Chapter 16: Double and Triple Integrals

Double Integrals

$$M_{ij}(\text{ area of } R_{ij}) = M_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) = M_{ij}\,\Delta x_i \Delta y_j$$

is called the P *upper sum* for f:

(16.2.1)

$$\boxed{U_f(P) = \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}(\text{ area of } R_{ij}) = \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\,\Delta x_i\,\Delta y_j.}$$

(16.2.2)

$$\boxed{L_f(P) = \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\text{area of } R_{ij}) = \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\,\Delta x_i\,\Delta y_j.}$$

Double Integrals

DEFINITION 16.2.3 THE DOUBLE INTEGRAL OVER A RECTANGLE R

Let f be continuous on a closed rectangle R. The unique number I that satisfies the inequality

$$L_f(P) \leq I \leq U_f(P) \quad \text{for all partitions } P \text{ of } R$$

is called the *double integral* of f over R, and is denoted by

$$\iint_R f(x, y)\, dxdy.\dagger$$

Definition 16.2.3, p. 948

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Double Integrals

If f is continuous and nonnegative on the rectangle R, the equation

$$z = f(x, y)$$

(16.2.4)

$$\boxed{\text{volume of } T = \iint_R f(x, y)\; dxdy.}$$

The double integral as volume, p. 949

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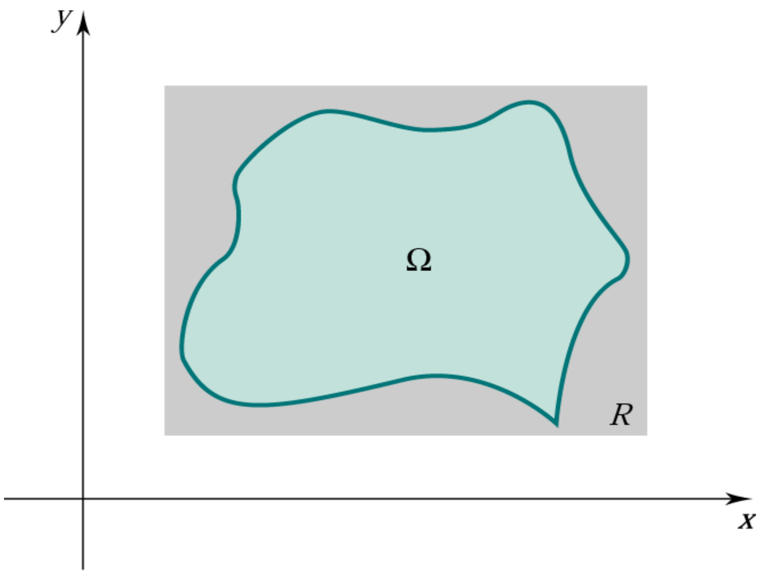
Double Integrals

(16.2.6)

$$\boxed{\iint_{\Omega} f(x,y)\ dxdy = \iint_{R} f(x,y)\ dxdy.}$$



(16.2.6), p. 953. Figure 16.2.12

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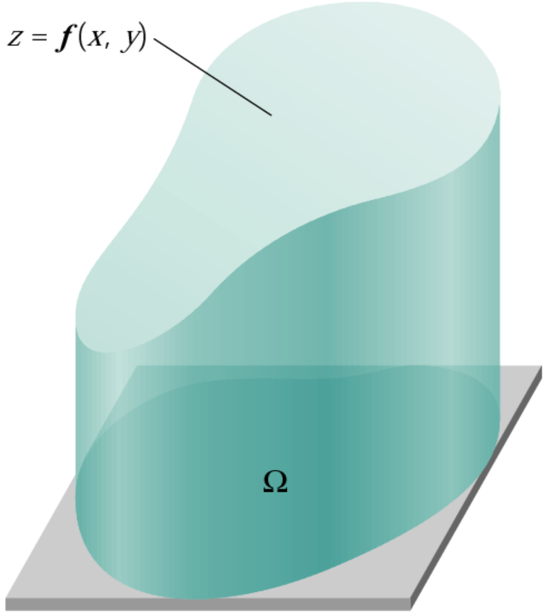
Double Integrals

If f is continuous and nonnegative over Ω,

(16.2.7)

$$\boxed{\text{volume of } T = \iint_{\Omega} f(x,y)\, dxdy.}$$



(16.2.7), p. 953. Figure 16.2.13

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Double Integrals

I. Linearity: The double integral of a linear combination is the linear combination of the double integrals:

$$\iint_{\Omega} [\alpha f(x,y) + \beta g(x,y)]\ dxdy = \alpha \iint_{\Omega} f(x,y)\ dxdy + \beta \iint_{\Omega} g(x,y)\ dxdy.$$

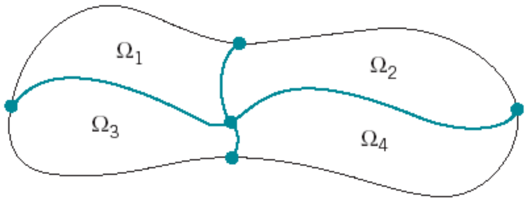
II. Order: The double integral preserves order:

$$\text{if } f \geq 0 \text{ on } \Omega, \text{ then } \iint_{\Omega} f(x,y)\ dxdy \geq 0;$$

$$\text{if } f \leq g \text{ on } \Omega, \text{ then } \iint_{\Omega} f(x,y)\ dxdy \leq \iint_{\Omega} g(x,y)\ dxdy.$$

$$\iint_{\Omega} f(x,y)\ dxdy = \iint_{\Omega_1} f(x,y)\ dxdy + \cdots + \iint_{\Omega_n} f(x,y)\ dxdy.$$

See, for example, Figure 16.2.14.



Properties of double integrals, p. 954

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Double Integrals

THEOREM 16.2.10 MEAN-VALUE THEOREM FOR DOUBLE INTEGRALS

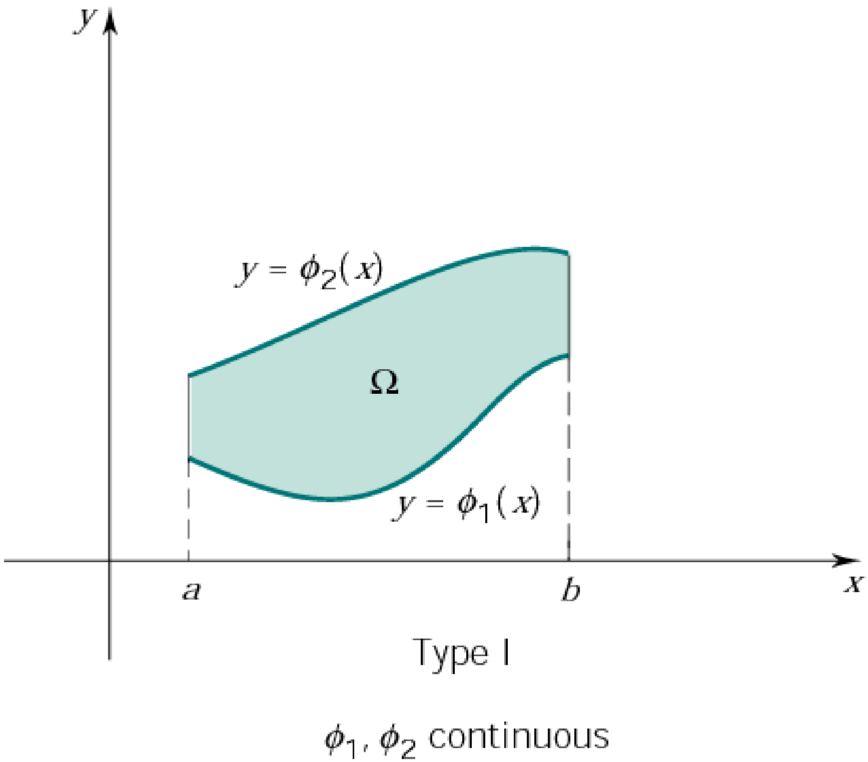
Let f and g be functions continuous on the basic region Ω. If g is nonnegative on Ω, then there exists a point (x_0, y_0) in Ω for which

$$\iint_{\Omega} f(x,y)\, g(x,y)\;dxdy = f(x_0,y_0)\iint_{\Omega} g(x,y)\;dxdy.^{\dagger}$$

We call $f(x_0, y_0)$ the *g-weighted average of f on Ω*.

Theorem 16.2.10, p. 955

Repeated Integrals



(16.3.1)

$$\boxed{\iint_{\Omega} f(x,y)\ dxdy = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\, dy\right) dx.}$$

Type I regions, (16.3.1), pp. 957, 958, figure 16.3.1-Type I

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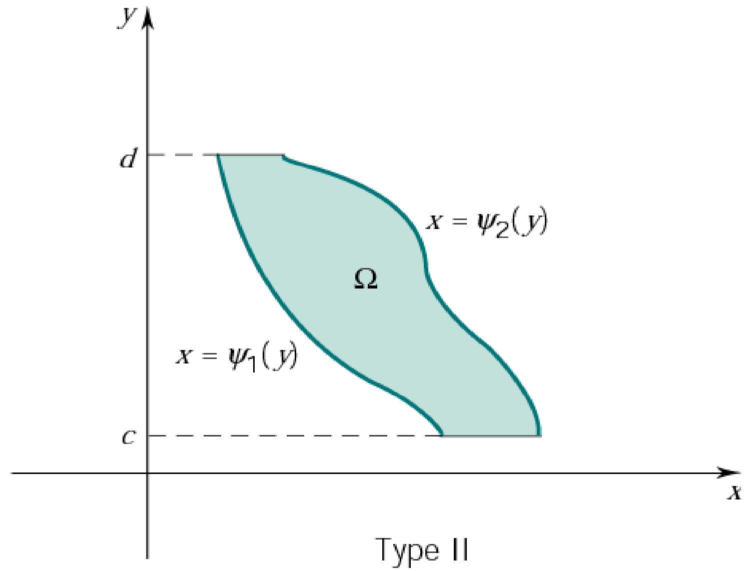
Repeated Integrals



ψ_1, ψ_2 continuous

Type II Region. The *projection* of Ω onto the y-axis is a closed interval $[c, d]$ and Ω consists of all points (x, y) with

$$c \leq y \leq d \qquad \text{and} \qquad \psi_1(y) \leq x \leq \psi_2(y).$$

Then

(16.3.2)
$$\boxed{\iint\limits_{\Omega} f(x,y)\ dxdy = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\ dx \right) dy.}$$

Type II regions, (16.3.2), pp. 958, figure 16.3.1-Type II

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The Double Integral as a Limit of Riemann Sums; Polar Coordinates

We can write this as a double integral over Γ:

(16.4.2)

$$\boxed{\text{area of } \Omega = \iint_{\Gamma} r\, dr d\theta.}$$

(16.4.2), pp. 970, 971, figure 16.4.3

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Some Applications of Double Integration

If the density varies continuously from point to point, say $\lambda = \lambda(x, y)$,

(16.5.1)

$$\boxed{M = \iint_{\Omega} \lambda(x, y)\ dxdy.}$$

Mass of a plate, (16.5.1), p. 976

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Some Applications of Double Integration

The Center of Mass of a Plate

The center of mass x_M of a rod is a density-weighted average of position taken over the interval occupied by the rod:

(16.5.2)

$$\boxed{x_M M = \iint_\Omega x\lambda(x,y)\ dxdy, \quad y_M M = \iint_\Omega y\lambda(x,y)\ dxdy.}$$

Center of mass of a plate, (16.5.2), p. 976

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Triple Integrals

DEFINITION 16.6.1 THE TRIPLE INTEGRAL OVER A BOX Π

Let f be continuous on a closed box Π. The unique number I that satisfies the inequality

$$L_f(P) \le I \le U_f(P) \quad \text{for all partitions } P \text{ of } \Pi$$

is called the *triple integral* of f over Π and is denoted by

$$\iiint\limits_{\Pi} f(x,y,z)\ dxdydz.^{\dagger}$$

Definition (16.6.1), p. 984

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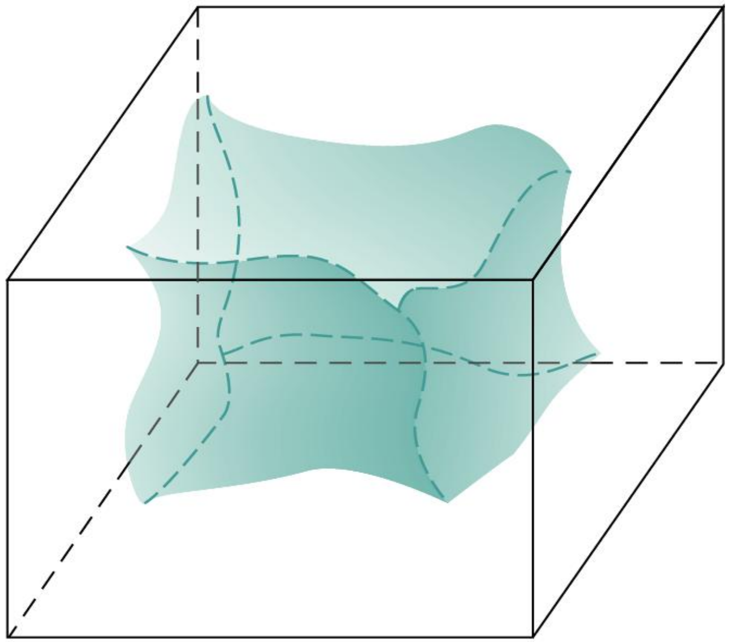
Triple Integrals



We define the triple integral over T by setting

(16.6.2)

$$\boxed{\iiint_T f(x,y,z)\ dxdydz = \iiint_\Pi f(x,y,z)\ dxdydz.}$$

(16.6.2), p. 985, figure 16.6.4

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Triple Integrals

(16.6.3)

$$\boxed{\text{volume of } T = \iiint\limits_T dxdydz.}$$

Volume, (16.6.3), p. 985

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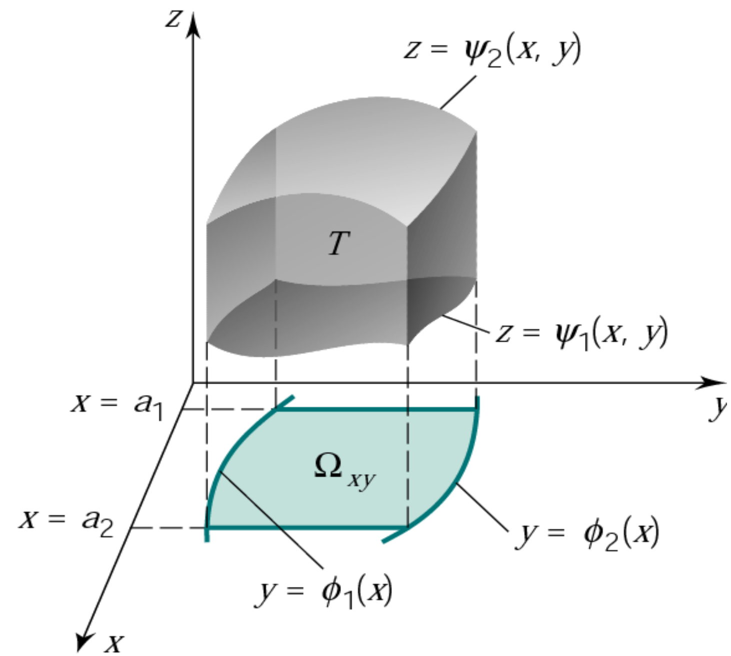
Reduction to Repeated Integrals



(16.7.1)

$$\boxed{\iiint\limits_T f(x,y,z)\ dxdydz = \int_{a_1}^{a_2}\int_{\phi_1(x)}^{\phi_2(x)}\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dz\,dy\,dx.}$$ †

(16.7.1), p. 989, figure 16.7.2

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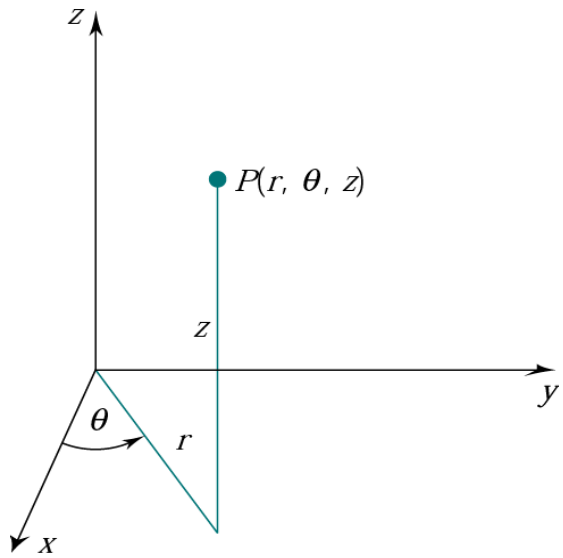
Cylindrical Coordinates



cylindrical coordinates (r, θ, z):
$r \geq 0,\ 0 \leq \theta < 2\pi,\ z$ real

Rectangular coordinates (x, y, z) can be obtained from cylindrical coordinates (r, θ, z) by means of the equations

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

Conversely, with the obvious exclusions, cylindrical coordinates (r, θ, z) can be obtained from rectangular coordinates (x, y, z) by means of the equations

$$r = \sqrt{x^2 + y^2}, \qquad \tan\theta = \frac{y}{x}, \qquad z = z.$$

Rectangular coordinates/cylindrical coordinates, p. 998, figure 16.8.1

Cylindrical Coordinates

(16.8.1)

$$\boxed{\iiint\limits_{T} f(x, y, z)\ dxdydz = \iiint\limits_{S} f(r\cos\,\theta, r\sin\,\theta, z)\, r\, drd\theta dz.}$$

(16.8.1), p.999

Cylindrical Coordinates

If $f(x, y, z) = 1$ for all (x, y, z) in T, then (16.8.1) reduces to

$$\iiint_T dxdydz = \iiint_S r\,dr d\theta\,dz.$$

(16.8.2)

$$\boxed{V = \iiint_S r\,dr d\theta\,dz.}$$

Volume in cylindrical coordinates, (16.8.2), p. 1000

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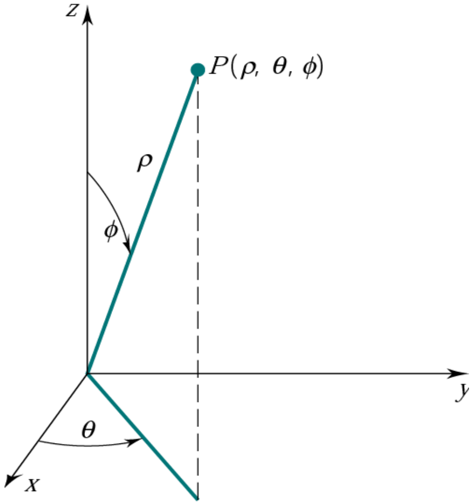
Spherical Coordinates



spherical coordinates (ρ, θ, ϕ):
$\rho \geq 0,\ 0 \leq \theta < 2\pi,\ 0 \leq \phi \leq \pi$

Rectangular coordinates (x, y, z) are related to spherical coordinates (ρ, θ, ϕ) by the following equations:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho\cos\phi.$$

You can verify these relations by referring to Figure 16.9.3. (Note that the factor $\rho \sin\phi$ appearing in the first two equations is the r of cylindrical coordinates: $r = \rho\sin\phi$.) Conversely, with obvious exclusions, we have

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \tan\theta = \frac{y}{x}, \quad \cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

$$\Pi: \quad a_1 \leq \rho \leq a_2, \quad b_1 \leq \theta \leq b_2, \quad c_1 \leq \phi \leq c_2.$$

The volume of this wedge is given by the formula

(16.9.2) $$\boxed{V = \iiint_{\Pi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.}$$

Spherical coordinates, pp. 1004, 1005, figure 16.9.1

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Spherical Coordinates

Suppose that T is a basic solid in xyz-space with spherical coordinates in some basic solid S of $\rho\theta\phi$-space. Then

(16.9.3)

$$\iiint_T f(x, y, z)\ dxdydz = \iiint_S f(\rho \sin\phi \cos\theta,\ \rho \sin\phi \sin\theta,\ \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho d\theta d\phi.$$

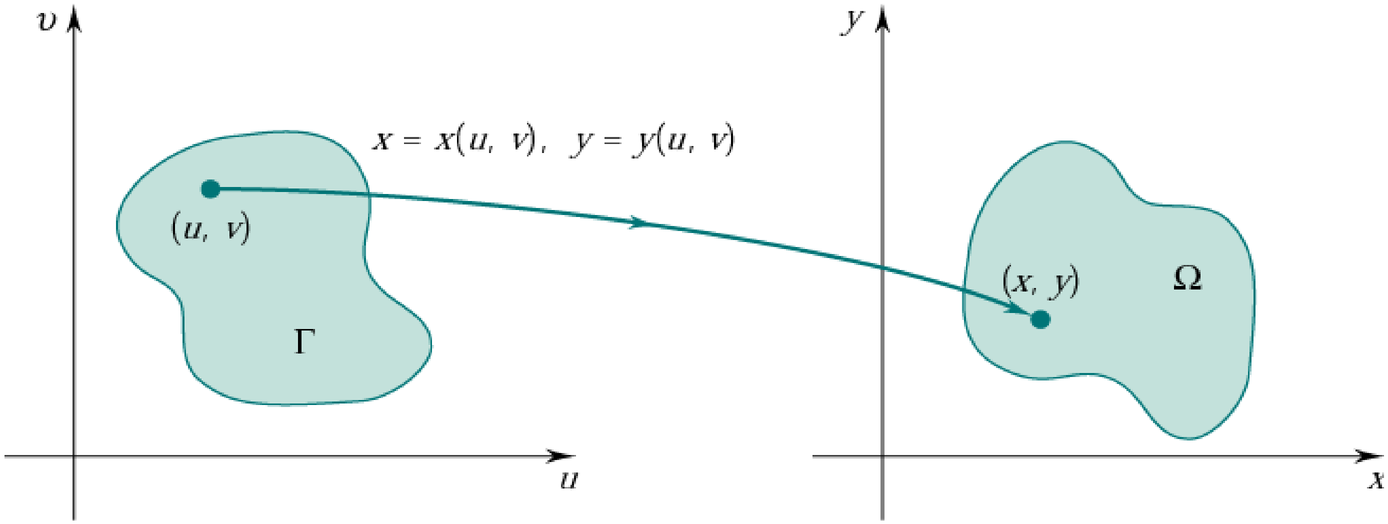
(16.9.4)

$$V = \iiint_S \rho^2 \sin\phi \, d\rho d\theta d\phi.$$

(16.9.3), (16.9.4), p. 1007

Jacobians, Changing Variables

$$x = x(u, v), \quad y = y(u, v)$$



Change of variables, pp. 1011, 1012, figure 16.10.1

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Jacobians, Changing Variables

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\[2ex] \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u},$$

(16.10.1)

$$\boxed{\text{area of } \Omega = \iint_{\Gamma} |J(u,v)|\, du dv.}$$

Jacobian, (16.10.1), p.1012

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Jacobians, Changing Variables

(16.10.2)

$$\boxed{\iint\limits_{\Omega} f(x,y)\ dxdy = \iint\limits_{\Gamma} f(x(u,v),y(u,v))|J(u,v)|\ dudv.}$$

(16.10.2), p. 1012