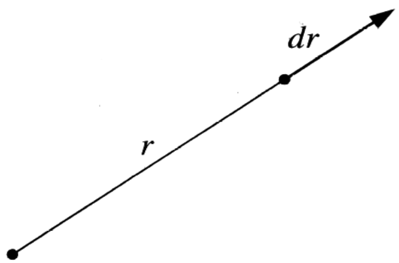
$$d\vec{l} = dr\hat{r} + r d\theta\,\hat{\theta} + r\sin\theta\, d\varphi\,\hat{\varphi}$$

$$d\vec{a}_1 = dl_\theta dl_\varphi \hat{r} = r^2 \sin\theta\, d\theta\, d\varphi\,\hat{r}$$

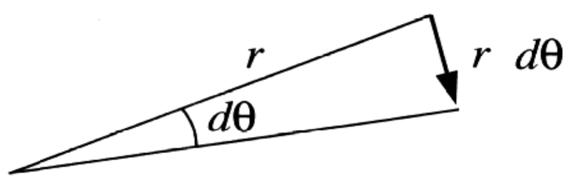
$$d\vec{a}_2 = dl_r dl_\varphi \hat{\theta} = r\, dr\, d\varphi\,\hat{\theta}$$

$$d\vec{a}_3 = dl_r dl_\theta \hat{\varphi} = r\, dr\, d\theta\,\hat{\varphi}$$

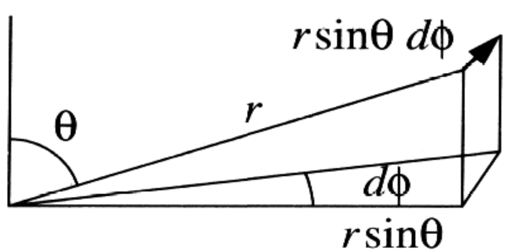
$$d\tau = dl_r dl_\theta dl_\varphi = r^2 \sin\theta\, dr\, d\theta\, d\varphi$$



(a)



(b)



(C) Figure 1.38

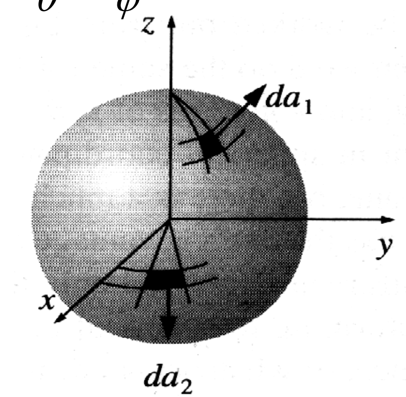


Figure 1.39

1.4.1 (4)

In summary:

$$d\vec{l} = \sum_i h_i \, dq_i \, \hat{e}_i$$

$$d\vec{a} = \in_{ijk} \, h_i dq_i h_j dq_j \hat{e}_k$$

$$d\tau = h_1 h_2 h_3 dq_1 dq_2 dq_3$$

1.4.2 Gradient

Gradient

$$dT = (\nabla T)\cdot d\vec{l}$$

$$\Rightarrow \frac{\partial T}{\partial q_i} dq_i = (\nabla T)\cdot (h_i dq_i \hat{e}_i)$$

$$\Rightarrow \nabla T = \hat{e}_i \left[\frac{(\partial T / \partial q_i) dq_i}{h_i dq_i} \right] = \hat{e}_i \frac{1}{h_i} \frac{\partial T}{\partial q_i}$$

So for $(x, y, z),\quad \nabla T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$

for $(r, \phi, z),\quad \nabla T = \hat{e}_r \frac{\partial T}{\partial r} + \hat{e}_\phi \frac{\partial T}{r\,\partial \phi} + \hat{e}_z \frac{\partial T}{\partial z}$

for $(r, \theta, \phi),\quad \nabla T = \hat{e}_r \frac{\partial T}{\partial r} + \hat{e}_\theta \frac{\partial T}{r\,\partial \theta} + \hat{e}_\phi \frac{\partial T}{r \sin\theta\, \partial \phi}$

1.4.3 Divergence

$$\int_V (\nabla \cdot \vec{V}) d\tau = \oint_s \vec{V} \cdot d\vec{a}$$

$$(\nabla \cdot \vec{V}) \prod h_i dq_i = \vec{V} \cdot d\vec{a}$$

$$\sum_k \left| \varepsilon_{ijk} \right| V_k h_i dq_i h_j dq_j$$

$$(\nabla \cdot \vec{V}) h_1 dq_1 h_2 dq_2 h_3 dq_3 = (V_1 \cdot A_1 + V_2 \cdot A_2 + V_3 \cdot A_3)$$

$$= V_1 h_2 dq_2 h_3 dq_3 + V_2 h_3 dq_3 h_1 dq_1 + V_3 h_1 dq_1 h_2 dq_2$$

$$\nabla \cdot \vec{V} = \frac{V_1 h_2 dq_2 h_3 dq_3 + V_2 h_3 dq_3 h_1 dq_1 + V_3 h_1 dq_1 h_2 dq_2}{h_1 dq_1 h_2 dq_2 h_3 dq_3}$$

$$= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \underbrace{(V_1 h_2 h_3)}_{A_1} + \frac{\partial}{\partial q_2} \underbrace{(V_2 h_1 h_3)}_{A_2} + \frac{\partial}{\partial q_3} \underbrace{(V_3 h_1 h_2)}_{A_3} \right]$$

1.4.3 (2)

$$\frac{d}{d\tau}\int(\nabla\cdot\vec{V})d\tau=\frac{d}{d\tau}\oint_S\vec{V}\cdot d\vec{a}$$

$$\nabla\cdot\vec{V}=\frac{\partial}{\prod_l h_l\partial q_l}\int\sum_k V_k\cdot(da)_k$$

$$=\frac{\partial}{\prod_l h_l\partial q_l}\int\sum_k V_k(h_i\,\partial q_i\,h_j\,\partial q_j\,\varepsilon_{ijk})$$

$$=\frac{\partial}{h_1h_2h_3\partial q_1\partial q_2\partial q_3}\int(V_1h_2\partial q_2\,h_3\partial q_3+V_2\,h_3\partial q_3\,h_1\partial q_1+V_3\,h_1\partial q_1\,h_2\partial q_2)$$

$$=\frac{1}{h_1h_2h_3}\left[\frac{\partial V_1h_2h_3}{\partial q_1}+\frac{\partial V_2h_3h_1}{\partial q_2}+\frac{\partial V_3h_1h_2}{\partial q_3}\right]$$

$$q_i=x,y,z\quad \nabla\cdot\vec{V}=\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}$$

$$q_i=r,\phi,z\quad \nabla\cdot\vec{V}=\frac{1}{r}\left[\frac{\partial}{\partial r}(V_r\cdot r)+\frac{\partial}{\partial\phi}(V_\phi\cdot 1)+\frac{\partial}{\partial z}(V_z\cdot r)\right]$$

$$q_i=r,\theta,\phi\quad \nabla\cdot\vec{V}=\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(V_r r^2\sin\theta)+\frac{\partial}{\partial\theta}(V_\theta r\sin\theta)+\frac{\partial}{\partial\phi}(V_\phi r)\right]$$

1.4.4 Curl

$$\int_s (\nabla \times \vec{V}) \cdot d\vec{a} = \oint_c \vec{V} \cdot d\vec{l} \Rightarrow \frac{d}{d\vec{a}} \int (\nabla \times \vec{V}) \cdot d\vec{a} = \frac{d}{da} \int (\vec{V} \cdot d\vec{l})$$

$$(\nabla \times \vec{V}) \cdot \varepsilon_{ijk} h_i \partial q_i h_j \partial q_j \hat{e}_k = \sum_j V_j \cdot h_j \partial q_j$$

$$\nabla \times \vec{V} = \frac{1}{h_i h_j h_k} \varepsilon_{ijk} [\frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} (\partial q_j h_j V_j)] h_k \hat{e}_k$$

$$= \frac{1}{h_i h_j h_k} \varepsilon_{ijk} \frac{\partial (V_j h_j)}{\partial q_i} h_k \hat{e}_k$$

1.4.4 (2)

x, y, z

$$\nabla \times \vec{v} = (\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z})\hat{x} + (\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x})\hat{y}$$

$$+ (\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y})\hat{z}$$

r, ϕ, z

$$\nabla \times \vec{v} = (\frac{1}{r}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z})\hat{r} + (\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r})\hat{\phi}$$

$$+ \frac{1}{r}[\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial \phi}]\hat{z}$$

r, θ, ϕ

$$\nabla \times \vec{v} = \frac{1}{r\sin\theta}[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}]\hat{r}$$

$$+ \frac{1}{r}[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)]\hat{\theta}$$

$$+ \frac{1}{r}[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}]\hat{\phi}$$

1.5 The Direc Delta Function

1.5.1 The Definition of the Delta Function

1.5.2 Some Properties of the Delta Function

1.5.3 The Three-Dimensional Delta Function

1.5.4 The Divergence of $\hat{r}/r^2$

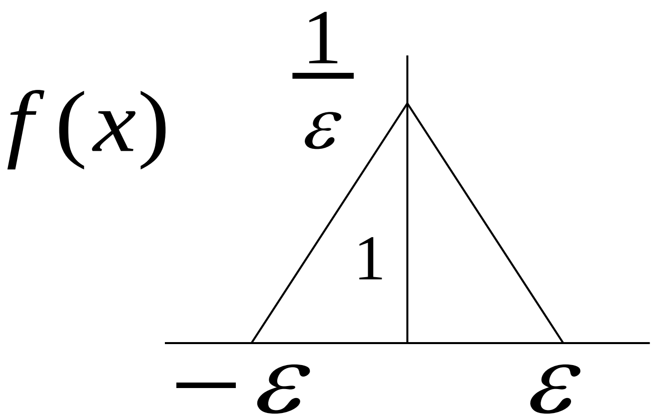
1.5.1 The Definition of the Delta Function

The definition of Delta function ：

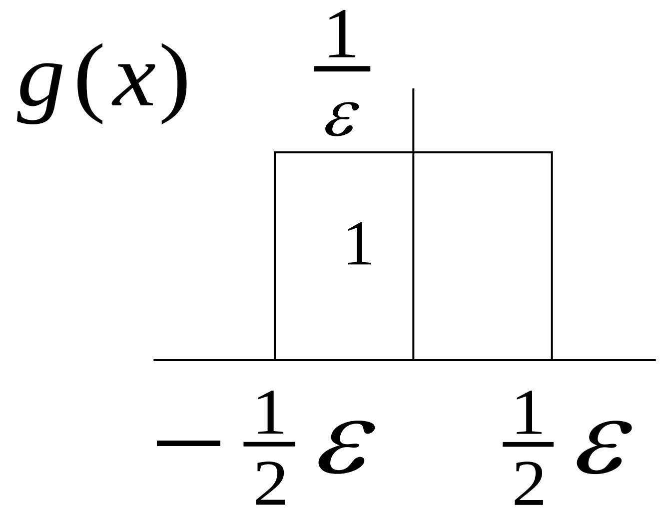
$$\delta(x) = 0 \qquad x \neq 0$$

$$\delta(x) = \infty \qquad x = 0$$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$



$$\lim_{\varepsilon \to 0} f(x) = \delta(x)$$



$$\lim_{\varepsilon \to 0} g(x) = \delta(x)$$

Figure 1.46

1.5.1 (2)

- **Definition with shifted variable.**

$$\delta(x) \overset{x=y-a}{=} \delta(y-a) = 0 \qquad y \neq a$$

$$\delta(x) \overset{x=y-a}{=} \delta(y-a) = \infty \qquad y = a$$

$$\int_{-\infty}^{\infty} \delta(y-a)d(y-a) = 1 = \int_{-\infty}^{\infty} \delta(y-a)dy$$

1.5.2 Some Properties of the Delta Function

- $$\delta(ky) = \frac{1}{|k|}\delta(y)$$

$$1 = \int_{-\infty}^{\infty} \delta(x)dx = \int_{-\infty}^{\infty} \delta(ky)\cdot d(ky) = |k|\int_{-\infty}^{\infty} \delta(ky)dy$$

$$1 = \int_{-\infty}^{\infty} \delta(y)dy = |k|\int_{-\infty}^{\infty} \delta(ky)dy = 1$$

- $$\delta(y) = |k|\,\delta(k\,y) \Rightarrow \delta(k\,y) = \frac{1}{|k|}\delta(y)$$

- $$\int_{-\infty}^{\infty} f(x)\delta(x)dx = \int_{-\infty}^{\infty} f(0)\delta(x)dx = f(0)\int_{-\infty}^{\infty} \delta(x)dx = f(0)$$

- $$\int_{-\infty}^{\infty} f(x)\delta'(x)dx = \underbrace{f(x)\delta(x)\Big|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} f'(x)\delta(x)dx = -f'(0)$$

- $$f(x)\delta(x-a) = f(a)\delta(x-a)$$

- $$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

1.5.3 The Three-Dimensional Delta Function

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

$$\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

$$\int_{all\ space} \delta^3(\vec{r})d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)dxdydz$$

$$= 1$$

$$\int_{all\ space} f(\vec{r})\delta^3(\vec{r}-\vec{a})d\tau = f(\vec{a})$$

1.5.4 The Divergence of $\hat{r}/r^2$

$$\vec{V} = \frac{\hat{r}}{r^2} \qquad (r \neq 0)$$

$$\nabla \cdot \vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

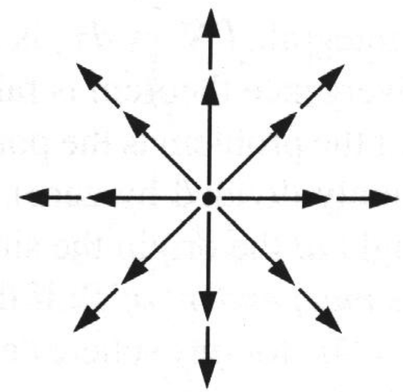


Figure 1.44

$$\int_V \nabla \cdot \vec{V}\, d\tau = \oint_S \vec{V} \cdot d\vec{a}$$

$$= \oint_S \left(\frac{1}{R^2}\right) \cdot (R^2 \sin\theta\, d\theta\, d\phi) = 4\pi$$

1.5.4 (2)

$$\int_V \frac{1}{4\pi} \nabla \cdot \vec{V}\, d\tau = 1$$

$$\Rightarrow \frac{1}{4\pi} \nabla \cdot \vec{V} = \delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

$$\Rightarrow \frac{1}{4\pi} \nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = \delta^3(\vec{r})$$

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$$

$$\nabla \cdot \left(\frac{\hat{r}'}{r'^2}\right) = 4\pi\delta^3(\vec{r}') \qquad \vec{r}' = \vec{r} - \vec{r}_0$$

$$\because \nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2} \qquad \therefore \nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{r})$$

1.6 Theory of Vector Fields

1.6.1 The Helmholtz Throrem

1.6.2 Potentials

1.6.1 The Helmholtz Theorem

Helmholtz theorem:

$$\nabla \cdot \vec{F} = D$$

$$\nabla \times \vec{F} = \vec{C} \qquad \vec{F} = -\nabla V + \nabla \times \vec{A}$$

requirement：

$$\nabla \cdot \vec{C} = 0 \quad , \quad \lim_{r \to 0} r^2 C = \mathbf{0} \quad , \quad \lim_{r \to 0} r^2 D = \mathbf{0}$$

Proof **:**

Assume

$$V = \frac{1}{4\pi} \int_V \frac{D(r')}{|\vec{r} - \vec{r}\,'|} d\tau' \qquad \vec{A} = \frac{1}{4\pi} \int_V \frac{C(r')}{|\vec{r} - \vec{r}\,'|} d\tau'$$

$$\nabla \cdot \vec{F} = \nabla \cdot (-\nabla V) + \underbrace{\nabla \cdot \nabla \times \vec{A}}_{0} = -\frac{1}{4\pi} \nabla^2 \int_V \frac{D(r')}{|\vec{r} - \vec{r}\,'|} d\tau'$$

$$= \int_V \delta^3 (\vec{r} - \vec{r}\,') D(r') d\tau' = D$$

1.6.1 (2)

$$\nabla\times\vec{F}=\nabla\times(\overset{0}{\cancel{-\nabla V}})+\nabla\times(\nabla\times\vec{A})=-\nabla^2\vec{A}+\nabla(\nabla\cdot\vec{A})=\vec{C}$$

$$-\nabla^2\vec{A}=\frac{-1}{4\pi}\nabla^2\int_V\frac{\vec{C}(r')}{|\vec{r}-\vec{r}'|}d\tau'=\vec{C}$$

$$\left(\nabla\frac{1}{|\vec{r}-\vec{r}'|}d\tau'=-\nabla'\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

$$4\pi\nabla\cdot\vec{A}=\int_V\nabla\cdot\frac{\vec{C}(r')}{|\vec{r}-\vec{r}'|}d\tau'=\int_V\vec{C}(r')\cdot\nabla\frac{1}{|\vec{r}-\vec{r}'|}d\tau'$$

$$=\int_S\vec{C}\cdot\frac{1}{|\vec{r}-\vec{r}'|}d\vec{a}'-\int_V\nabla'\cdot\vec{C}(r')\frac{1}{|\vec{r}-\vec{r}'|}d\tau'=0$$

$$\nabla\cdot\frac{\vec{C}(r')}{|\vec{r}-\vec{r}'|}=e_i\partial_i\cdot\frac{e_ic_i}{|\vec{r}-\vec{r}'|}=c_i\cdot\partial_i\frac{1}{|\vec{r}-\vec{r}'|}$$

$$=c_i\hat{e}_i\cdot\hat{e}_i\partial_i\frac{1}{|\vec{r}-\vec{r}'|}=\vec{c}\cdot\nabla\frac{1}{|\vec{r}-\vec{r}'|}$$

1.6.1 (3)

$$\nabla^2 \frac{1}{\left|\vec{r}-\vec{r}\,'\right|} = -4\pi\delta^3\left|\vec{r}-\vec{r}\,'\right|$$

$$\int \delta(x-x_0) f(x)dx = f(x_0)$$

$$s \to \infty , r \to \infty \Rightarrow |c| \to 0 , r'|c| \to 0$$

$$\lim_{s\to\infty} \int_s dr \frac{c}{r} \cdot r^2 \qquad c=\frac{1}{r}\to\infty$$

$$c = \frac{1}{r^2} \to \int_{s\to\infty} dr \frac{1}{r} = \ln r\Big|_{s\to\infty} = \infty$$

$$c = \frac{1}{r^3} \to 0$$

$$\Rightarrow c \cdot r^2 \xrightarrow[r\to\infty]{} 0$$

$$\Rightarrow c = \frac{1}{r^n} \xrightarrow[r\to\infty]{} 0 \quad \text{When n >2}$$

1.6.2 Potentials

$$\nabla\times\vec{F}=0 \Leftrightarrow \vec{F}=-\nabla V \quad V \text{ is a scalar potential.}$$

Curl-less fields :

$$(1)\ \nabla\times\vec{F}=\mathbf{0}$$

$$(2)\ \vec{F}=-\nabla V$$

$$(3)\ \mathbf{0}=\int_s(\nabla\times\vec{F})\cdot d\vec{a}$$

$$=\oint_l \vec{F}\cdot d\vec{l}=\mathbf{0}$$

$$(4)\ \int_a^b \vec{F}\cdot d\vec{l}=\Phi(b)-\Phi(a)$$

1.6.2 (2)

$$\nabla \cdot \vec{F} = 0 \Leftrightarrow \vec{F} = \nabla \times \vec{A} \quad \vec{A}\, is\ a\ vector\ \ potential\,.$$

Divergence-less fields :

$$(1)\nabla \cdot \vec{F} = 0$$

$$(2)\vec{F} = \nabla \times \vec{A}$$

$$(3)\oint_S \vec{F} \cdot d\vec{a} = \mathbf{0}$$

$$(4)\int_S \vec{F} \cdot d\vec{a}$$ **is independent of surface**