CHAPTER 0

Review

The basic concepts and rules of probability

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Key words:

Probability, objective Probability, subjective Probability, equally likely Mutually exclusive, multiplicative rule, Conditional Probability, independent events, Bayes theorem.

3.1 INTRODUCTION

- The concept of probability is frequently encountered in everyday communication. For example, a physician may say that a patient has a 50-50 chance of surviving a certain operation.
- Another physician may say that she is 95 percent certain that a patient has a particular disease.
- Most people express probabilities in terms of percentages. But, it is more convenient to express probabilities as fractions. Thus, we may measure the probability of the occurrence of some event by a number between 0 and 1.
- The more likely event, has a probability closer to one.
- An event that can't occur has a probability of zero,
- and an event that is certain(مؤكد) to occur has a probability of one
- **Probability:**

  It is a number used to measure the chance of the occurrence of some event. The number between 0 and 1.

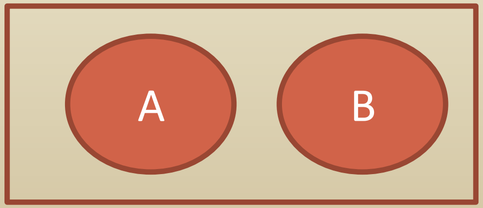
- **<u>Some definitions</u>:**

<u>1.Equally likely outcomes (النتائج المتساوية الاحتمال):</u>

Are the outcomes that have the same chance of occurring.

<u>2.Mutually exclusive (النتائج المتعارضة):</u>

Two events are said to be mutually exclusive if they **cannot occur simultaneously** such that $A \cap B = \Phi$.



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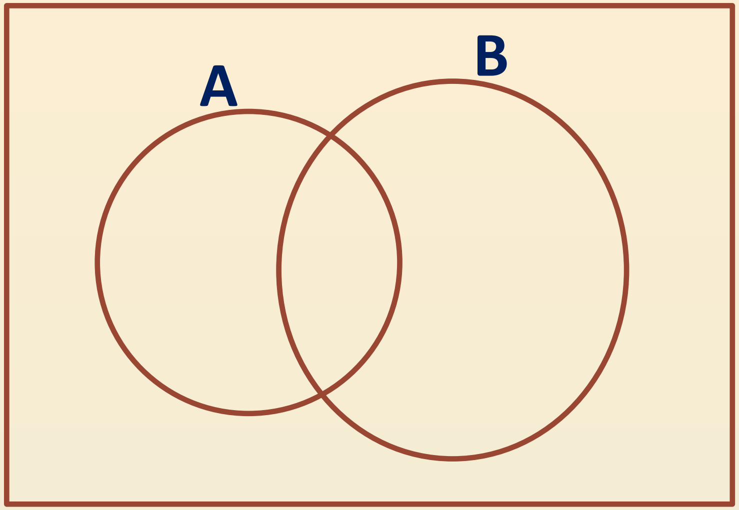
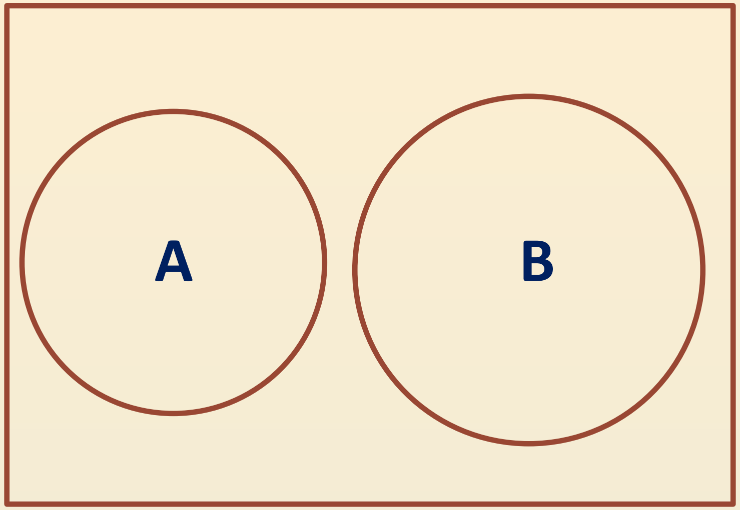




$$A \cap B \neq \phi$$

A **and** ***B*** **are not mutually exclusive**

$$A \cap B = \phi$$

A **and** ***B*** **are mutually exclusive (disjoint).**
A and B can not occur in the same time

- **The universal Set (S)**: The set of all possible outcomes.
- **The empty set Φ** : Contain no elements.
- **The event ,E** : is a set of outcomes in S which has a certain characteristic.
- **Classical Probability**

Classical Probability of an event:

If the experiment has **n(S) equally likely outcomes**, then the probability of the event E is denoted by P(E) and is defined by:

$$P(A) = \frac{n(A)}{n(S)}$$

Where: $n(A)$ no. of outcomes in E ,

$n(S)$ no. of outcomes in S ,

Therefore:

1. $P(\Phi) = \frac{n(\Phi)}{n(S)} = 0$
2. $P(S) = \frac{n(S)}{n(S)} = 1$
3. $0 \leq P(A) \leq 1,$

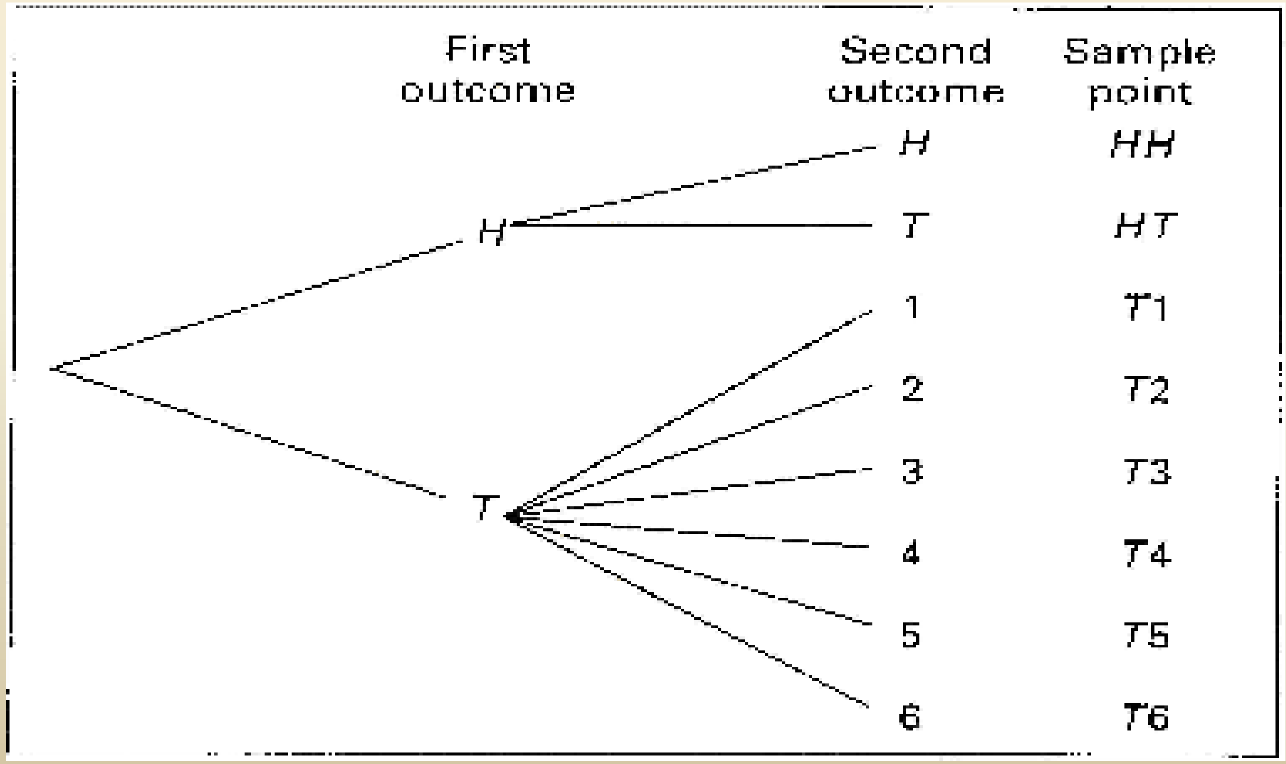
- **For Example:** in the rolling of the die , each of the six sides is equally likely to be observed . So, the probability that a 4 will be observed is equal to 1/6.

- **Relative Frequency Probability (optional):**
- **Def:** If some posses is repeated a large number of times, n, and if some resulting event E occurs m times , the relative frequency of occurrence of E , m/n will be approximately equal to probability of E . P(E) = m/n .
- *** **Subjective Probability** :
- Probability measures the confidence that a particular individual has in the truth of a particular proposition or occurrence of something.
- **For Example** :
- The probability of success of an operation is 0.85

Tree diagram

- A **Tree diagram** can be used to list the elements of the sample space systematically.
- **Example 1**. Flip a coin first. If a head occurs, flip it again; otherwise, toss a die.

$$S = \{HH, HT, T1, T2, T3, T4, T5, T6\}$$



•**Example 2**. Three items are selected at random from a manufacturing process (D is refers to defective, N (not defective)).

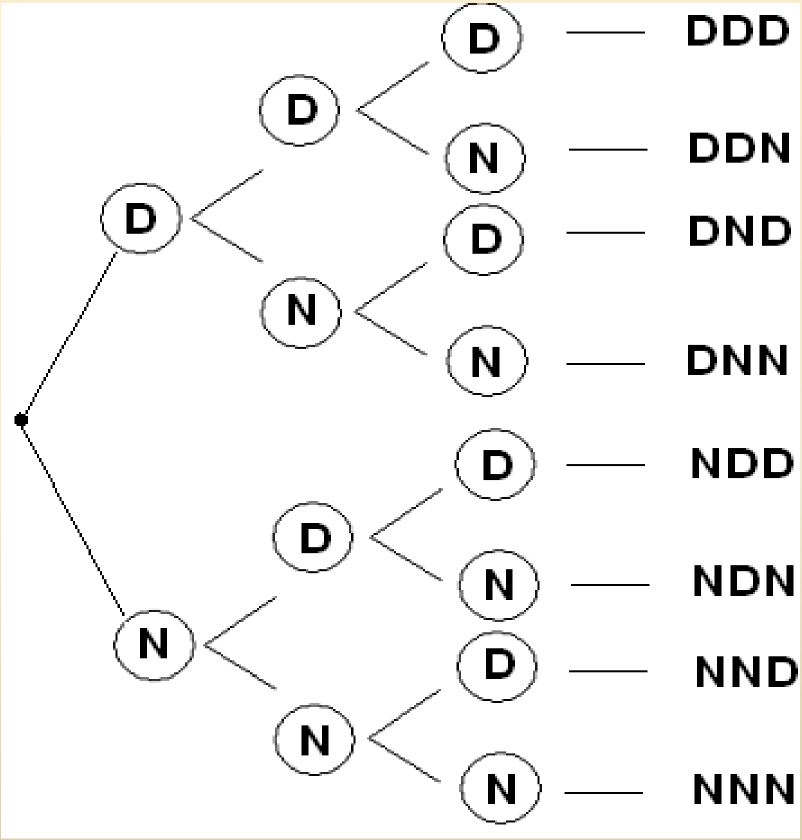
$$S = \{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\}$$

•**P (Two items are defective) = 3/8**

•**P (No items are defective) = 1/8**

•**P (no. of defectives = no. of not defectives) = 0**

•**P (one item is defective) = 3/8**



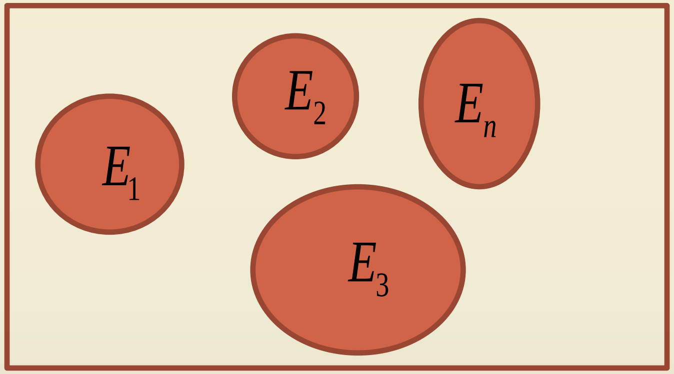
Elementary Properties of Probability:

- Given some process (or experiment) with n mutually exclusive events $E_1, E_2, E_3, \ldots\ldots\ldots\ldots, E_n$,

Then

- $1 \geq P(E_i) \geq 0, \; i = 1,2,3,\ldots\ldots n$
- $P(E_1) + P(E_2) + \ldots\ldots + P(E_n) = 1$
- $P(E_i \cup E_j) = P(E_i) + P(E_j)$

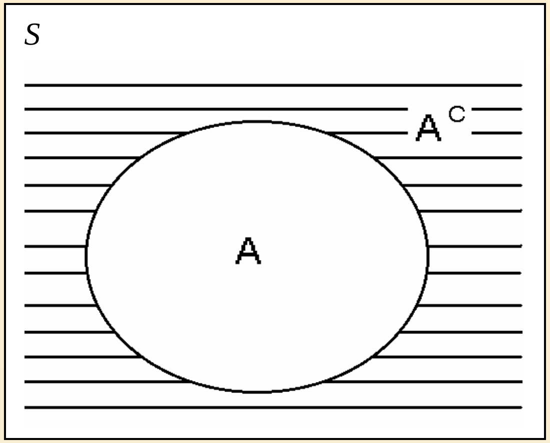
E_i, E_j are mutually exclusive

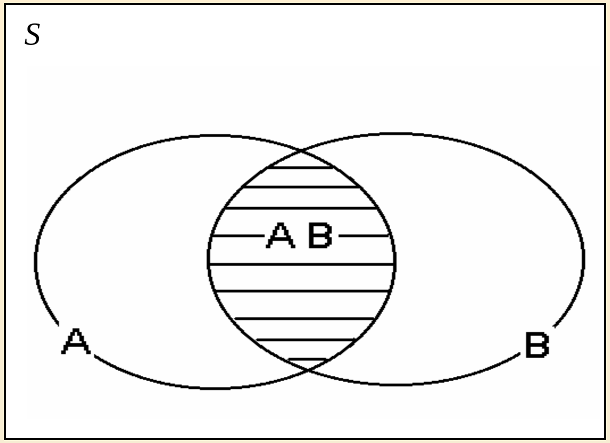


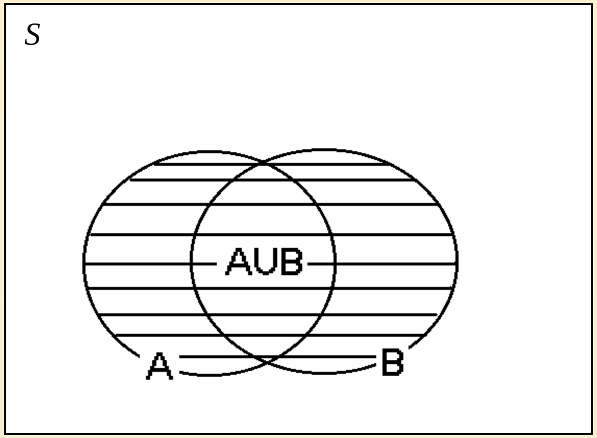
Rules of Probability

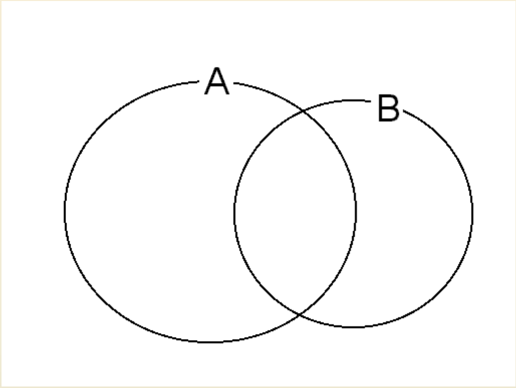
- **Addition Rule**
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

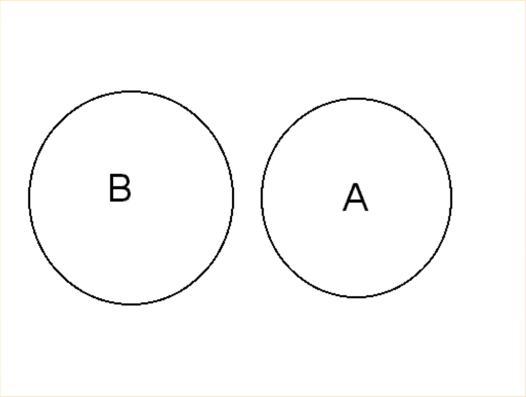
- If A and B are mutually exclusive (**disjoint**) ,then
  $P(A \cap B) = 0$
- Then , addition rule is
  $P(A \cup B) = P(A) + P(B)$.
- **Complementary Rule**
- $P(A') = 1 - P(A)$
- where, A' = = complement event
- Consider example 3.4.1 Page 63











$A \cap B \neq \phi$

***A* and *B* are not mutually exclusive**

$A \cap B = \phi$

***A* and *B* are mutually exclusive (disjoint)**

Example 3

التاريخ العائلي لاضطراب المزاج Family history of Mood Disorders	Early =< 18 **(E)**	Later >18 **(L)**	Total
Negative **(A)**	**28**	**35**	**63**
(اضطراب ذي قطبين) Bipolar Disorder **(B)**	**19**	**38**	**57**
أحادي Unipolar **(C)**	**41**	**44**	**85**
Unipolar and Bipolar **(D)**	**53**	**60**	**113**
Total	**141**	**177**	**318**

**Answer the following questions:

Suppose we pick a person at random from this sample.

1-The probability that this person will be 18-years old or younger?

2-The probability that this person has family history of mood orders Unipolar (C)?

3-The probability that this person has no family history of mood orders Unipolar ($\overline{C}$)?

4-The probability that this person is 18-years old or younger <u>or</u> has no family history of mood orders Unipolar (C))?

5-The probability that this person is more than18-years old <u>and</u> has family history of mood orders Unipolar and Bipolar(D)?

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P(E)=141/318=0.44

3-The probability that this person has no family history of mood orders Unipolar ($\overline{C}$)? $\overline{C}$

4-The probability that this person is 18-years old or younger <u>or</u> has no family history of mood orders Unipolar (C))?

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P($\bar{C}$)=1- (85/318) = 0.73

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$$P(E \cup \overline{C}) = P(E) + P(\overline{C}) - P(E \cap \overline{C})$$

$$= \frac{141}{318} + \frac{63+57+113}{318} - (\frac{28+19+53}{318})$$

$$= 0.44 + 0.73 - 0.31 = 0.86$$

5-The probability that this person is more than18-years old <u>and</u> has family history of mood orders Unipolar and Bipolar(D)?

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5-The probability that this person is more than18-years old <u>and</u> has family history of mood orders Unipolar and Bipolar(D)?

$$P(L \cap D) = \frac{60}{318} = 0.19$$

Conditional Probability:

P(A\B) is the probability of A assuming that B has happened.

- P(A\B)= $\frac{P(A\cap B)}{P(B)}$, P(B)≠ 0
- P(B\A)=$\frac{P(A\cap B)}{P(A)}$, P(A)≠ 0

Example 4

From example 3

- suppose we pick a person at random and find he is 18 years or younger (E),what is the probability that this person will be one with Negative family history of mood disorders (A)?

- suppose we pick a person at random and find he has family history of mood (D) what is the probability that this person will be 18 years or younger (E)?

Example 4

From example 3

- suppose we pick a person at random and find he is 18 years or younger (E),what is the probability that this person will be one with Negative family history of mood disorders (A)?

$$P(A|E)=\frac{P(A\cap E)}{P(E)}=\frac{\frac{28}{318}}{\frac{141}{318}}=\frac{28}{141}=0.199$$

- suppose we pick a person at random and find he has family history of mood (D) what is the probability that this person will be 18 years or younger (E)?

Example4

From example3

- suppose we pick a person at random and find he is 18 years or younger (E),what is the probability that this person will be one with Negative family history of mood disorders (A)?

$$P(A|E) = \frac{P(A \cap E)}{P(E)} = \frac{\frac{28}{318}}{\frac{141}{318}} = \frac{28}{141} = 0.199$$

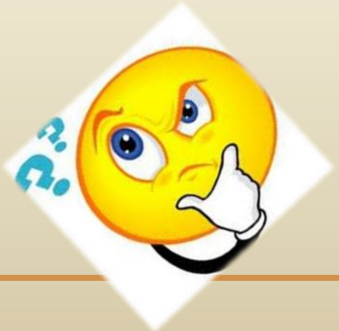
- suppose we pick a person at random and find he has family history of mood (D) what is the probability that this person will be 18 years or younger (E)?

$$P(E|D) = \frac{P(E \cap D)}{P(D)} = \frac{\frac{53}{318}}{\frac{113}{318}} = \frac{53}{113} = 0.47$$

Calculating a joint (مشترك)Probability :

<u>Example 5</u>

- In example 3, suppose we pick a person at random from the 318 subjects. Find the probability that he will **early (E)** **<u>and</u>** **has no family history of mood disorders A. i.e (A′).**



Calculating a joint (مشترك)Probability :

Example 5

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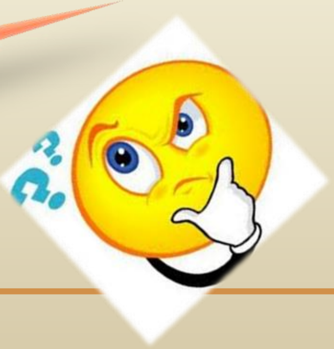
$P(E \cap A')$



Multiplicative Rule:

Joint probability.

- $P(A \cap B) = P(A \setminus B)\, P(B)$
- $P(A \cap B) = P(B \setminus A)\, P(A)$
- Where,
- $P(A)$: **marginal** probability of A.
- $P(B)$: **marginal** probability of B.
- $P(B \setminus A)$:The **conditional** probability.

Example 6

- From previous example no. 3, we wish to compute the joint probability of Early age at on set(E) and a negative family history of mood disorders(A) from a knowledge of an appropriate marginal probability and an appropriate conditional probability.

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$$P(E \cap A) = P(E \mid A)P(A) = \frac{28}{63} * \frac{63}{318} = \frac{28}{318}$$

$$P(E \cap A) = P(A \mid E)P(E) = \frac{28}{141} * \frac{141}{318} = \frac{28}{318}$$

Independent Events:

- If A has no effect on B, we said that A,B are independent events.
- Then,
- $$P(A \cap B) = P(B)P(A)$$
- $$P(A \backslash B) = P(A)$$
- $$P(B \backslash A) = P(B)$$

Example7

- In a certain high school class consisting of **60** girls and **40 boys**, it is observed that **24** girls and **16 boys** wear eyeglasses . If a student is picked at random from this class ,the probability that the student wears eyeglasses , P(E), is 40/100 or 0.4 .
- What is the probability that a student picked at random wears eyeglasses given that the student is a boy?
- What is the probability of the joint occurrence of the events of wearing eye glasses and being a boy?

Example 7

	G	B	Total
E	24	16	40
E'	36	24	60
Total	60	40	100

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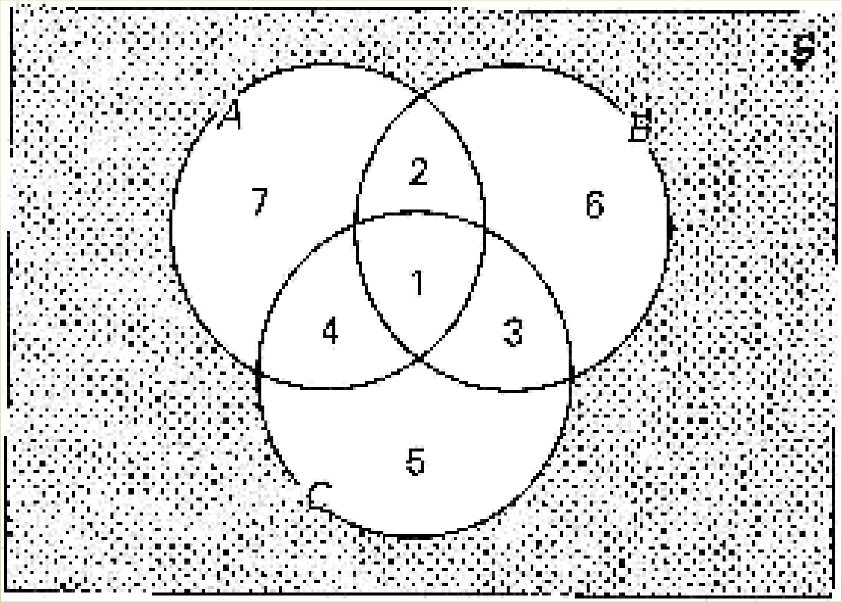
- What is the probability of the joint occurrence of the events of wearing eye glasses and being a boy?

$$\boxed{P(E \cap B) = P(E \mid B) * P(B) = 0.4 * 0.4 = 0.16}$$

Example 8

- Suppose that of 1200 admission to a general hospital during a certain period of time,750 are private admissions. If we designate these as a set A, then compute P(A) , P($\overline{A}$).

- For the following diagram find the probabilities of the following Events



- A ∩ B
- A U B
- A’
- B’
- A ∩ B ∩ C

Marginal Probability:

- **Definition:**

  Given some variable that can be broken down into m categories designated

by $A_1, A_2, \ldots\ldots, A_i, \ldots\ldots, A_m$ and another jointly occurring variable that is broken down into n categories

designated by $B_1, B_2, \ldots\ldots, B_j, \ldots\ldots, B_n$,

the marginal probability of A_i with all the categories of B. That is,

$$P(A_i) = \sum P(A_i \cap B_j), \quad \text{for all value of j}$$

Example 9

Use data of Table in example 3, and rule of marginal Probabilities to calculate P(E).

Marginal Probability:

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Example 9

Use data of Table in example 3, and rule of marginal Probabilities to calculate P(E)

$$P(E) = P(E \cap A) + P(E \cap B) + P(E \cap C) + P(E \cap D)$$

$$= \frac{28}{318} + \frac{19}{318} + \frac{41}{318} + \frac{53}{318} = \frac{141}{318}$$

Exercises

Q3.4.1: In a study of violent victimization of women and men, Porcelli et al. (A-2) collected information from 679 women and 345 men aged **18 to 64** years at several family practice centers in the metropolitan Detroit area. Patients filled out a health history questionnaire that included a question about victimization.

The following table shows the sample subjects cross-classified by sex and type of violent victimization reported. The victimization categories are defined as **no victimization**, **partner victimization** (and not by others), **victimization by persons other than partners** (friends, family members, or strangers), and those who reported **multiple victimization** partners (friends, family members, or strangers), and those who reported multiple victimization.

	No Victimization	Partners	Nonpartners	Multiple Victimization	Total
Women	611	34	16	18	679
Men	308	10	17	10	345
Total	919	44	33	28	**1024**

(a) Suppose we pick a subject at random from this group. What is the probability that this subject will be a women?

(b) What do we call the probability calculated in part a?

(c) Show how to calculate the probability asked for in part a by two additional methods.

(d) If we pick a subject at random, what is probability that the subject will be a women and have experienced partner abuse?

(e) What do we call the probability calculated in part d?

(f) Suppose we picked a man at random. Knowing this information, what is the probability that he experienced abuse from nonpartners?

(g) What do we call the probability calculated in part f?

(h) Suppose we pick a subject at random. What is the probability that it is a man or someone who experienced abuse from a partner?

(i) What do we call the method by which you obtained the probability in part h?

Q3.4.3: Fernando et al. (A-3) studied drug-sharing among injection drug users in the South Bronx in New York City. Drug users in New York City use the term "split a bag" or "get down on a bag" to refer to the practice of diving a bag of heroin or other injectable substances. A common practice includes splitting drugs after they are dissolved in a common cooker, a procedure with considerable HIV risk. Although this practice is common, little is known about the prevalence of such practices. The researchers asked injection drug users in four neighborhoods in the South Bronx if they ever “got down on” drugs in bags or shots. The results classified by gender and splitting practice are given below:

Gender	Split Drugs	Never Split Drugs	Total
Male	349	324	673
Female	220	128	348
Total	569	452	1021

State the following probabilities in words and calculate:

(a) $P(Male \cap Split\ Drugs)$ Ans: 0.3418

(b) $P(Male \cup Split\ Drugs)$ Ans: 0.8746

(c) $P(Male \mid Split\ \ Drugs)$ Ans: 0.6134

(d) $P(Male)$ Ans: 0.6592

Q3.4.4: Laveist and Nuru-Jeter (A-4) conducted a study to determine if doctor-patient race concordance was associated with greater satisfaction with care. Toward that end, they collected a national sample of African-American, Caucasian, Hispanic, and Asian-American respondents. The following table classifies the race of the subjects as well as the race of their physician:

Patient Race					
Physician's Race	Caucasian	African-American	Hispanic	Asian-American	Total
White	779	436	406	175	1796
African-American	14	162	15	5	196
Hispanic	19	17	128	2	166
Asian/Pacific-Island	68	75	71	203	417
Other	30	55	56	4	145
Total	910	745	676	389	2720

(a) What is the probability that a randomly selected subject will have an Asian/Pacific-Islander physician? Ans: 0.1533

(b) What is the probability that an African-American subject will have an African- American physician?

Ans: 0.2174

(c) What is the probability that a randomly selected subject in the study will be Asian-American and have an Asian/Pacific-Islander physician? Ans: 0.075

(d) What is the probability that a subject chosen at random will be Hispanic or have a Hispanic physician? Ans: 0.2625

(e) Use the concept of complementary events to find the probability that a subject chosen at random in the study does not have a white physician? Ans: 0.3397

Q3.4.5:

If the probability of left-handedness in acertain group of people is 0.5, what is the probability of right-handedness (assuming no ambidexterity)?

Q3.4.6:

The probability is 0.6 that a patient selected at random from the current residents of a certain hospital will be a male. The probability that the patient will be a male who is in for surgery is 0.2. A patient randomly selected from current residents is found to be a male; what is the probability that the patient is in the hospital for surgery?

Ans: 0.3333

Q3.4.7:

In a certain population of hospital patients the probability is 0.35 that a randomly selected patient will have heart disease. The probability is 0.86 that a patient with heart disease is a smoker. What is the probability that a patient randomly selected from the population will be a smoker and have heart disease?

Ans: 0.301

Example: Our sample space S is the population of adults in a small town. They can be categorized according to gender and employment status (see Table 2.1).

Table 2.1: Categorized adult population in a small town.

	Employed	Unemployed	Total
Male	460	40	500
Female	140	260	400
Total	600	300	900

- One individual is to be selected at random for a publicity tour.
- The concerned events
  - M : a man is chosen
  - E : the one chosen is employed
  - ***F:*** a Female is chosen
  - ***U:*** The one chosen is unemployed

$$P(M|E) = P(M \cap E)/P(E) = \frac{n(P(M \cap E)/n(S)}{n(E)/n(S)} = \frac{n(P(M \cap E))}{n(E)} = 460/600$$

2.8 Bayes' Rule:

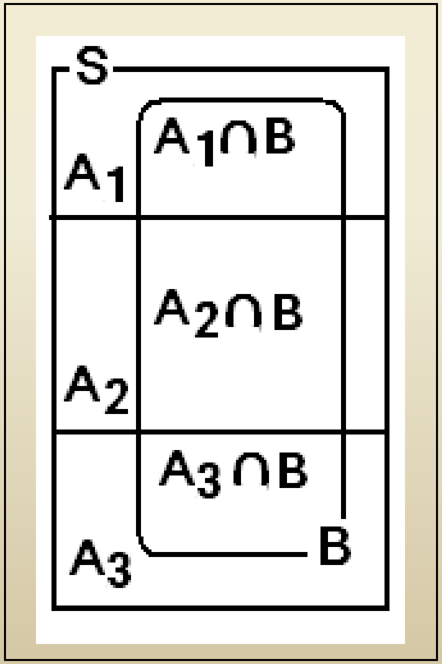
Definition:

The events $A_1, A_2, \ldots,$ and A_n constitute a partition of the sample space S if:

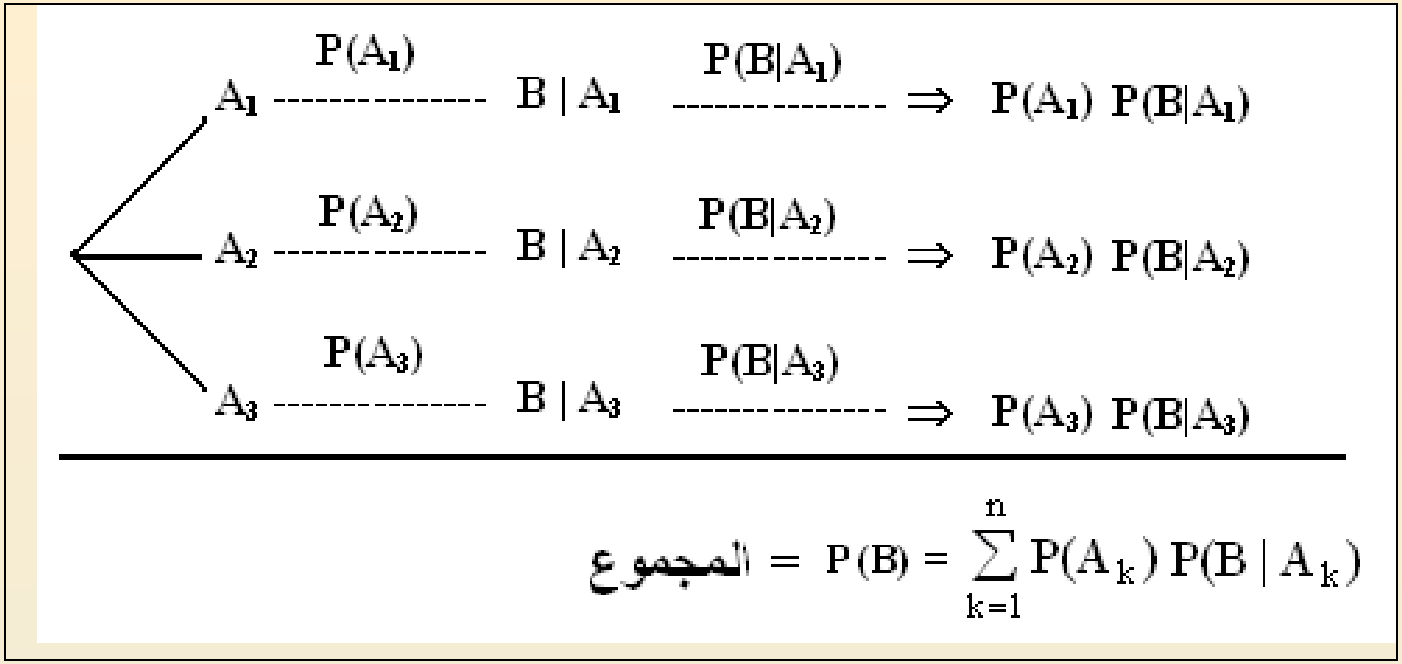
- $$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n = S$$
- $A_i \cap A_j = \phi, \quad \forall\, i \neq j$

Theorem 1: (Total Probability)

If the events $A_1, A_2, \ldots,$ and A_n constitute a partition of the sample space S such that $P(A_k) \neq 0$ for $k=1, 2, \ldots, n$, then for any event B:



$$P(B) = \sum_{k=1}^{n} P(A_k) P(B|A_k)$$
$$= \sum_{k=1}^{n} P(A_k \cap B)$$



Example 10:

Three machines A_1, A_2, and A_3 make 20%, 30%, and 50%, respectively, of the products. It is known that 1%, 4%, and 7% of the products made by each machine, respectively, are defective. If a finished product is randomly selected, what is the probability that it is defective?

Solution:

Define the following events:

B = {the selected product is defective}

A_1 = {the selected product is made by machine A_1}

A_2 = {the selected product is made by machine A_2}

A_3 = {the selected product is made by machine A_3}

$$P(A1) = \frac{20}{100} = 0.2; \quad P(B \mid A1) = \frac{1}{100} = 0.01$$

$$P(A2) = \frac{30}{100} = 0.3; \quad P(B \mid A2) = \frac{4}{100} = 0.04$$

$$P(A3) = \frac{50}{100} = 0.5; \quad P(B \mid A3) = \frac{7}{100} = 0.07$$

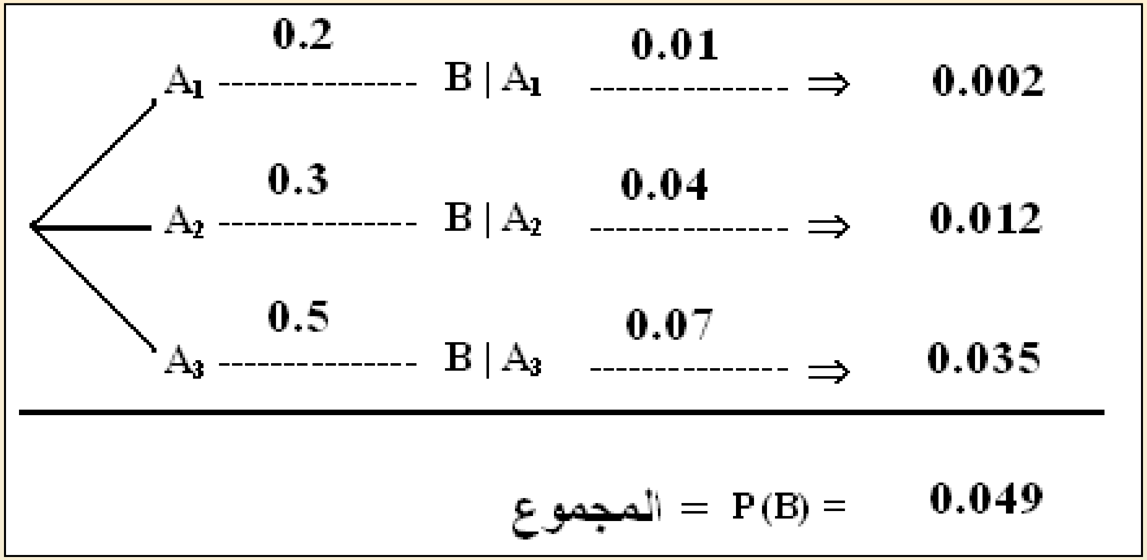
$$P(B) = \sum_{k=1}^{3} P(A_k)\,P(B \mid A_k)$$

$= P(A_1)\ P(B|A_1) + P(A_2)\ P(B|A_2) + P(A_3)\ P(B|A_3)$

$= 0.2\times 0.01 + 0.3\times 0.04 + 0.5\times 0.07$

$= 0.002 + 0.012 + 0.035$

$= 0.049$



Question:

If it is known that the selected product is defective, what is the probability that it is made by machine A_1?

Answer:

$$P(A1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{0.2 \times 0.01}{0.049} = \frac{0.002}{0.049} = 0.0408$$

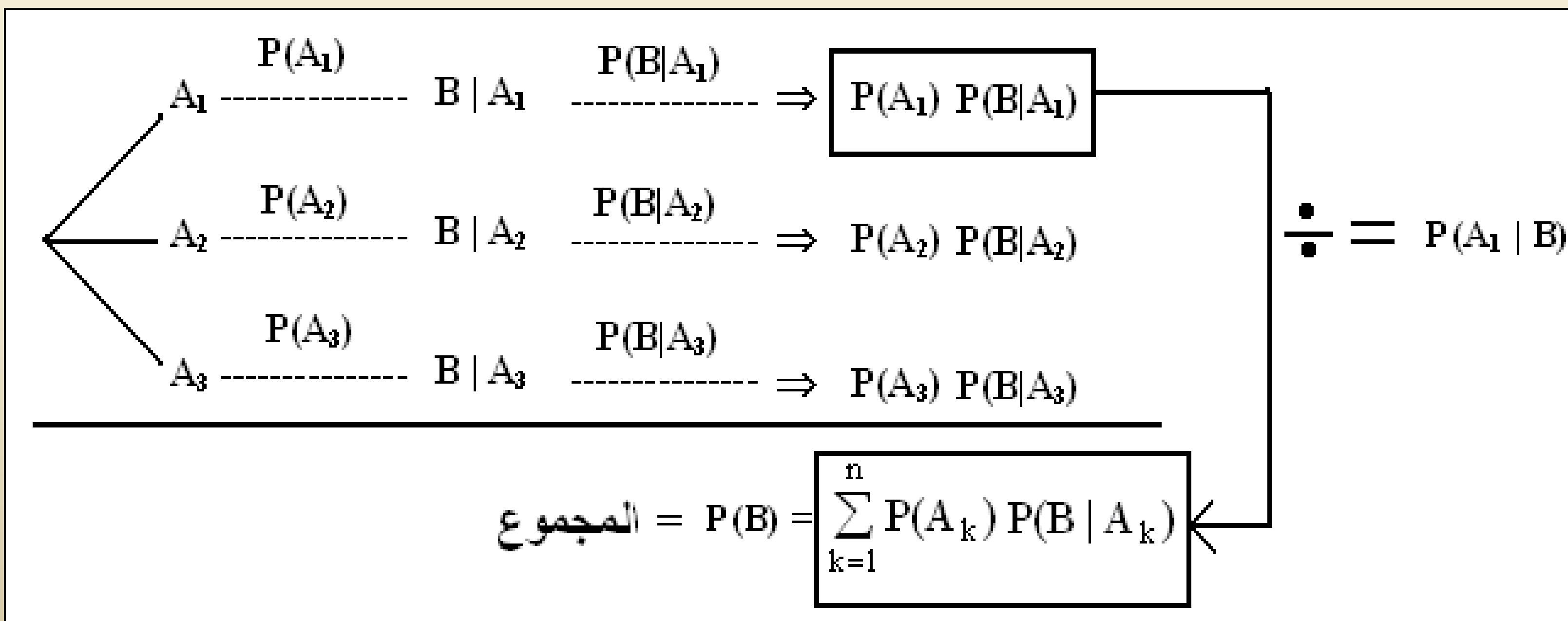
This rule is called Bayes' rule.

Theorem 2 : (Bayes' rule)

If the events $A_1, A_2, \ldots,$ and A_n constitute a partition of the sample space S such that $P(A_k) \neq 0$ for k=1, 2, …, n, then for any event B such that $P(B) \neq 0$:

$$P(A_i | B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{k=1}^{n} P(A_k)P(B|A_k)} = \frac{P(A_i)P(B|A_i)}{P(B)}$$

for i = 1, 2, …, *n*.



Example 11:

In Example 2.38, if it is known that the selected product is defective, what is the probability that it is made by:

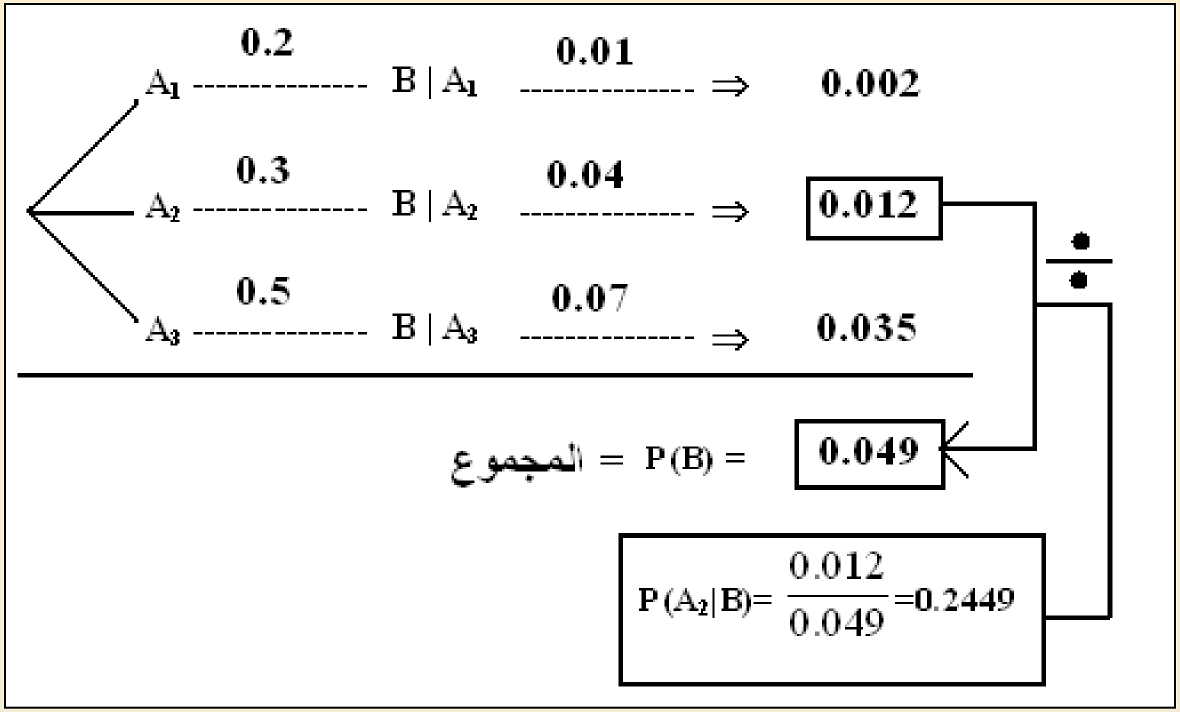
(a) machine A_2?

(b) machine A_3?

Solution:

$$(a)P(A_2|B)=\frac{P(A_2)P(B|A_2)}{\sum_{k=1}^{n}P(A_k)P(B|A_k)}=\frac{P(A_2)P(B|A_2)}{P(B)}$$

$$=\frac{0.3\times 0.04}{0.049}=\frac{0.012}{0.049}=0.2449$$



$$(b)P(A_3|B)=\frac{P(A_3)P(B|A_3)}{\sum_{k=1}^{n}P(A_k)P(B|A_k)}=\frac{P(A_3)P(B|A_3)}{P(B)}$$

$$=\frac{0.5\times 0.07}{0.049}=\frac{0.035}{0.049}=0.7142$$

Note:

$P(A_1|B) = 0.0408$, $P(A_2|B) = 0.2449$, $P(A_3|B) = 0.7142$

- $$\sum_{k=1}^{3} P(A_k | B) = 1$$

- If the selected product was found defective, we should check machine A_3 first, if it is ok, we should check machine A_2, if it is ok, we should check machine A_1.