ESSENTIAL CALCULUS

CH12 Multiple integrals

In this Chapter:

- 12.1 Double Integrals over Rectangles
- 12.2 Double Integrals over General Regions
- 12.3 Double Integrals in Polar Coordinates
- 12.4 Applications of Double Integrals
- 12.5 Triple Integrals
- 12.6 Triple Integrals in Cylindrical Coordinates
- 12.7 Triple Integrals in Spherical Coordinates
- 12.8 Change of Variables in Multiple Integrals

Review

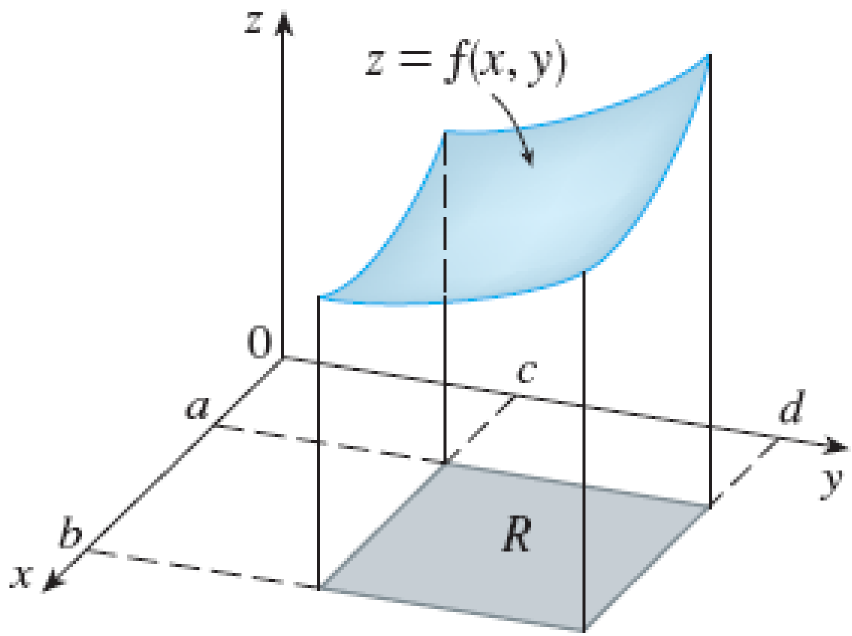


FIGURE 2

$$s=\{(x,y,z)\in R^3 \mid 0\le z\le f(x,y),(x,y)\in R\}$$

(See Figure 2.) Our goal is to find the volume of *S*.

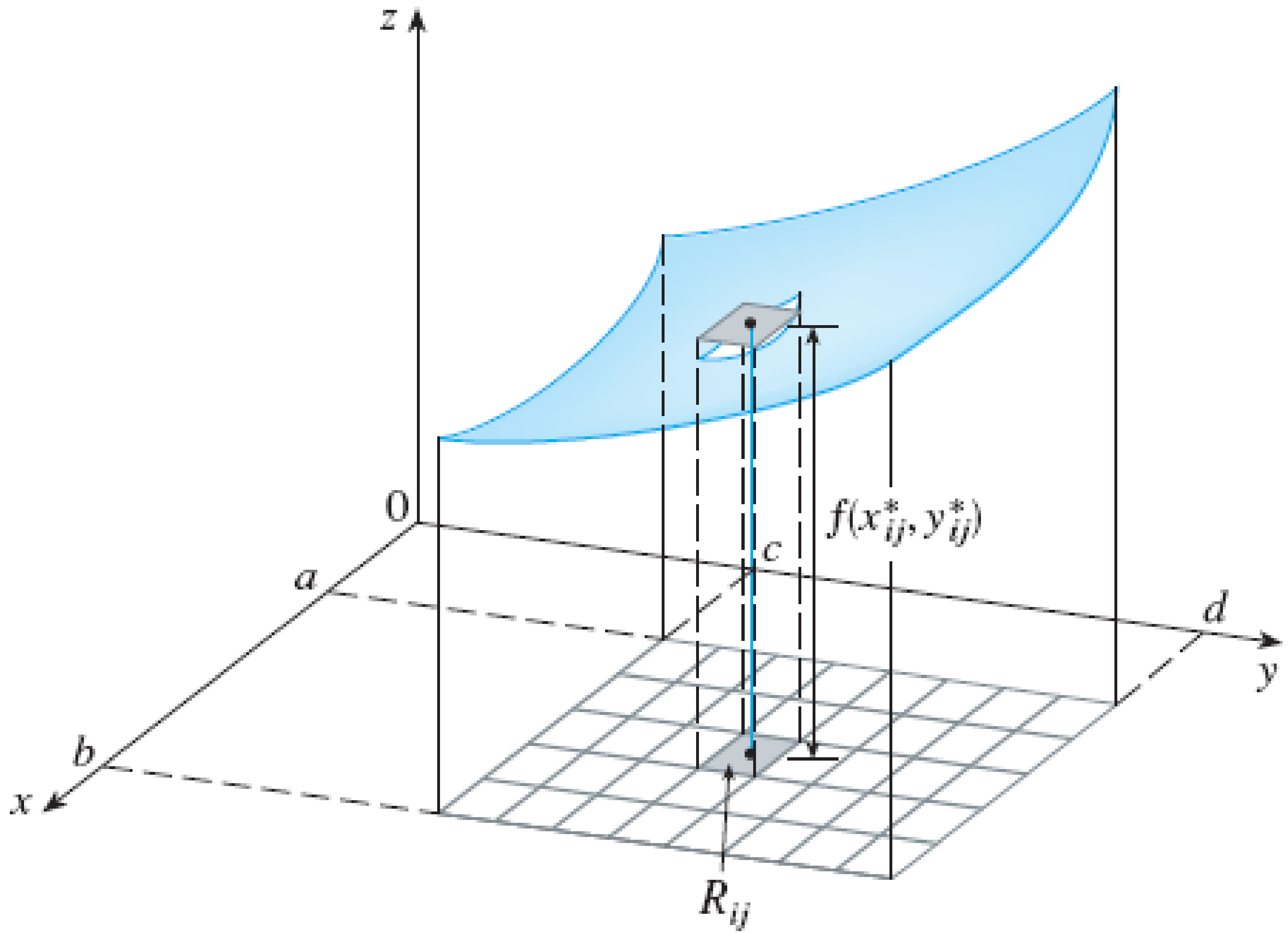


FIGURE 4

Chapter 12, 12.1, P667

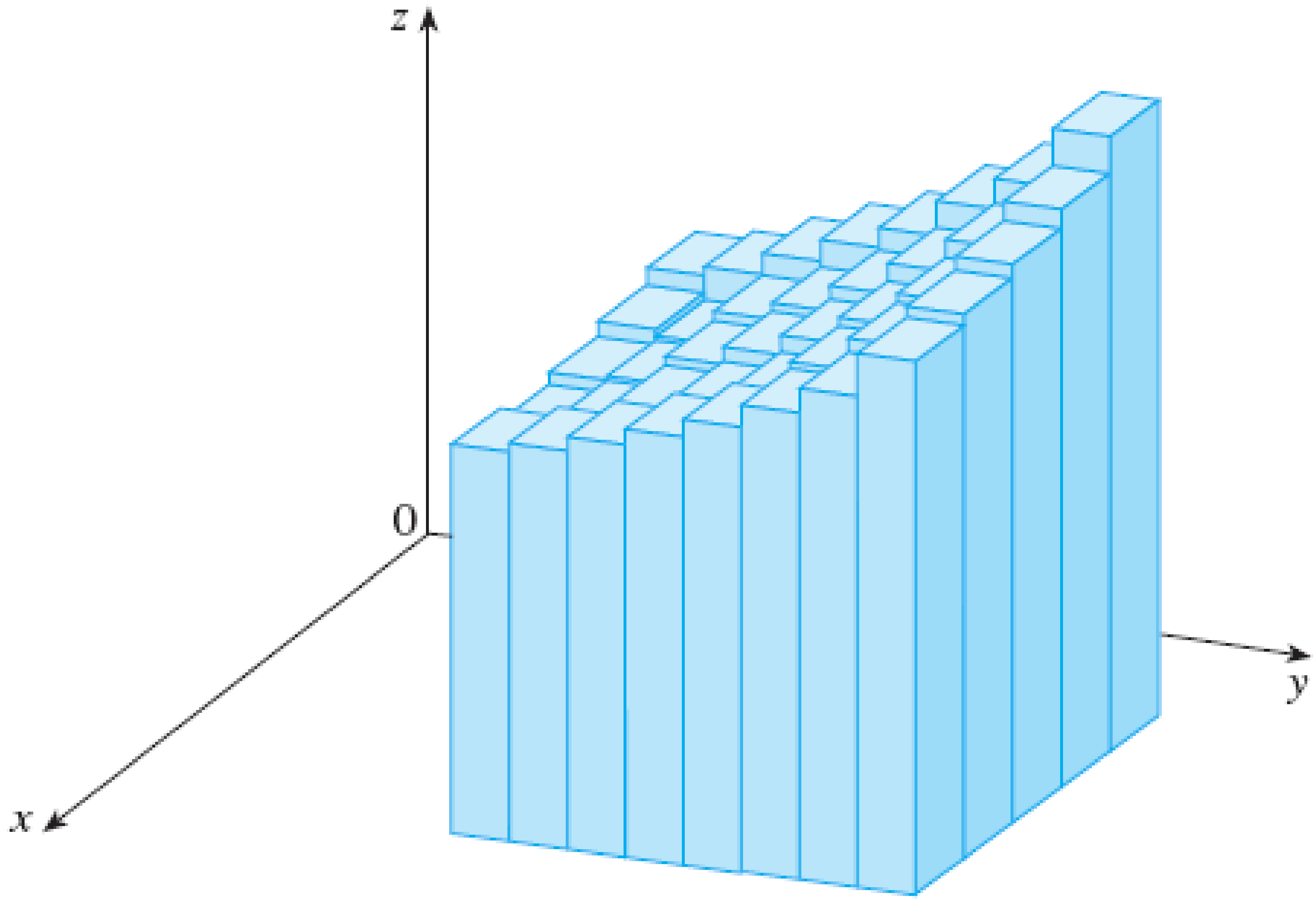


FIGURE 5

Chapter 12, 12.1, P667

5. DEFINITION The **double integral** of f over the rectangle R is

$$\iint_R f(x,y)\,dA = \lim_{\max \Delta x_i, \Delta y_i \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(xi\overset{*}{j}, yi\overset{*}{j}) \Delta A_{ij}$$

if this limit exists.

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\,\Delta A$$

Chapter 12, 12.1, P668

If f (x, y) ≥ 0, then the volume V of the solid that lies above the rectangle R and below the surface z=f (x, y) is

$$v = \iint_R f(x, y)\,dA$$

MIDPOINT RULE FOR DOUBLE INTEGRALS

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j)\,\Delta A$$

where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

10. FUBINI'S THEOREM If f is continuous on the rectangle R={(x, y} │a ≤ x≤ b, c ≤ y ≤d} , then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

More generally, this is true if we assume that f is bounded on R, is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

$$\iint_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$ where R $$R = [a,b] \times [c,d]$$

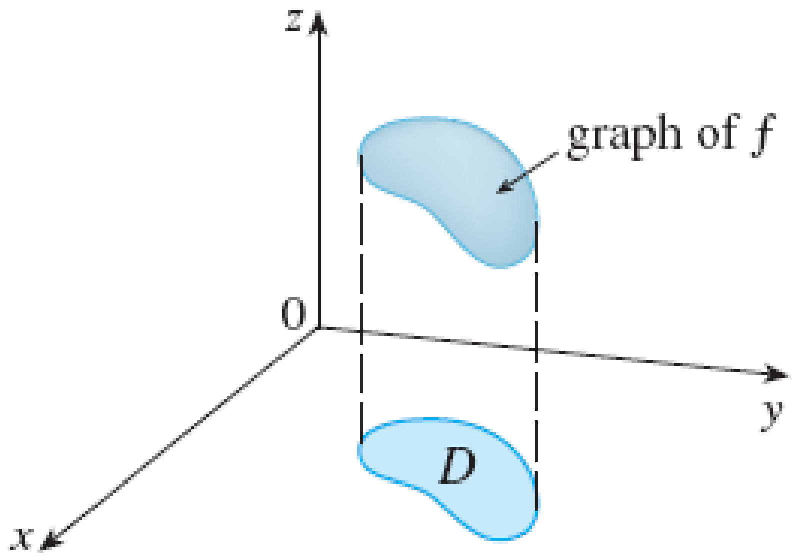


FIGURE 3

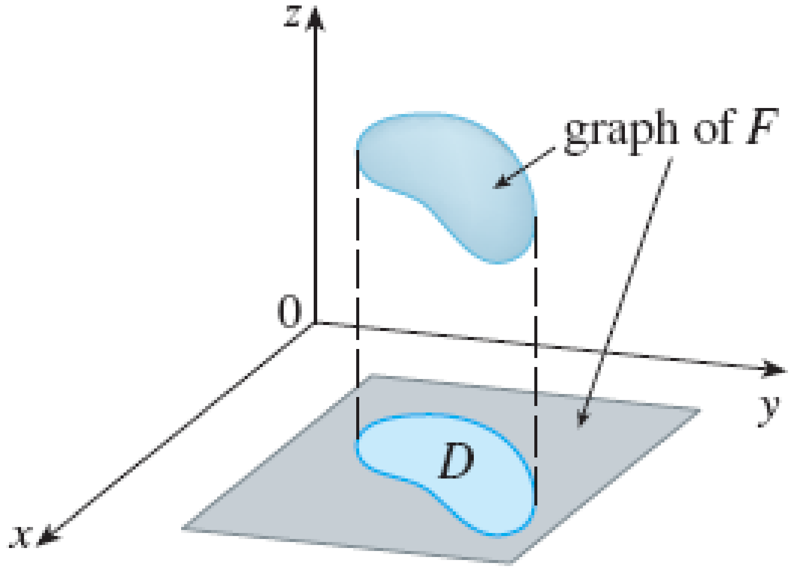
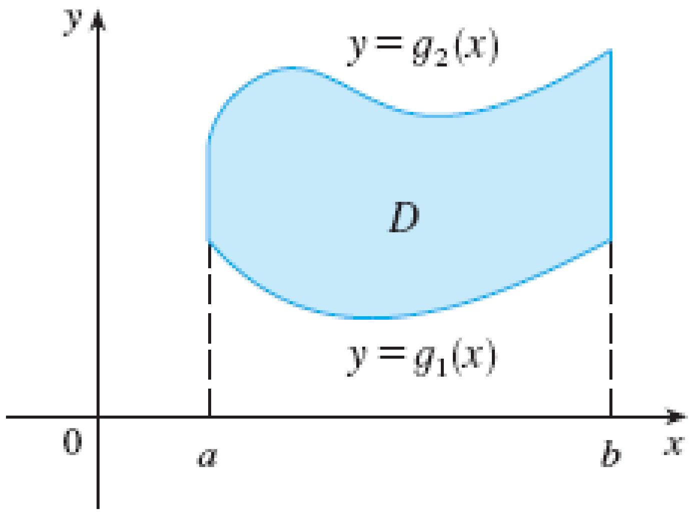


FIGURE 4

Chapter 12, 12.2, P676



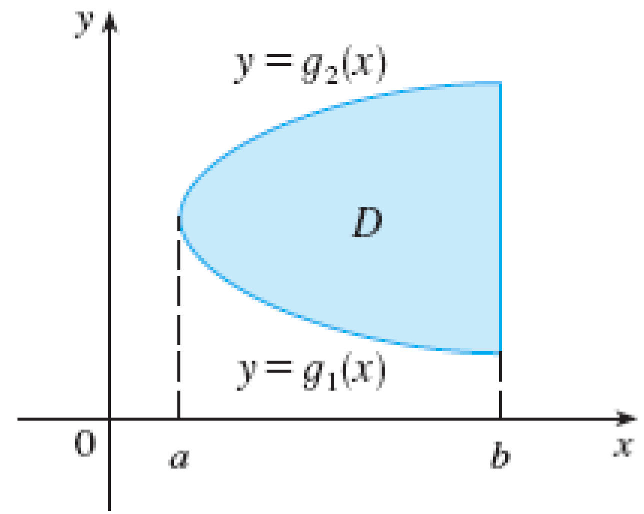
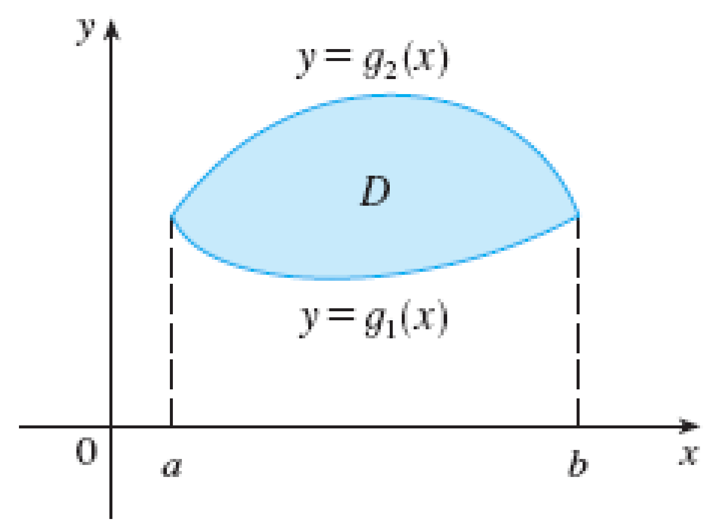


FIGURE 5 Some type I regions



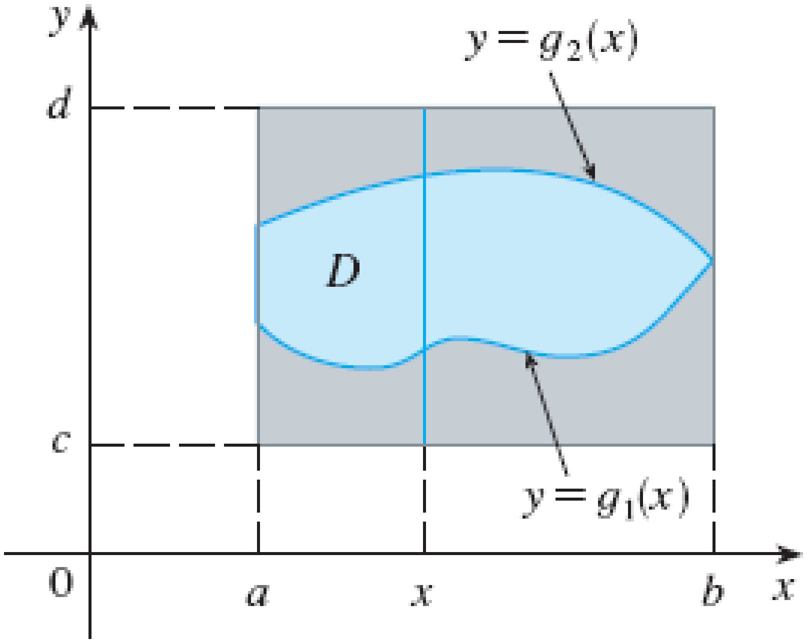


FIGURE 6

Chapter 12, 12.2, P677

3.If f is continuous on a type I region *D* such that

$$D = \{(x, y) | a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

then

$$\iint_D f(x, y)\,dA = \int_a^b \int_{g_1 x}^{g_2 x} f(x, y)\,dy\,dx$$

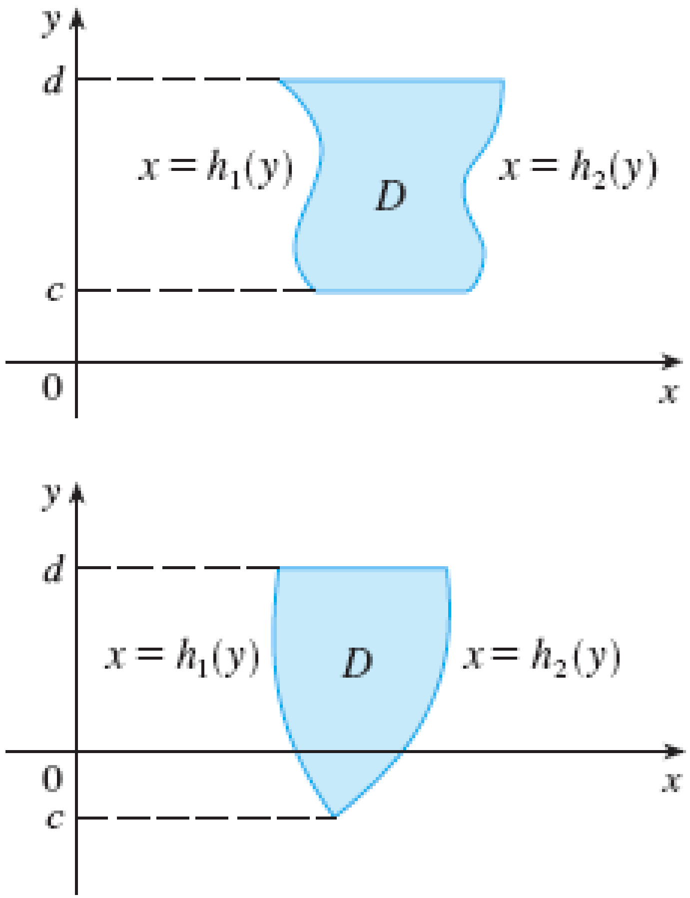


FIGURE 7
Some type II regions

$$\iint_D f(x, y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy$$

where *D* is a type II region given by Equation 4.

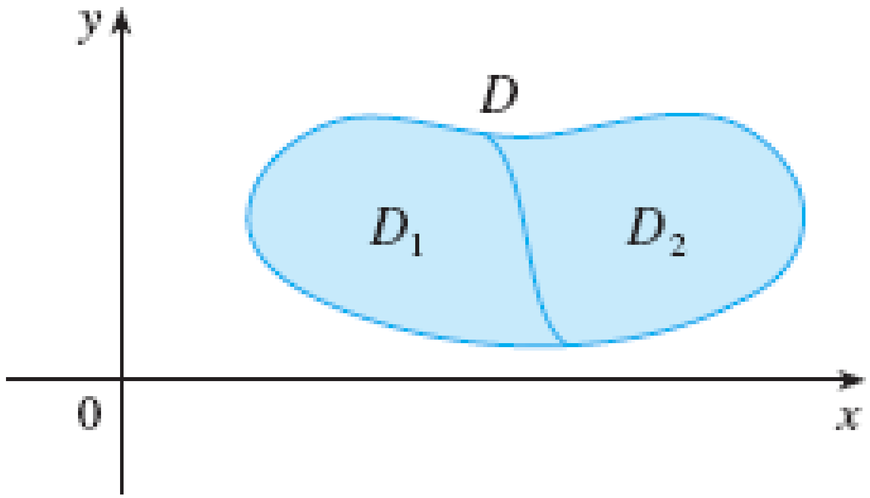


FIGURE 17

6. $\iint_D [f(x,y)+g(x,y)]dA = \iint_D f(x,y)dA + \iint_D g(x,y)dA$

7. $\iint_D cf(x,y)dA = c\iint_D f(x,y)dA$

If f (x, y) ≥ g (x, y) for all (x, y) in D, then

8. $\iint_D f(x,y)dA \geq \iint_g g(x,y)dA$

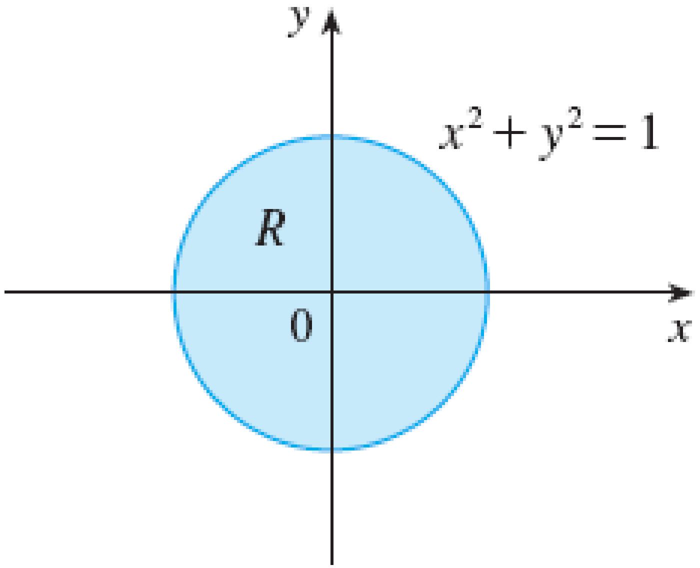
If $D=D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$

$$\iint_D 1\,dA = A(D)$$

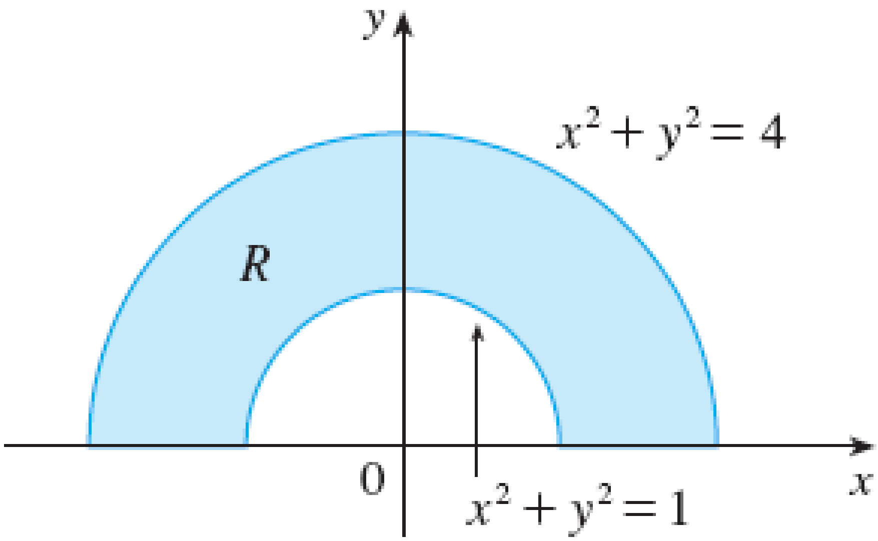
11. If $m \le f(x, y) \le M$ for all (x, y) in D, then

$$mA(D) \le \iint_D f(x, y)\,dA \le MA(D)$$



(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$

Chapter 12, 12.3, P684



(b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

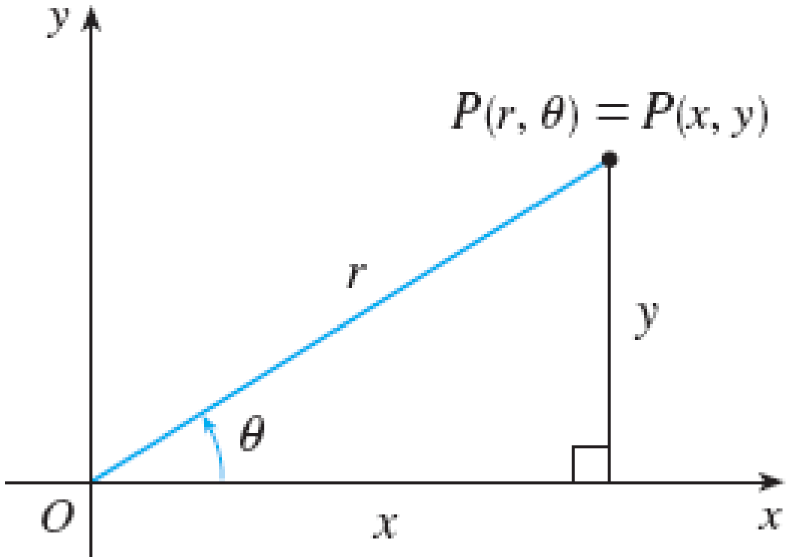


FIGURE 2

$r^2=x^2+y^2 \quad x=r\cos\theta \quad y=r\sin\theta$

Chapter 12, 12.3, P684

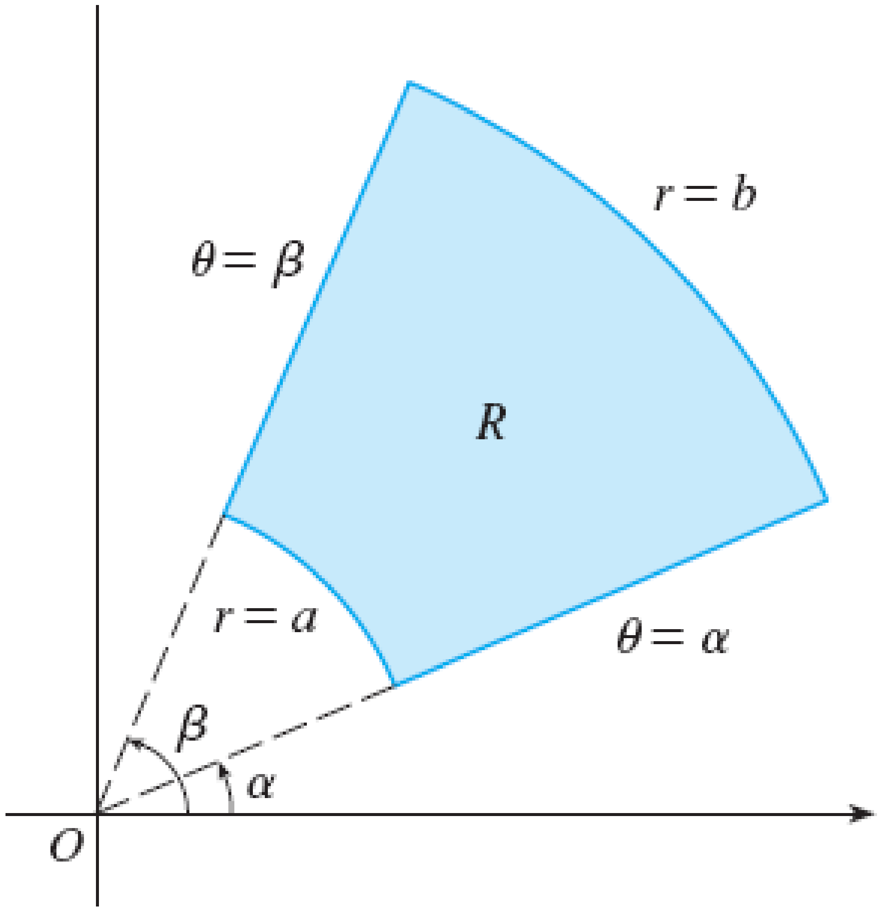


FIGURE 3 Polar rectangle

2. CHANGE TO POLAR COORDINATES IN A DOUBLE INTEGRAL If f is continuous on a polar rectangle R given by 0≤a≤r≤b, a≤θ≤ β , where 0≤ β - α ≤2π , then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

3. If f is continuous on a polar region of the form

$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)$$

then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

3. DEFINITION The **triple integral** of f over the box B is

$$\iiint_B f(x,y,z)\,dV = \lim_{\max \Delta x_i, \Delta y_i, \Delta z_k \to 0} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V_{ijk}$$

if this limit exists.

$$\iiint_B f(x,y,z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i, y_j, z_k)\Delta V$$

4. FUBINI'S THEOREM FOR TRIPLE INTEGRALS If f is continuous on the rectangular box B=[a, b]╳ [c, d]╳[r, s], then

$$\iiint\limits_B f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz$$

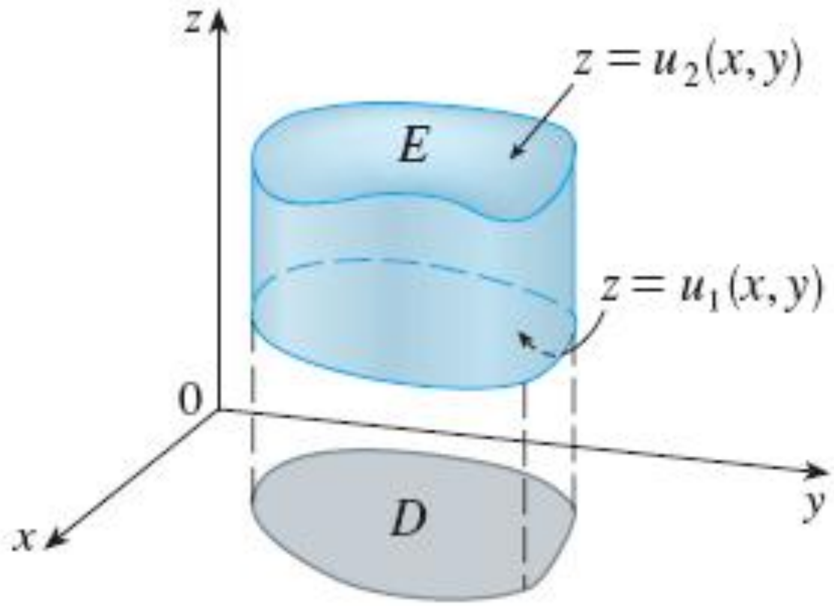


FIGURE 2

A type 1 solid region

$$\iiint_E f(x, y, z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,dz\right] dA$$

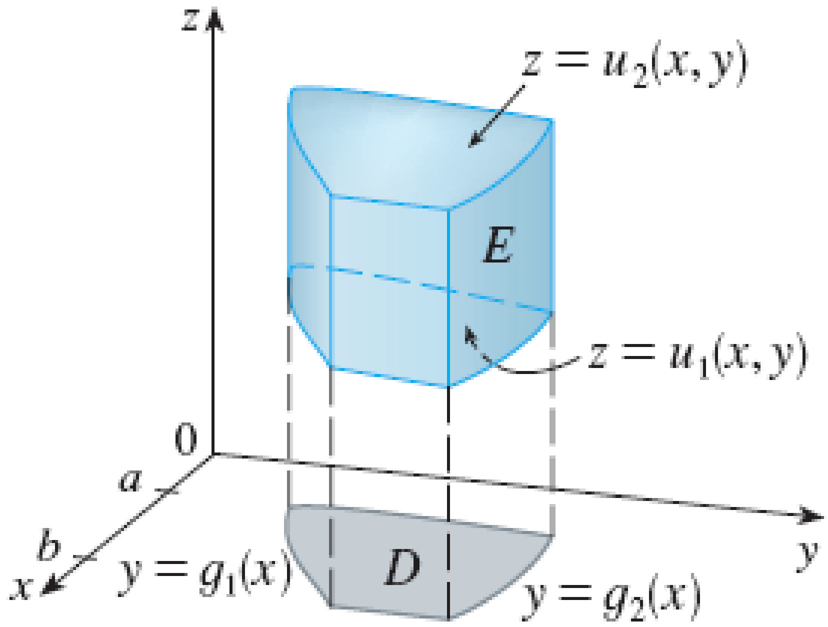


FIGURE 3

A type 1 solid region

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Chapter 12, 12.5, P697

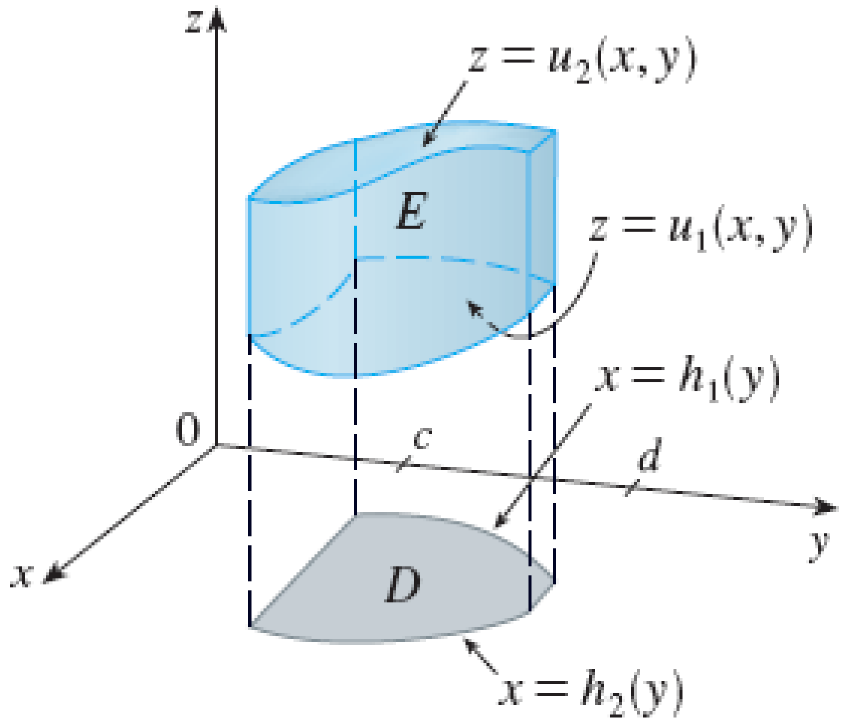


FIGURE 4

Another type 1 solid region

$$\iiint_E f(x, y, z)\,dV = \int_c^d \int_{h_1(x)}^{h_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,dz\,dx\,dy$$

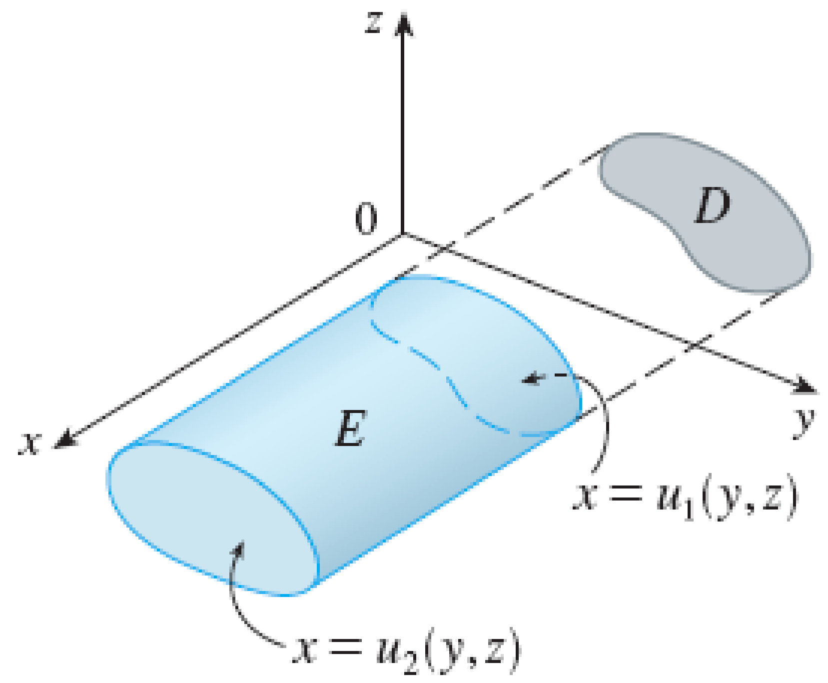


FIGURE 7
A type 2 region

$$\iiint_E f(x, y, z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,dx\right] dA$$

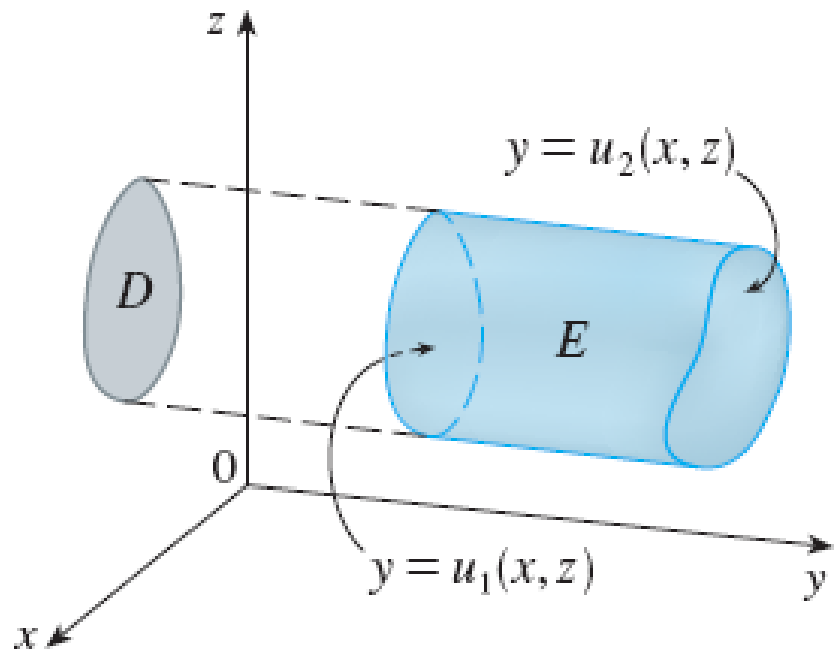


FIGURE 8
A type 3 region

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dy \right] dA$$

$$V(E) = \iiint_E dV$$

Chapter 12, 12.5, P700

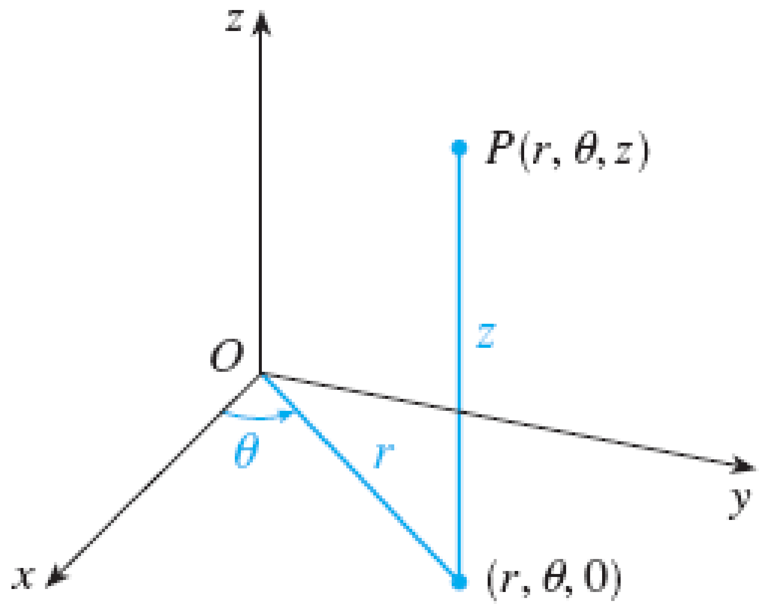


FIGURE 2

The cylindrical coordinates of a point

Chapter 12, 12.6, P705

To convert from cylindrical to rectangular coordinates, we use the equations

1 $x=r\cos\theta \quad y=r\sin\theta \quad z=z$

whereas to convert from rectangular to cylindrical coordinates, we use

2. $r^2=x^2+y^2 \quad \tan\theta=\dfrac{y}{x} \quad z=z$

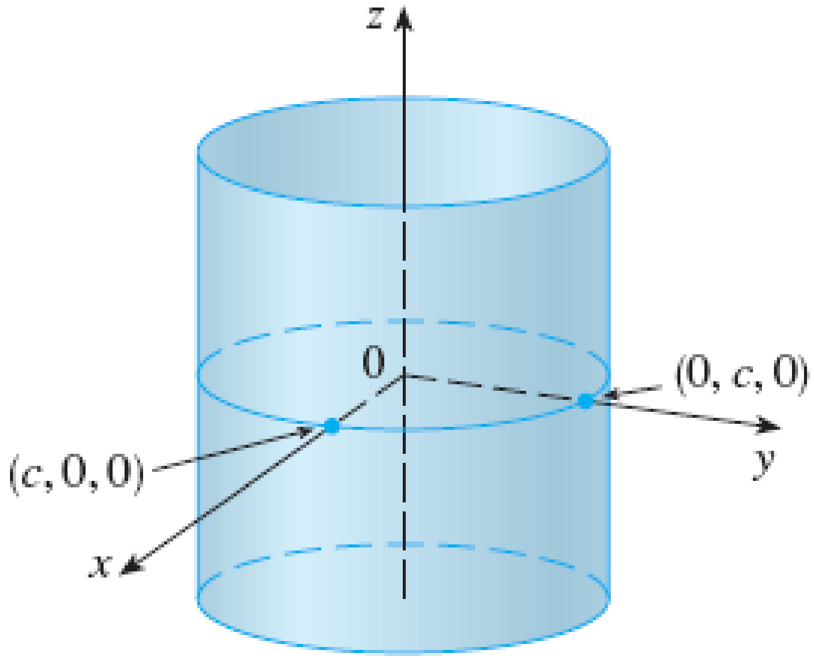


FIGURE 4

$r = c$, a cylinder

Chapter 12, 12.6, P706

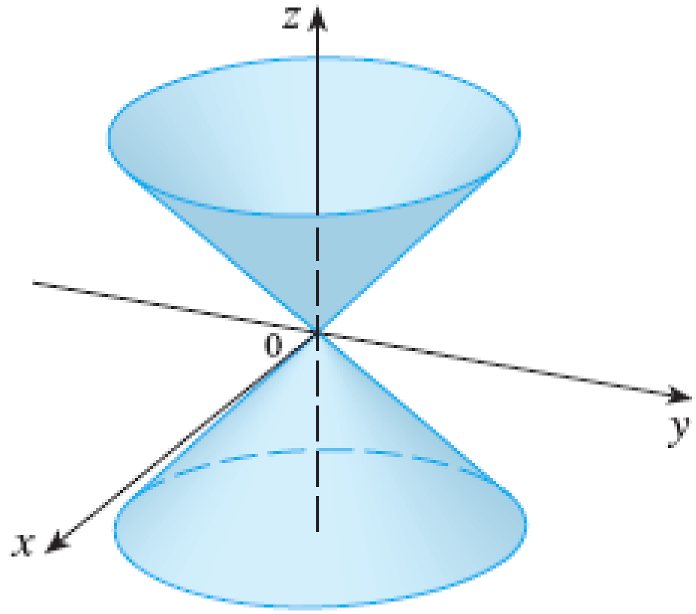


FIGURE 5

$z = r$, a cone

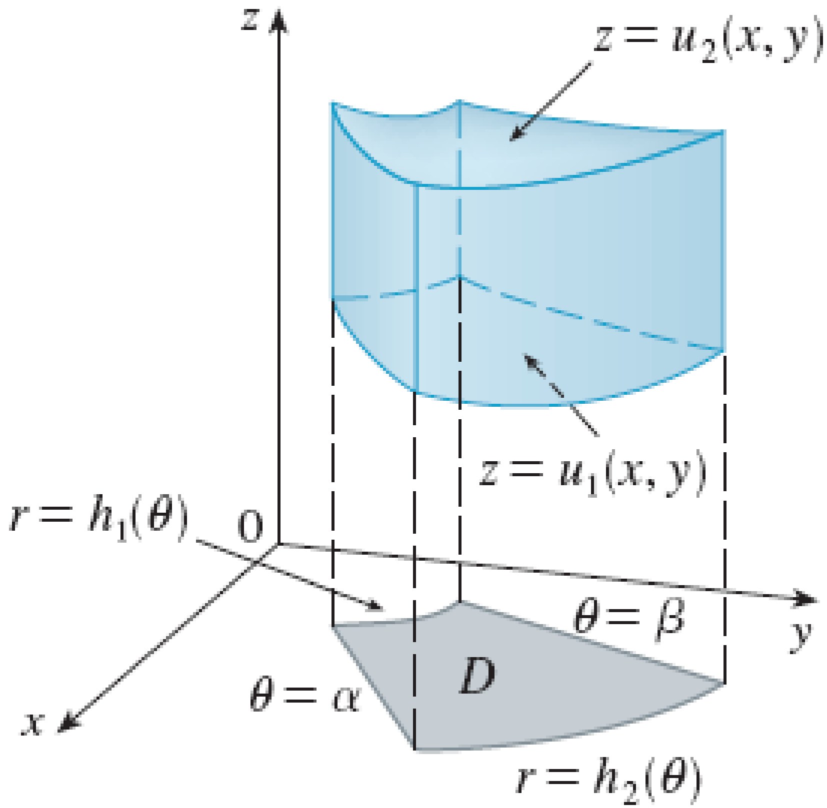

z
$z = u_2(x, y)$
$z = u_1(x, y)$
$r = h_1(\theta)$
0
$\theta = \beta$
y
D
$\theta = \alpha$
x
$r = h_2(\theta)$

FIGURE 6

formula for triple integration in cylindrical coordinates.

$$\iiint_E f(x,y,z)\,dV = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)}\int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

Chapter 12, 12.6, P707

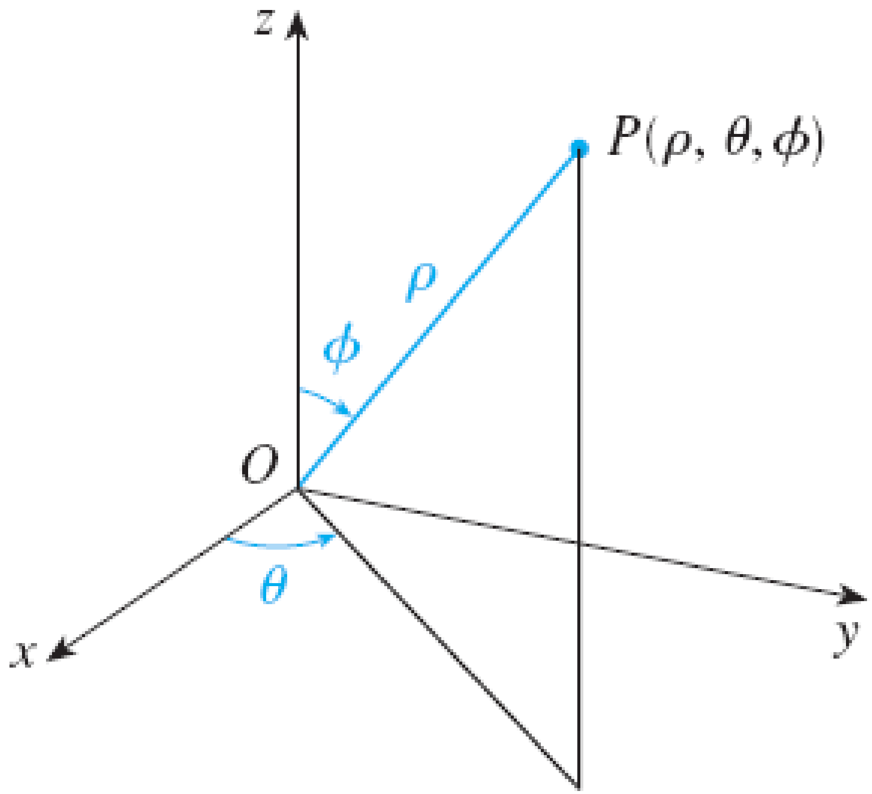


FIGURE 1

The spherical coordinates of a point

$$p \geq 0 \qquad 0 \leq \phi \leq \pi$$

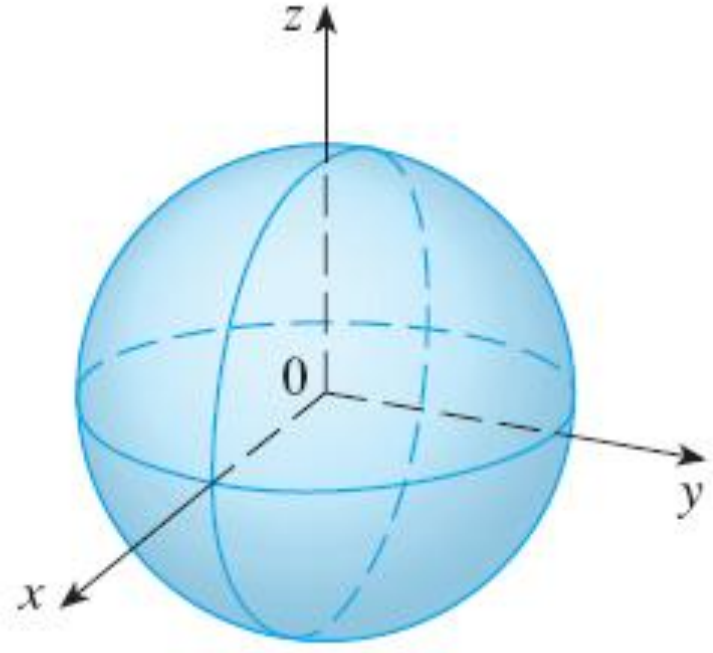


FIGURE 2 $\rho = c$, a sphere

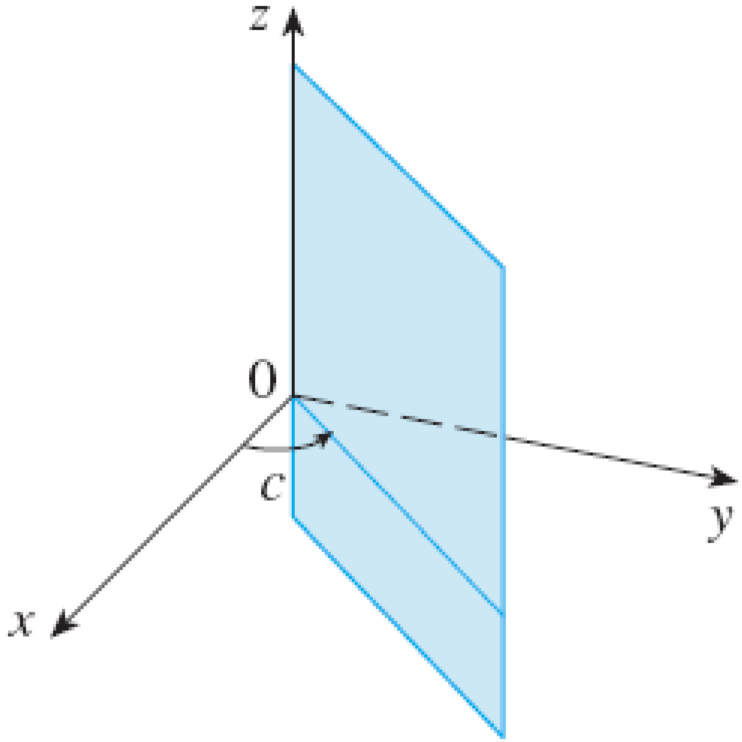
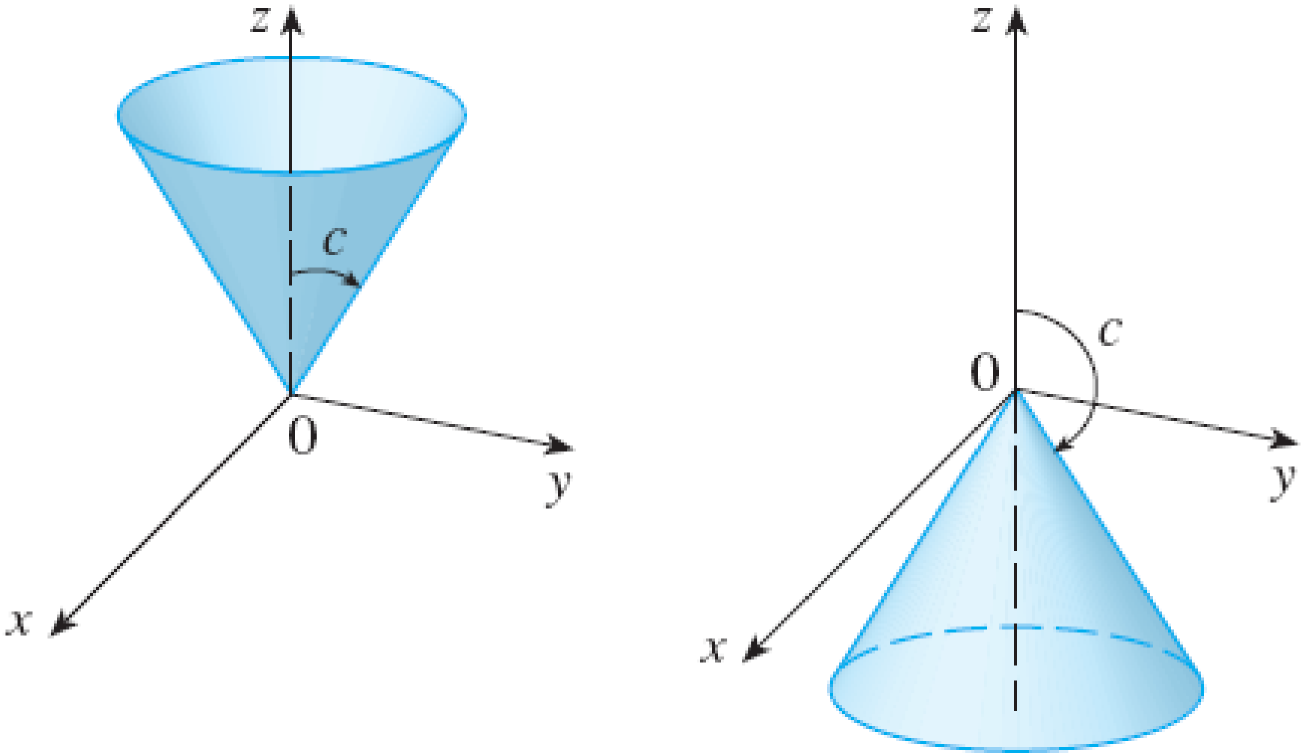


FIGURE 3 $\theta = c$, a half-plane



$0 < c < \pi/2$ $\qquad$ $\pi/2 < c < \pi$

FIGURE 4 $\phi = c$, a half-cone

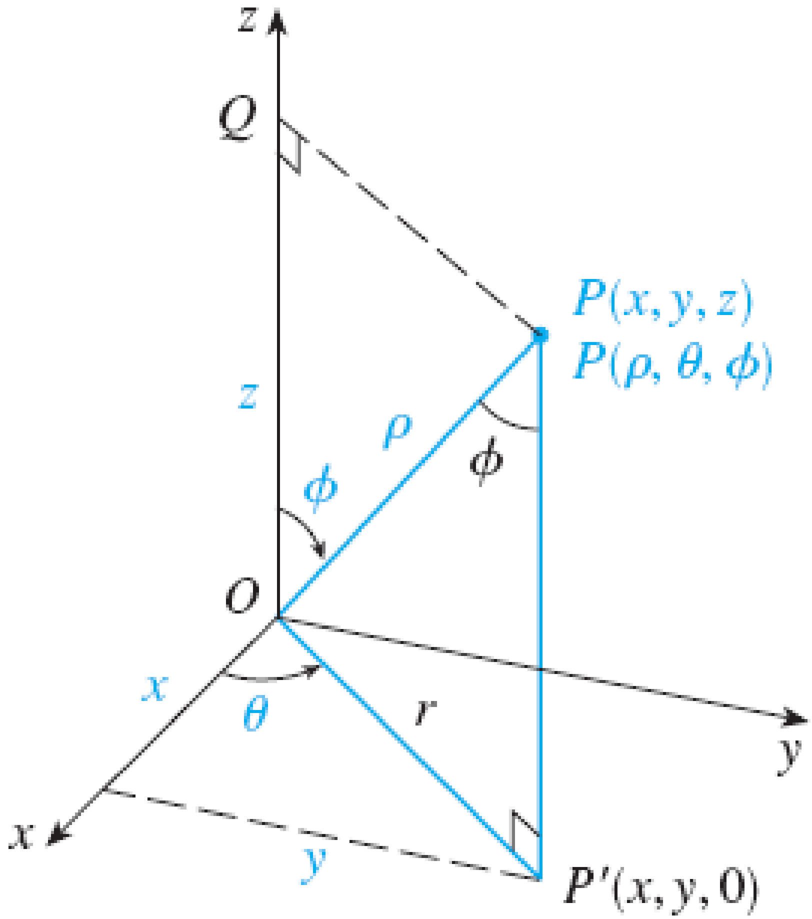


FIGURE 5

Chapter 12, 12.7, P710

$$x = p\sin\phi\cos\theta$$

$$y = p\sin\phi\sin\theta$$

$$z = p\cos\phi$$

$$p^2 = x^2 + y^2 + z^2$$

Chapter 12, 12.7, P710

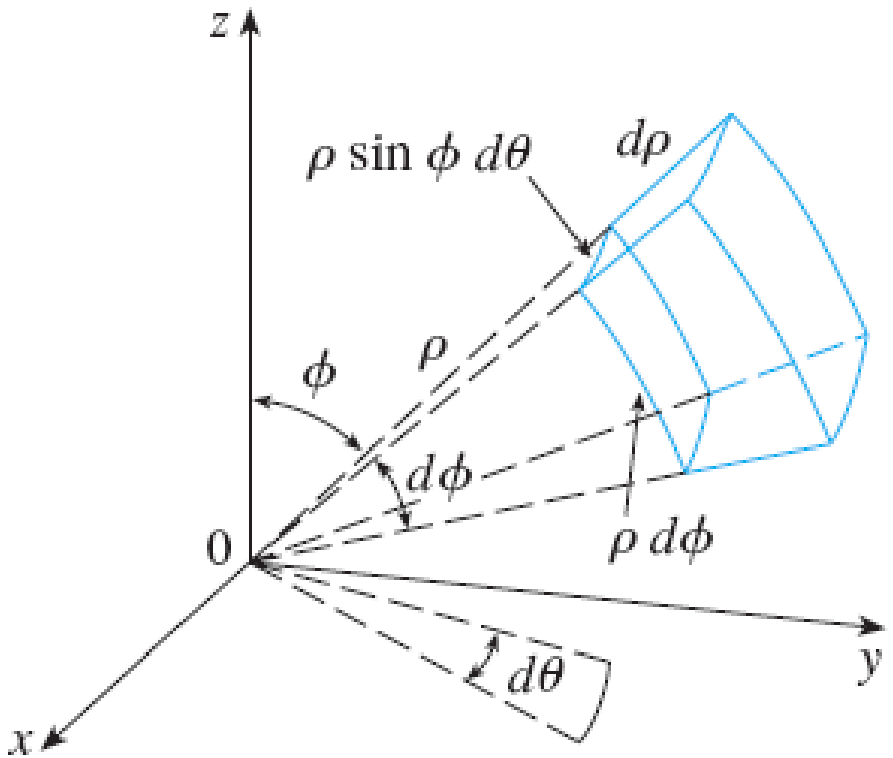


FIGURE 8
Volume element in spherical coordinates: dV=p²sinødpdΘdø

Chapter 12, 12.7, P711

Formula for triple integration in spherical coordinates

$$\iiint_E f(x,y,z)dV$$

$$=\int_c^d \int_\alpha^\beta \int_a^b f(p\sin\phi\cos\theta, p\sin\phi som\theta, p\cos\phi)p^2\sin\phi dp d\theta d\phi$$

where E is a spherical wedge given by

$$E=\{(p,\theta,\phi)|a\le p\le b, \alpha\le\theta\le\beta, c\le\phi\le d\}$$

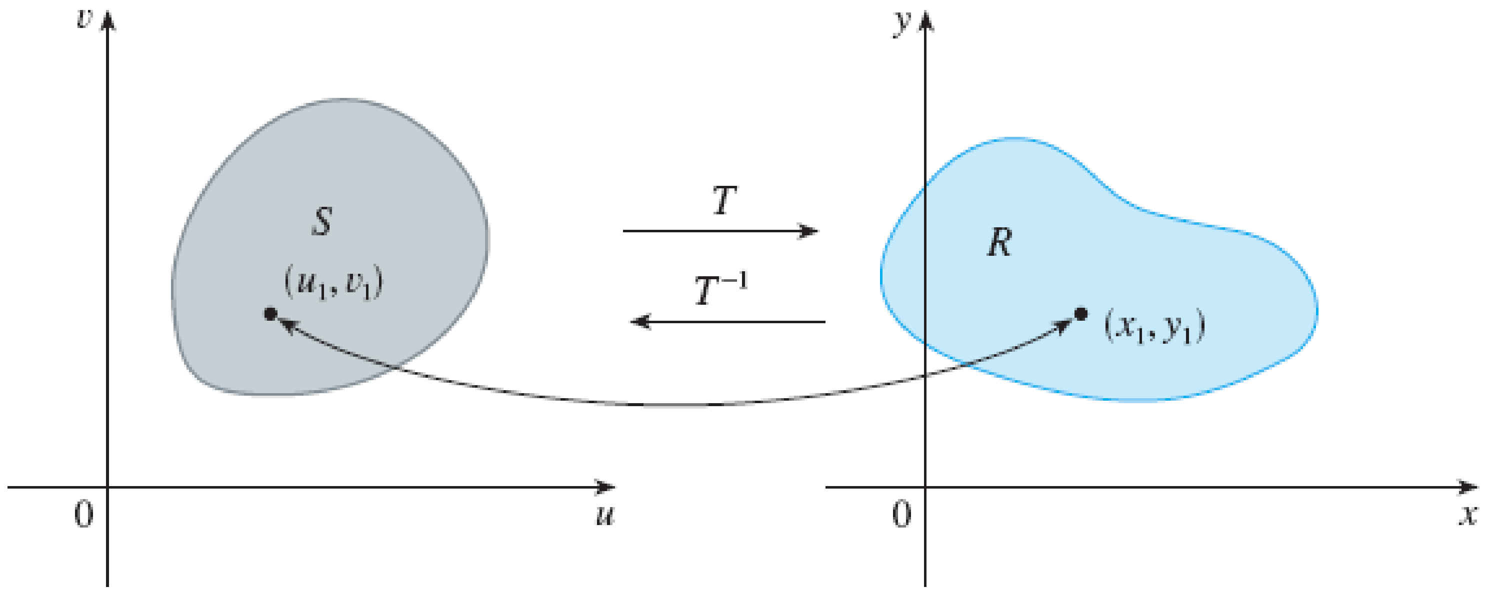


FIGURE 1

Chapter 12, 12.8, P716

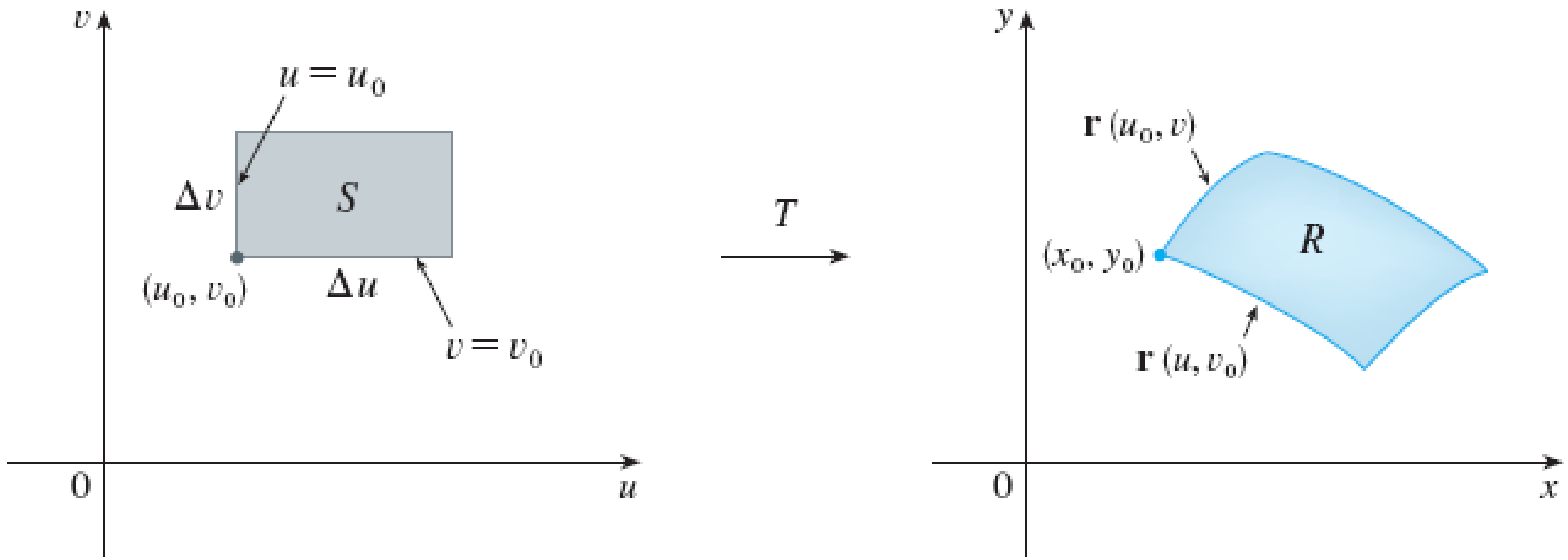


FIGURE 3

Chapter 12, 12.8, P717

7. DEFINITION The **Jacobian** of the transformation T given by x= g (u, v) and y= h (u, v) is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

9. CHANGE OF VARIABLES IN A DOUBLE INTEGRAL Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of . S. Then

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space by means of the equations

x=g (u, v, w)   y=h (u, v, w) z=k (u, v, w)

The **Jacobian** of T is the following 3X3 determinant:

12.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

13. $$\iiint\limits_R f(x,y,z)\,dV$$

$$= \iiint\limits_s f(x(u,v,w), y(u,v,w), z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$