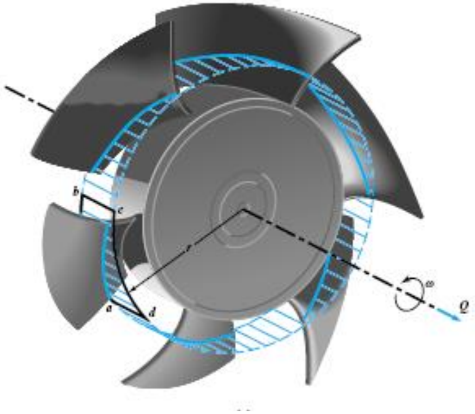
ME376
Design of Thermal Fluid Systems



Fundamentals of Heat Transfer

6.1 Conduction of Heat Through a Plane Wall

$$q_x = kA\left(-\frac{dT}{dx}\right)$$ k is known as the **thermal conductivity** W/(m·K)

$$q_x = kA\,\frac{T_0 - T_1}{L}$$

$$q_x = \frac{T_0 - T_1}{L/kA} = \frac{T_0 - T_1}{R}$$

$L/kA = R$ is introduced as a resistance

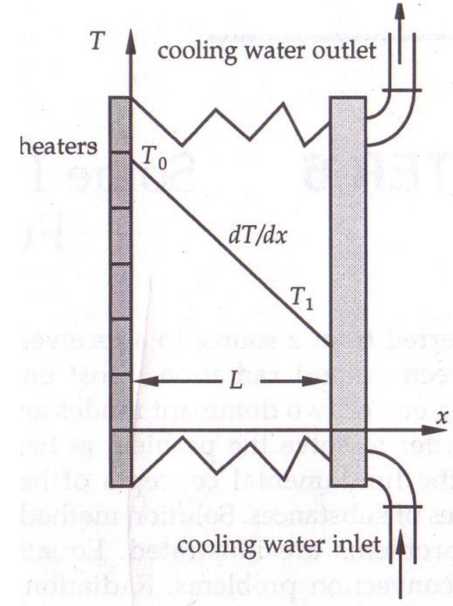


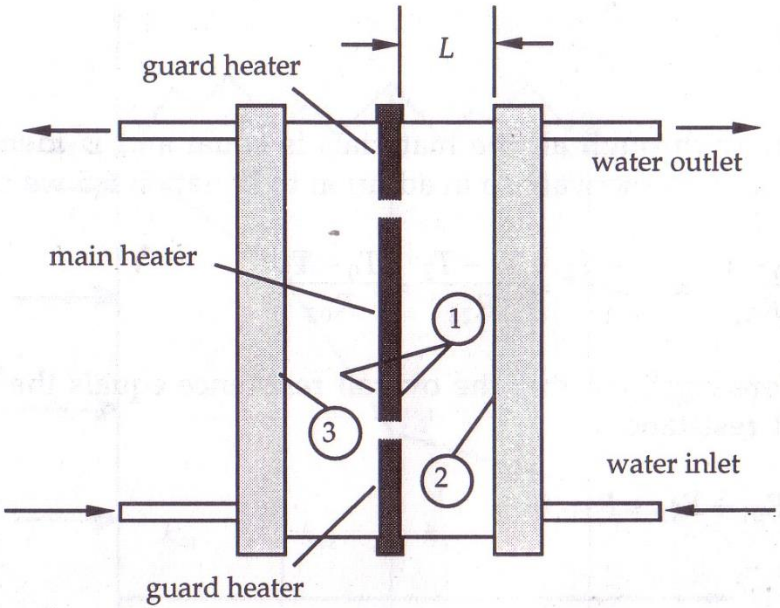
TABLE 6.1. *Thermal properties of selected metals and alloys.*

Material	Specific Gravity	Specific Heat C_p $\frac{J}{kg \cdot K}$	Specific Heat C_p $\frac{BTU}{lbm \cdot °R}$	Thermal Conductivity, k $\frac{W}{m \cdot K}$	Thermal Conductivity, k $\frac{BTU}{hr \cdot ft \cdot °R}$	Diffusivity $m^2 \times 10^6$	Diffusivity $ft^2 \times 10^3$
Selected Metals at 293 K = 20°C = 528°R = 68°F							
Aluminum	2.702	896	0.214	236	163	97.5	1.05
Bronze	8.666	343	0.0819	26	15.0	8.59	0.0925
Brass	8.522	385	0.0920	111	64.1	34.12	0.3673
Cast Iron	7.272	420	0.100	52	30.0	17.02	0.1832
Copper	8.933	383	0.0915	399	231	116.6	1.26
Wrought iron	7.849	460	0.110	59	34.1	16.26	0.1750
Carbon steel	7.801	473	0.113	43	24.8	11.72	0.1262
Chrome steel	7.865	460	0.110	61	35.2	16.65	0.1792
Silicon steel	7.769	460	0.110	42	24.3	11.64	0.1254
Stainless steel	7.817	461	0.110	14.3	8.26	3.87	0.0417

Material	Specific Gravity	$\frac{J}{kg \cdot K}$	$\frac{BTU}{lbm \cdot °R}$	$\frac{W}{m \cdot K}$	$\frac{BTU}{hr \cdot ft \cdot °R}$	$m^2 \times 10^5$	$ft^2 \times 10^6$
Selected Building Materials at 293 K = 20°C = 528°R = 68°F							
Asbestos	0.383	816	0.195	0.113	0.0653	0.036	3.88
Asphalt	2.120			0.698	0.403		
Brick							
Common	1.800	840	0.201	0.45	0.26	0.031	3.33
Masonry	1.700	837	0.200	0.658	0.38	0.046	5.0
Silica	1.9			1.07	0.618		
Cardboard				0.25	0.14		
Concrete	0.500	837	0.200	0.128	0.074	0.049	5.3
Cork	0.120	1880	0.449	0.042	0.0243	0.03	3.15
Ebonite				0.163	0.0093		
Glass fiber	0.220			0.035	0.02		
Glass wool				0.040	0.023		
Glass (window)	2.800	800	0.191	0.81	0.47	0.034	3.66
Ice at 0°C	0.913	1830	0.437	2.22	1.28	0.124	13.3
Kapok	0.025			0.035	0.02		
Wood							
Fir, pine, spruce	0.444	2720	0.650	0.15	0.087	0.0124	1.33
Oak	0.705	2390	0.571	0.19	0.11	0.0113	1.22
Wool	0.200			0.038	0.022		

EXAMPLE 6.1. Figure 6.2 shows a cross section of a **guarded hot-plate** apparatus. It is used to measure the thermal conductivity of a planar shaped material, such as plywood, insulation, sheet rock, etc. The guarded hot-plate apparatus consists of a main heater and a guard heater. The guard heater completely surrounds the main heater. Two samples of the material to be tested are required. One sample is placed on each side of the heaters. Cooling water jackets are made to contact the samples. To operate the device, both heaters are activated. Heat from the main heater is made to flow in one dimension through each sample to the cooling water jackets. The guard heater supplies energy to the outer perimeter of the samples so that heat flowing from the main heater will not flow in any lateral direction. Thus a one-dimensional flow of heat from the main heater to the cooling water jackets is set up.

The heaters are both heated electrically, so readings of voltage and amperage on the wires to the main heater provide data from which input power to *both* samples can be calculated. Thermocouples are used to make temperature readings, which are needed for several purposes. To ensure that one-dimensional heat flow exists, temperature at surface 1 (Figure 6.2) of the samples must be the same at the main and the guard heater once steady state is achieved. At that time, the temperature at surface 1 of the main heater and at surfaces 2 and 3 of the cooling water jackets are recorded.



A guarded hot-plate apparatus is used to measure the thermal conductivity of 3/8-in.-thick plywood. The electrical input to the main heater is 110 V x 1 A. The temperature at the main heater surface is 210°F, while the surface temperature of the cooling jackets is 80°F. The cross sectional area through which heat flows is 0.75 ft². (a) Determine the thermal conductivity of the plywood, and (b) calculate the value of the resistance as defined in Equation 6.4.

Solution: Half the electrical power input goes into each piece of plywood. The heat flow into each piece is given by Equation 6.4:

$$q_x = \frac{T_0 - T_1}{L/kA} = \frac{T_0 - T_1}{R} \tag{6.4}$$

in which q_x is calculated to be

$$q_x = 0.5(110)(1) = 55\ \text{W}/0.2928 = 187\ \text{BTU/hr}$$

The conversion factor was obtained from Appendix Table A.2. Rearranging Equation 6.4 to solve for thermal conductivity and substituting gives

$$k = \frac{Lq_x}{A(T_0 - T_1)} = \frac{3/(8(12))(187)}{0.75(210 - 80)}$$

(a) $\underline{k = 0.060\ \text{BTU/(hr·ft·°R)}}$

The resistance is found from the definition

$$R = \frac{L}{kA} = \frac{3/(8(12))}{0.060(0.75)}$$

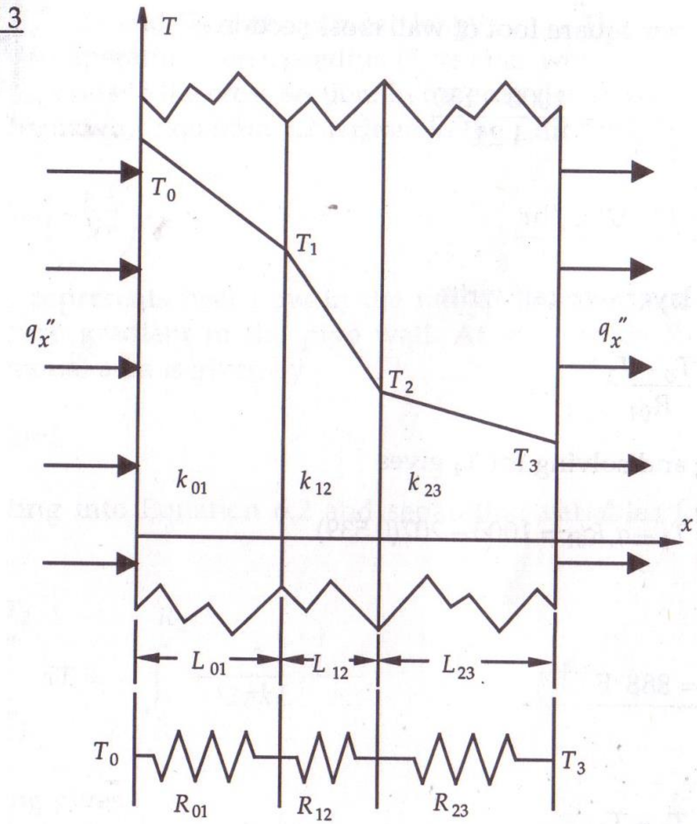
(b) $\underline{R = 0.692\ \text{hr·°R/BTU}}$

$$q_x = \frac{T_0 - T_3}{R_{03}} \qquad q_x = \frac{T_0 - T_1}{R_{01}} = \frac{T_1 - T_2}{R_{12}} = \frac{T_2 - T_3}{R_{23}} = \frac{T_0 - T_3}{R_{03}}$$

$$R_{03} = R_{01} + R_{12} + R_{23}$$

$$R_{03} = \frac{L_{01}}{Ak_{01}} + \frac{L_{12}}{Ak_{12}} + \frac{L_{23}}{Ak_{23}}$$

$$q_x = \frac{T_0 - T_3}{R_{03}} = \frac{T_0 - T_3}{\dfrac{L_{01}}{Ak_{01}} + \dfrac{L_{12}}{Ak_{12}} + \dfrac{L_{23}}{Ak_{23}}}$$



EXAMPLE 6.2. An oven wall consists of three layers of brick arranged as in Figure 6.3. The inside wall is made of silica brick, 4 in. thick, covered with masonry brick, 8 in. thick, while the outside layer is of common brick, 6 in. thick. During operation, the inside oven wall temperature reaches 1000°F and the outside surface temperature is 130°F. Calculate the heat transferred through the wall per square foot. Determine also the interface temperatures.

From Table 6.1, we read the following values for thermal conductivity:

Silica brick $k_{01} = 0.618$ BTU/hr·ft·°R
Masonry brick $k_{12} = 0.38$ BTU/hr·ft·°R
Common brick $k_{23} = 0.26$ BTU/hr·ft·°R

We now calculate the resistance offered by each layer assuming a cross-sectional area A of 1 ft^2:

Silica brick $R_{01} = L_{01}/Ak_{01} = (4/12)/(1(0.618)) = 0.539$ hr·°R/BTU
Masonry brick $R_{12} = L_{12}/Ak_{12} = (8/12)/(1(0.38)) = 1.75$ hr·°R/BTU
Common brick $R_{23} = L_{23}/Ak_{23} = (6/12)/(1(0.26)) = 1.92$ hr·°R/BTU

The total resistance for the three layers is the sum of these:

$$R_{03} = R_{01} + R_{12} + R_{23} = 0.539 + 1.75 + 1.92 \tag{6.6a}$$

or $R_{03} = 4.21$ hr·°R/BTU

The heat loss per square foot of wall cross section is

$$q_x = \frac{T_0 - T_3}{R_{03}} = \frac{1000 - 130}{4.21} \tag{6.5}$$

or $\underline{q_x = 207 \text{ BTU/hr}}$

For the first layer, we can write

$$q_x = \frac{T_0 - T_1}{R_{01}}$$

Rearranging and solving for T_1 gives

$$T_1 = T_0 - q_x R_{01} = 1000 - 207(0.539)$$

Solving,

$$\underline{T_1 = 888°\text{F}}$$

Similarly,

$$q_x = \frac{T_1 - T_2}{R_{12}}$$

and $T_2 = T_1 - q_x R_{12} = 888 - 207(1.75)$

Solving,

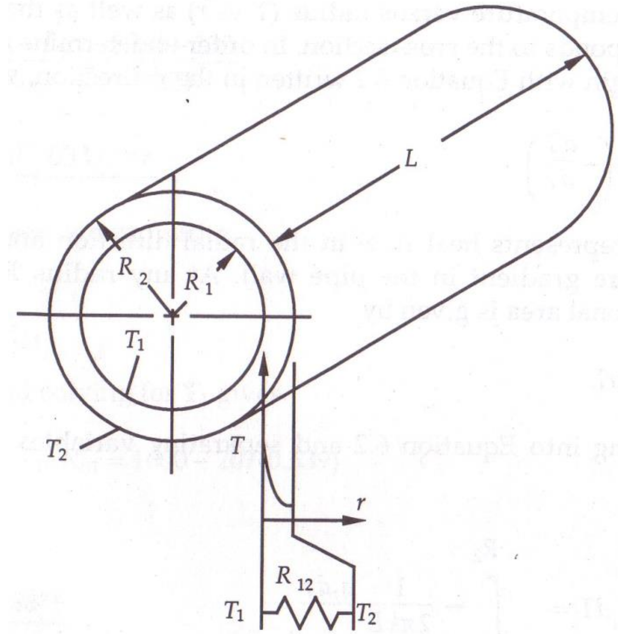
$$\underline{T_2 = 526°\text{F}}$$

6.2 Conduction of Heat Through a Cylindrical Wall

$$q_r = kA\left(-\frac{dT}{dr}\right) \qquad A = 2\pi rL$$

$$q_r = \frac{T_1 - T_3}{R_{13}} \qquad q_r = \frac{T_1 - T_2}{R_{12}} = \frac{T_2 - T_3}{R_{23}}$$

$$q_r = \frac{T_1 - T_3}{\dfrac{1}{2\pi k_{12} L}\ln\dfrac{R_2}{R_1} + \dfrac{1}{2\pi k_{23} L}\ln\dfrac{R_3}{R_2}}$$



EXAMPLE 6.3. A steel pipe [k = 40 W/(m·K)] is insulated with kapok insulation, similar in cross section to the sketch of Figure 6.5. The pipe carries a fluid that maintains the inside surface at 100°C. The outside surface of the insulation is at 25°C. The pipe is 4-nom sch 40 and the insulation is 6 cm thick. Determine the heat transferred per unit length through the cylindrical wall, and the temperature at the interface between the two materials.

From Appendix Table D.1, we read the following dimensions of 4-nom sch 40 pipe:

$$OD = 11.43 \text{ cm} \qquad ID = 10.23 \text{ cm}$$

From Table 6.1, the thermal conductivity of kapok is 0.035 W/(m·K). In terms of the notation of Figure 6.5, we have for each radius

$$R_1 = 10.23/2 = 5.12 \text{ cm}$$
$$R_2 = 11.43/2 = 5.72 \text{ cm}$$
$$R_3 = R_2 + 6 = 11.72 \text{ cm}$$

Also, for each material,

steel $\quad k_{12}$ = 40 W/(m·K)
kapok $\quad k_{23}$ = 0.035 W/(m·K)

Assuming a unit length, the resistances are calculated to be

$$\text{steel} \quad R_{12} = \frac{1}{2\pi k_{12} L} \ln \frac{R_2}{R_1} = \frac{1}{2\pi\,(40)(1)} \ln \frac{5.72}{5.12} = 0.000\,44\ \text{K/W}$$

$$\text{kapok} \quad R_{23} = \frac{1}{2\pi k_{23} L} \ln \frac{R_3}{R_2} = \frac{1}{2\pi\,(0.035)(1)} \ln \frac{11.72}{5.72} = 3.26\ \text{K/W}$$

As seen from these figures, the insulation offers a much greater resistance to the flow of heat than does steel. The total resistance then is

$$R_{13} = R_{12} + R_{23} = 3.26\ \text{K/W}$$

The heat transfer rate becomes

$$q_r = \frac{T_1 - T_3}{R_{13}} = \frac{100 - 25}{3.26} \tag{6.9}$$

or $\underline{q_r = 23\ \text{W}}$

In order to find the interface temperature we apply the heat flow equation to either material. For the steel,

$$q_r = \frac{T_1 - T_2}{R_{12}}$$

Rearranging and solving for the interface temperature gives

$$T_2 = T_1 - q_r R_{12} = 100 - 23(0.000\,44)$$

Solving,

$$\underline{T_2 \approx 100°\text{C}}$$

The temperature drop across the steel is virtually negligible. In many practical problems, temperature within a metal is often assumed to be a constant throughout.

6.3 Convection Heat Transfer—The General Problem

Heat transfer by convection occurs when a solid surface is in contact with a moving fluid and a temperature difference exists between the two. We identify two different ways convection heat transfer takes place: **forced convection** and **natural convection.** Forced convection occurs when the fluid motion is due to an external motive force. Natural convection (also traditionally known as **free convection**) occurs if fluid motion is induced by the transfer of heat.

The heat transferred by convection is calculated by use of a **convection coefficient** $\bar{h}$. The dimensions of the convection coefficient are $F{\cdot}L/(T{\cdot}L^2{\cdot}t)$ [$W/(m^2{\cdot}K)$ or $BTU/hr{\cdot}ft^2{\cdot}{}^\circ R$)]. The overbar denotes that the convection coefficient for the problem of interest is an average value, valid over the entire surface or geometry. In this text, the overbar is not used, in order to simplify the notation and because the average value of the convection coefficient h is all that will be used here.

Measurements of heat transfer rates and temperatures must be made in order to calculate the convection coefficient. It has been found that the convection coefficient is a function of temperature difference and of actual temperatures. Therefore, the convection coefficient (also called the **surface coefficient**) cannot be calculated except by trial-and-error methods. Once the convection coefficient h is known, the heat transfer rate can be found with

$$q = hA(T_s - T_\infty)$$

$$q = \frac{T_s - T_\infty}{R_{s\infty}} = \frac{T_s - T_\infty}{1/hA}$$

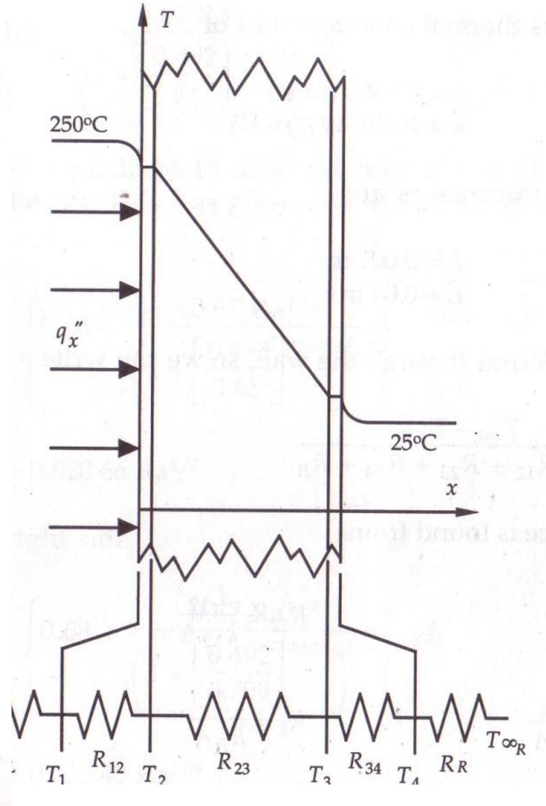
$$R_{s\infty} = \frac{1}{hA}$$

6.4 Convection Heat Transfer Problems: Formulation and Solution

TABLE 6.2. *Some commonly encountered dimensionless groups.*

Ratio	Symbol	Name
hL/k	**Bi**	Biot number
$\mu V^2/[k_f(T_s - T_\infty)]$	**Br**	Brinkman number
$2D_f g_c/\rho V^2 D^2$	C_D	Drag coefficient
$2\,\Delta p D/\rho V^2 L$	f	Friction factor
V^2/gL	**Fr**	Froude number
$g\beta(T_s - T_\infty)L^3/\nu^2$	**Gr**	Grashof number
hL/k_f	**Nu**	Nusselt number
VL/α	**Pe = Re · Pr**	Peclet number
$C_p\mu/k_f = \nu/\alpha$	**Pr**	Prandtl number
$2\,\Delta p g_c/\rho V^2$	C_p	Pressure coefficient
$g\beta(T_s - T_\infty)L^3/\nu\alpha$	**Ra = Gr · Pr**	Rayleigh number
$\rho VD/\mu g_c$	**Re**	Reynolds number
$h/\rho VC_p$	$\mathbf{St} = \dfrac{\mathbf{Nu}}{\mathbf{Re \cdot Pr}}$	Stanton number
$\rho V^2 L/\sigma g_c$	**We**	Weber number

EXAMPLE 6.4. A 1-m-tall vertical wall of a kitchen oven consists of three materials placed in series—sheet metal, insulation, and sheet metal. The sheet metal pieces are made of carbon steel and are 1 mm thick, while the (glass fiber) insulation is 4 cm thick. Inside the oven, the air temperature is 250°C and heat is convected to the wall. Determine the heat transferred through the wall if the outside surface is in contact with air at 25°C.



Reference to a heat transfer textbook shows that a number of equations are available to determine a natural convection coefficient for a vertical wall. Here we use the experimentally determined Churchill-Chu equation:

$$\mathbf{Nu} = \frac{hL}{k_f} = 0.68 + \frac{0.67\,\mathbf{Ra}^{1/4}}{\left(1 + \left[\dfrac{0.492}{\mathbf{Pr}}\right]^{9/16}\right)^{4/9}} \qquad (6.15)$$

where

$$\mathbf{Ra} = \frac{g\beta(T_s - T_\infty)L^3}{\nu\alpha} < 10^9; \qquad 0 < \mathbf{Pr} = \frac{\nu}{\alpha} = < \infty$$

and

$$\beta = 1/T_\infty \ = \text{coefficient of thermal expansion}$$

For convection heat transfer on the outside of each piece of sheet metal, we will need air properties. The properties of air vary with temperature, so we must first decide on the temperature to use for both cases. For the left-hand side (Figure 6.6), the air temperature is 250°C (+ 273 = 523 K), but near the surface of the sheet metal, the temperature is known to be somewhat lower. So for the left-hand side, we *intuitively* elect to use properties evaluated at 500 K. Similarly for the right-hand side, we have an air temperature of 25°C (= 298 K). We evaluate properties at 300 K. From Table C.2, we read for air at 500 K:

$\rho = 0.705\ \text{kg/m}^3$ $\qquad C_p = 1\,029.5\ \text{J/(kg·K)}$

$k_f = 0.040\,38\ \text{W/(m·K)}$ $\qquad \alpha = 0.556\,4 \times 10^{-4}\ \text{m}^2/\text{s}$

$\nu = 37.90 \times 10^{-6}\ \text{m}^2/\text{s}$ $\qquad \text{Pr} = 0.68$

For air at 300 K, we read

$\rho = 1.177\ \text{kg/m}^3$ $\qquad C_p = 1\,005.7\ \text{J/(kg·K)}$

$k_f = 0.026\,24\ \text{W/(m·K)}$ $\qquad \alpha = 0.221\,60 \times 10^{-4}\ \text{m}^2/\text{s}$

$\nu = 15.68 \times 10^{-6}\ \text{m}^2/\text{s}$ $\qquad \text{Pr} = 0.708$

Table 6.1 gives thermal conductivities of

sheet metal	$k = 43\ \text{W/(m·K)}$
glass fiber	$k = 0.035\ \text{W/(m·K)}$

The material thicknesses are

sheet metal	$L = 0.001\ \text{m}$
glass fiber	$L = 0.04\ \text{m}$

$$q = \frac{T_{\infty L} - T_{\infty R}}{R_L + R_{12} + R_{23} + R_{34} + R_R} \quad \text{(i)}$$

Each resistance is found from

$$R_L = \frac{1}{h_L A} \qquad R_{12} = \frac{L_{12}}{k_{12} A} \qquad R_{23} = \frac{L_{23}}{k_{23} A}$$

$$R_{34} = \frac{L_{34}}{k_{34} A} \qquad R_R = \frac{1}{h_R A}$$

It is apparent that the temperature within the sheet metal pieces is uniform due to how thin they are; therefore

$$T_1 \approx T_2 \qquad T_3 \approx T_4 \qquad R_{12} = 0 \qquad R_{34} = 0$$

Each resistance contains a cross-sectional area term A; and because area is unspecified, we assume an area of 1 m^2 and perform the calculations on a per-square-meter basis. The resistances are now determined as

$$R_L = \frac{1}{h_L} \quad \text{(evaluated with Equation 6.15)} \quad \text{(ii)}$$

$$R_{12} = 0 \qquad R_{23} = \frac{L_{23}}{k_{23} A} = \frac{0.04}{0.035(1)} = 1.143 \text{ K/W}$$

$$R_{34} = 0$$

$$R_R = \frac{1}{h_R} \quad \text{(evaluated with Equation 6.15)} \quad \text{(iii)}$$

To find the convection coefficients we start with the Churchill-Chu equation:

$$\mathbf{Nu} = \frac{hL}{k_f} = 0.68 + \frac{0.67\,\mathbf{Ra}^{1/4}}{\left(1 + \left[\frac{0.492}{\mathbf{Pr}}\right]^{9/16}\right)^{4/9}} \qquad (6.15)$$

The length term in Equation 6.15 does not refer to a wall thickness but instead to a wall height. This was given as $L = 1$ m. So for the left side of the wall,

$$h_L = \frac{0.040\,38}{1}\left\{0.68 + \frac{0.67\,\mathbf{Ra}^{1/4}}{\left(1 + \left[\frac{0.492}{0.68}\right]^{9/16}\right)^{4/9}}\right\}$$

$$\text{or} \quad h_L = 0.027\,46 + 0.020\,66\,\mathbf{Ra}^{1/4} \qquad \text{(iv)}$$

Likewise for the right side,

$$h_R = \frac{0.026\,24}{1}\left\{0.68 + \frac{0.67\,\mathbf{Ra}^{1/4}}{\left(1 + \left[\frac{0.492}{0.708}\right]^{9/16}\right)^{4/9}}\right\}$$

$$\text{or} \quad h_R = 0.017\,84 + 0.013\,49\,\mathbf{Ra}^{1/4} \qquad \text{(v)}$$

The Rayleigh number is calculated with

$$\mathbf{Ra} = \frac{g\beta(T_s - T_\infty)L^3}{\nu\alpha}; \qquad \text{where } \beta = 1/T_\infty$$

For the left side, $\beta = 1/T_{\infty L} = 1/(250 + 273) = 0.001\,912/\text{K}$, and for the right side, $\beta = 1/T_{\infty R} = 1/(25 + 273) = 0.003\,356/\text{K}$. The Rayleigh numbers are now calculated as

$$\mathbf{Ra}_L = \frac{(9.81)(0.001\,912)(250 - T_1)}{(37.90 \times 10^{-6})(0.556\,4 \times 10^{-4})} = 8.895 \times 10^6(250 - T_1) \qquad \text{(vi)}$$

$$\mathbf{Ra}_R = \frac{(9.81)(0.003\,356)(T_4 - 25)}{(15.68 \times 10^{-6})(0.221\,60 \times 10^{-4})} = 9.475 \times 10^7(T_4 - 25) \qquad \text{(vii)}$$

In addition to Equation i above, we can write other equations for the heat transferred through the wall:

$$q = \frac{T_{\infty L} - T_2}{R_L + R_{12}} \qquad q = \frac{T_3 - T_{\infty R}}{R_{34} + R_R}$$

Rearranging these equations gives

$$T_2 = T_{\infty L} - q\,(R_L + R_{12}) \qquad \text{(viii)}$$

$$T_3 = T_{\infty R} + q\,(R_{34} + R_R) \qquad \text{(ix)}$$

Interface temperatures can be determined with the above equations when heat transfer rate is known.

We now formulate an iterative procedure to determine the heat transferred through the wall. The steps are as follows:

- Assume temperatures T_1 (= T_2) and T_3 (= T_4). Calculate:
- Rayleigh numbers $\mathbf{Ra}_L$ and $\mathbf{Ra}_R$ with Equations vi and vii;
- Convection coefficients h_L and h_R with Equations iv and v;
- Resistances R_L and R_R with Equations ii and iii;
- Heat transfer rate q with Equation i;
- Refined values of T_2 and T_4 with Equations viii and ix; and
- Repeat the calculations with the new interface temperatures.

The procedure is repeated until convergence within a tolerable limit is achieved. The following tables summarize the results of these calculations.

$T_1 = T_2$ (assumed)	$\mathbf{Ra}_L$ (Eq. vi)	h_L (iv)	R_L (ii)	q (i)	T_2 (viii)
225°C	2.224 x 10^2	2.550 W/(m²·K)	0.392 1 K/W	115.1 W	205°C
205	4.003 x 10^2	2.950	0.339 0	127.4	207
207	3.825 x 10	2.917	0.342 9	125.6	207
207	3.825 x 10^2	2.917	0.342 9	126.0	206.8

$T_3 = T_4$ (assumed)	$\mathbf{Ra}_R$ (Eq. vii)	h_R (v)	R_R (iii)	q (i)	T_3 (ix)
35°C	9.475 x 10^8	2.385 W/(m²·K)	0.419 4 K/W	115.1 W	73.3°C
73.3	4.576 x 10^9	3.527	0.238 6	127.4	61.1
61.1	3.420 x 10^9	3.280	0.304 9	125.6	63.3
63.3	3.630 x 10^9	3.329	0.300 4	126.0	62.8
close enough					

The solution then is:

$\underline{q = 126 \text{ W}}$ (for each m² of surface)

EXAMPLE 6.5. A horizontally laid 2-nom sch 40 steel pipe (k = 25 BTU/hr·ft·°R) is lagged with fiberglas insulation (k = 0.02 BTU/hr·ft·°R) that is 1 in. thick. The pipe conveys steam that maintains the inside surface temperature at 250°F. Air outside the insulation is at 80°F. Determine the heat loss through the pipe and insulation.

Solution: Figure 6.7 shows a cross section of the insulated pipe, as well as a temperature profile and the appropriate resistances to the flow of heat. Heat is transferred by conduction through the pipe wall and through the insulation. Heat is transferred by natural convection from the outside surface of the insulation to the surrounding air. Heat is also transferred to the surroundings by radiation.

Assumptions:

1. The system is at steady state.
2. Properties of the materials are constant.
3. Air properties are constant and are evaluated at 80°F.
4. Radiation heat transfer is neglected.

Reference to a heat transfer textbook shows that a number of equations are available to determine a natural convection coefficient for a horizontal cylinder. Here we use another experimentally determined equation developed by Churchill-Chu:

$$\mathbf{Nu} = \frac{hD}{k_f} = \left\{0.60 + \frac{0.387\,\mathbf{Ra}^{1/6}}{\left(1 + \left[\dfrac{0.559}{\mathbf{Pr}}\right]^{9/16}\right)^{8/27}}\right\}^2 \qquad (6.16)$$

where

$$10^{-5} < \mathbf{Ra} = \frac{g\beta(T_s - T_\infty)L^3}{\nu\alpha} < 10^{12}; \qquad 0 < \mathbf{Pr} = \frac{\nu}{\alpha} < \infty$$

and

$\beta = 1/T_\infty$ = coefficient of thermal expansion

From Appendix Table D.1, we read the following for 2-nom sch 40 pipe:

OD = 2.375 in. $\qquad ID$ = 0.1723 ft

In terms of the notation of this problem,

R_1 = 0.1723 ft/2
R_2 = (2.375/12)/2 = 0.198 ft/2
R_3 = ((2.375 + 2)/12)/2 = 0.365 ft/2

$$\mathbf{Nu} = \frac{hD}{k_f} = \left\{0.60 + \frac{0.387\,\mathbf{Ra}^{1/6}}{\left(1 + \left[\dfrac{0.559}{\mathbf{Pr}}\right]^{9/16}\right)^{8/27}}\right\}^2 \qquad (6.16)$$

$$h = \frac{k_f}{2R_3}\left\{0.60 + \frac{0.387\,\mathbf{Ra}^{1/6}}{\left(1 + \left[\dfrac{0.559}{\mathbf{Pr}}\right]^{9/16}\right)^{8/27}}\right\}^2$$

Substituting gives

$$h = \frac{0.01516}{0.365}\left\{0.60 + \frac{0.387\,\mathbf{Ra}^{1/6}}{\left(1 + \left[\dfrac{0.559}{0.708}\right]^{9/16}\right)^{8/27}}\right\}^2$$

or $\quad h = 0.0415\,(0.6 + 0.321\,\mathbf{Ra}^{1/6})^2 \qquad$ (iv)

The iterative procedure is as follows:

- Assume T_3; then calculate:
- Rayleigh number **Ra** from Equation iii;
- Convection coefficient h from Equation iv;
- Resistance $R_{3\infty} = 1/hA = 1/(1.147h)$;
- Total resistance $\quad R_{1\infty} = R_{12} + R_{23} + R_{3\infty} = 0.000885 + 4.87 + R_{3\infty} = 4.87 + R_{3\infty}$
- Heat transferred $q = (T_1 - T_\infty)/R_{1\infty} = (250 - 80)/R_{1\infty} = 170/R_{1\infty}$
- Refined value of the surface temperature $T_3 = T_1 - q(R_{12} + R_{23})$
  $T_3 = 250 - 4.87q$ (from Equation ii); and
- Repeat the calculations until convergence is achieved.

The following table summarizes the results:

T_3	**Ra** (Eq iii)	h (iv)	$R_{3\infty} = 1/hA$	$R_{1\infty}$	q	T_3 (ii)
100°F	2.96 x 10^7	1.62	0.538	5.41	31.4	96.9
96.9	2.50 x 10^7	1.54	0.566	5.44	31.3	97.7
98	2.66 x 10^7	1.57	0.556	5.43	31.3	97.4

close enough

q = 31.3 BTU/hr (for each ft of pipe)

6.5 Optimum Thickness of Insulation

We can extend the results of the previous section to the problem of finding an optimum thickness of insulation for an insulated pipe or tube. The optimum thickness can be determined by straightforward calculation with suitable cost data. The procedure is to calculate heat loss for various insulation thicknesses. The annual cost of the heat loss for each thickness is expressed in terms of monetary units per heat loss unit ($/BTU or $/J): The annual installed cost of the insulation is found from the initial cost and the annual depreciation rate. The results are very similar to those developed earlier for optimum pipe diameter.

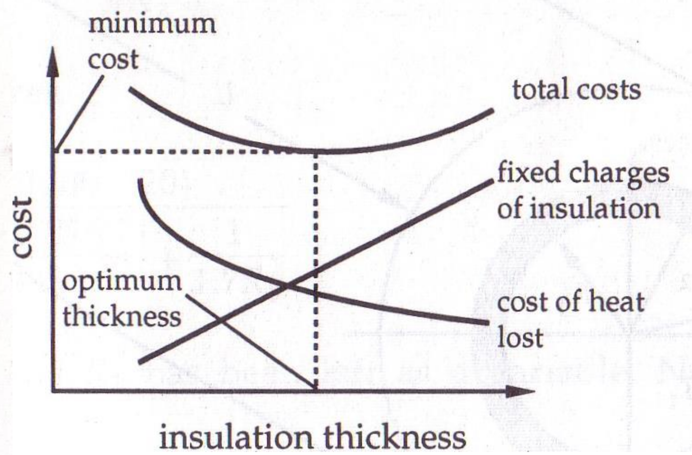


Figure 6.8 is a graph of costs for the optimum insulation thickness problem. The fixed charges increase with increasing insulation thickness. The cost associated with heat loss through the insulation decreases with increasing thickness. The total cost is the sum of these two costs, and it appears to have a minimum value that defines for us the optimum insulation thickness.

Critical Radius

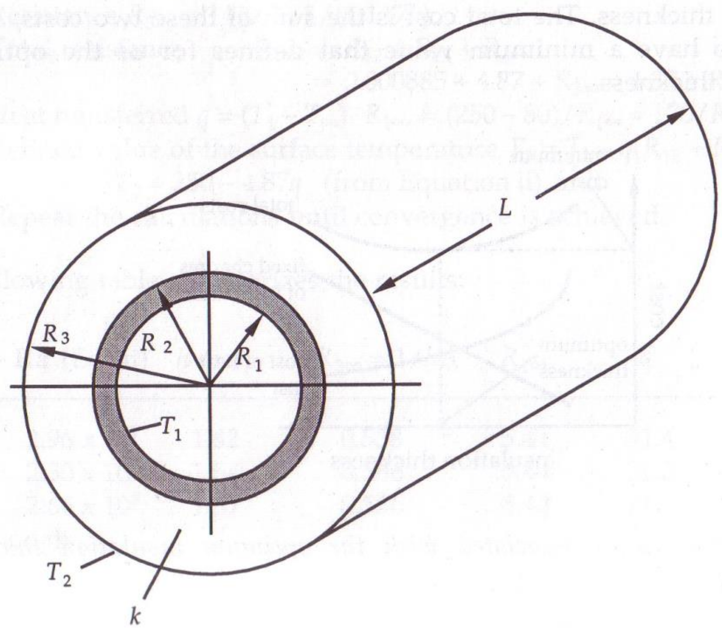
Under certain conditions, adding insulation to a heated pipe actually increases the heat transfer loss. As insulation is added, the conduction resistance increases, but so does the surface area. The increased surface area causes more heat to be transferred away by convection. We investigate this effect by an example.

Consider a heated 1-nominal pipe (OD = 3.340 cm; R = 1.67 cm) covered with kapok insulation [k = 0.035 W/(m·K)]. The outside diameter if the pipe is maintained at 100°C. The insulation transfers heat to the environment, which is at 20°C, and the convection coefficient is assumed constant at 1.7 W/(m²·K). We will determine the heat transferred to the ambient for various insulation thicknesses.

Figure 6.9 illustrates the insulated pipe. We define T_2 as the temperature of the surface exposed to the ambient. For no insulation, the heat loss is given by

$$q = \frac{T_2 - T_\infty}{1/hA_2} \tag{6.17a}$$

With $A_2 = 2\pi R_2 L$, we substitute to obtain the heat transfer per unit length as



$$\frac{q}{L} = \frac{T_2 - T_\infty}{1/h(2\pi R_2)} = 2\pi R_2 h(T_2 - T_\infty) \tag{6.17b}$$

Substituting,

$$\frac{q}{L} = 2\pi(1.67)(1.7)(100 - 20)$$

or $\frac{q}{L} = 14.3\ \text{W/m}$ (no insulation)

If insulation is added, we write

$$q = \frac{T_2 - T_\infty}{\dfrac{\ln(R_3/R_2)}{2\pi kL} + \dfrac{1}{hA_3}} \tag{6.18}$$

With $A_3 = 2\pi R_3 L$, Equation 6.18 becomes

$$q = \frac{T_2 - T_\infty}{\dfrac{\ln(R_3/R_2)}{2\pi kL} + \dfrac{1}{h2\pi R_3 L}} \tag{6.19a}$$

or
$$\frac{q}{L} = \frac{2\pi(T_2 - T_\infty)}{\dfrac{\ln(R_3/R_2)}{k} + \dfrac{1}{hR_3}} \tag{6.19b}$$

Substituting gives

$$\frac{q}{L} = \frac{2\pi(100 - 20)}{\dfrac{\ln(R_3/0.016\,7)}{0.035} + \dfrac{1}{1.7R_3}} \tag{6.20}$$

The outer radius R_3 has been left as a variable. Now the insulation thickness is

$$t = R_3 - R_2 = R_3 - 0.016\,7 \tag{6.21}$$

or $R_3 = t + 0.016\,7$

We select values for the insulation thickness and calculate R_3 and q/L using Equation 6.20. The results are given in the following table and graphed in Figure 6.10.

thickness t in m	R_3 in m	q/L in W/m
0	0	14.3
0.005	0.021 7	14.5
0.01	0.026 7	14.2
0.015	0.031 7	13.6
0.02	0.036 7	13.0
0.025	0.041 7	12.5

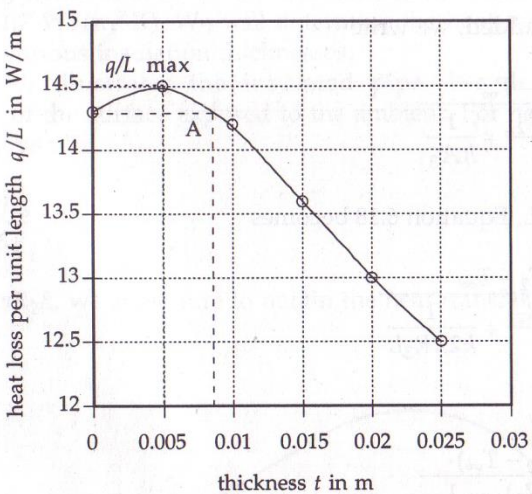


FIGURE 6.10. *Heat transfer variation with insulation thickness.*

As shown in the figure, adding insulation with a thickness of less than 1 cm causes an increase in the heat transfer rate. At 1 cm, the heat transfer rate just about equals that for no insulation. Beyond 1 cm thickness, the heat transfer rate decreases steadily. The maximum q/L occurs at an insulation thickness of about 0.5 cm.

To obtain a more general relationship between insulation thickness and heat transfer rate, we refer to Equation 6.19b:

$$\frac{q}{L} = \frac{2\pi(T_2 - T_\infty)}{\dfrac{\ln(R_3/R_2)}{k} + \dfrac{1}{hR_3}} \qquad (6.19)$$

Differentiating with respect to R_3 and setting the result equal to zero will give an equation for what is defined as the critical value of R_3 for which q/L is a maximum. This radius is called the **critical radius.** Differentiating Equation 6.19b, we get

$$\frac{d(q/L)}{dR_3} = -\frac{2\pi(T_2 - T_\infty)\left(\dfrac{1}{k}\dfrac{1}{R_3} - \dfrac{1}{hR_3^2}\right)}{\left(\dfrac{\ln(R_3/R_2)}{k} + \dfrac{1}{hR_3}\right)^2} = 0$$

Solving for the critical radius R_{cr} gives

$$R_{cr} = \frac{k}{h} \qquad (6.22)$$

This is the value of R_3 (the outside radius of the insulation) that gives a maximum value for the heat transferred. For the example just discussed,

$$R_{cr} = \frac{0.035}{1.7} = 0.020\,6 \text{ m} = 2.06 \text{ cm}$$

The critical insulation thickness is

$$t_{cr} = R_{cr} - R_2 = 0.020\,6 - 0.016\,7 = 0.003\,9 \text{ m}$$

or $t_{cr} = 0.39$ cm

The corresponding heat transfer per unit length is

$$\left.\frac{q}{L}\right|_{max} = \frac{2\pi(100 - 20)}{\dfrac{\ln(0.020\,6/0.016\,7)}{0.035} + \dfrac{1}{1.7(0.020\,6)}}$$

$$\left.\frac{q}{L}\right|_{max} = 14.5 \text{ W/m}$$

If the outer radius of insulation is less than k/h, then adding insulation increases the heat transferred. Conversely, if the outer radius of insulation is greater than k/h, then adding insulation decreases the heat transferred.

A more practical way to look at this problem is to find the insulation thickness that corresponds to the case of no insulation. Mathematically, the critical value of R_3 is found as k/h. Practically, however, we would rather know the thickness of insulation required for q/L to equal the no insulation case. Referring to Figure 6.10, we are seeking the insulation

thickness that corresponds to point A. To find this value, we set Equation 6.17b equal to Equation 6.19a:

$$q = \frac{T_2 - T_\infty}{1/h2\pi R_2 L} = \frac{T_2 - T_\infty}{\dfrac{\ln(R_3/R_2)}{2\pi kL} + \dfrac{1}{h2\pi R_3 L}}$$

Canceling $2\pi L$ and the temperature difference gives

$$\frac{1}{hR_2} = \frac{\ln(R_3/R_2)}{k} + \frac{1}{hR_3}$$

Clearing fractions,

$$kR_3 = R_2R_3h \ln(R_3/R_2) + kR_2$$

Rearranging and solving for R_3,

$$R_3 = \frac{R_2}{1 - (R_2h/k)\ln(R_3/R_2)} \qquad (6.23)$$

For the example previously discussed,

$$R_3 = \frac{0.016\,7}{1 - (0.016\,7(1.7)/0.035)[\ln(R_3/0.016\,7)]}$$

At first glance, it would appear that this equation could be solved iteratively by substituting a value for R_3 into the right-hand side to obtain an improved value, which again is substituted, and the procedure repeated until convergence is achieved. Although elegant, however, this method does not work. A more successful approach is merely to assume a value of R_3 and substitute it into the right-hand side. The result is compared to the left-hand side and the difference is taken. This procedure is repeated until the difference is zero or very small. The following table summarizes the calculations:

Trial	Assumed R_3 in m	RHS in m	R_3 – RHS
1	0.03	0.031 8	−0.001 8
2	0.029	0.030 2	−0.001 2
3	0.028	0.028 7	−0.000 7
4	0.026	0.027 3	0.000 056
5	0.025	0.026 0	0.000 17
6	0.025 5	0.024 8	0.000 6
7	0.025 7	0.025 4	0.000 2
8	0.025 8	0.025 8	0.000 004 close enough

This method takes only a few iterations, and it gives us the radius of insulation required so that q/L with insulation exceeds q/L for no insulation. We define this radius as the **practical radius**. The corresponding thickness for this example then is

$$t = R_3 - R_2 = 0.025\,8 - 0.016\,7 = 0.009\,1 \text{ m}$$

or $t = 0.91$ cm

It is this thickness that must be exceeded to yield a decrease in heat loss.

The effect of increasing heat transfer by adding insulation occurs usually for small pipe diameters and low values of the convection coefficient. For steam flowing in a pipe, it is usually not desirable to have a loss of energy through the pipe wall. Therefore, the critical (or practical) thickness should be calculated and the pipe insulated properly. On the other hand, current flowing through a wire could generate heat and raise the temperature of the wire. Insulation in this case could be added in order to help dissipate the heat and to keep the wire from overheating and from short-circuiting.

A vertical wall is made up of fiberglas (5 cm thick) attached to common brick (10 cm thick). The uninsulated side of the brick is at –40°C during winter months. The insulation receives energy by convection from the surrounding air, whose temperature is 20°C. Determine the interface temperature between the two materials under these conditions.

A new building material made of plywood with styrofoam insulation glued to it is to be tested. The plywood is 2 cm thick and the styrofoam is 7 cm thick. The test is to be performed while the sample is in a vertical configuration. On the plywood side, the sample is exposed to air at 35°C. On the styrofoam side, the sample is in contact with air at 0°C. Calculate the interface temperature between the two materials and the heat transferred through the wall.

A 1-nominal pipe is covered with glass wool insulation. The outside wall temperature of the pipe is 120°C, and the insulation is exposed to air at 25°C. The convection coefficient between the insulation and the air is 1 W/(m^2·K). Graph q/L as a function of the insulation thickness.