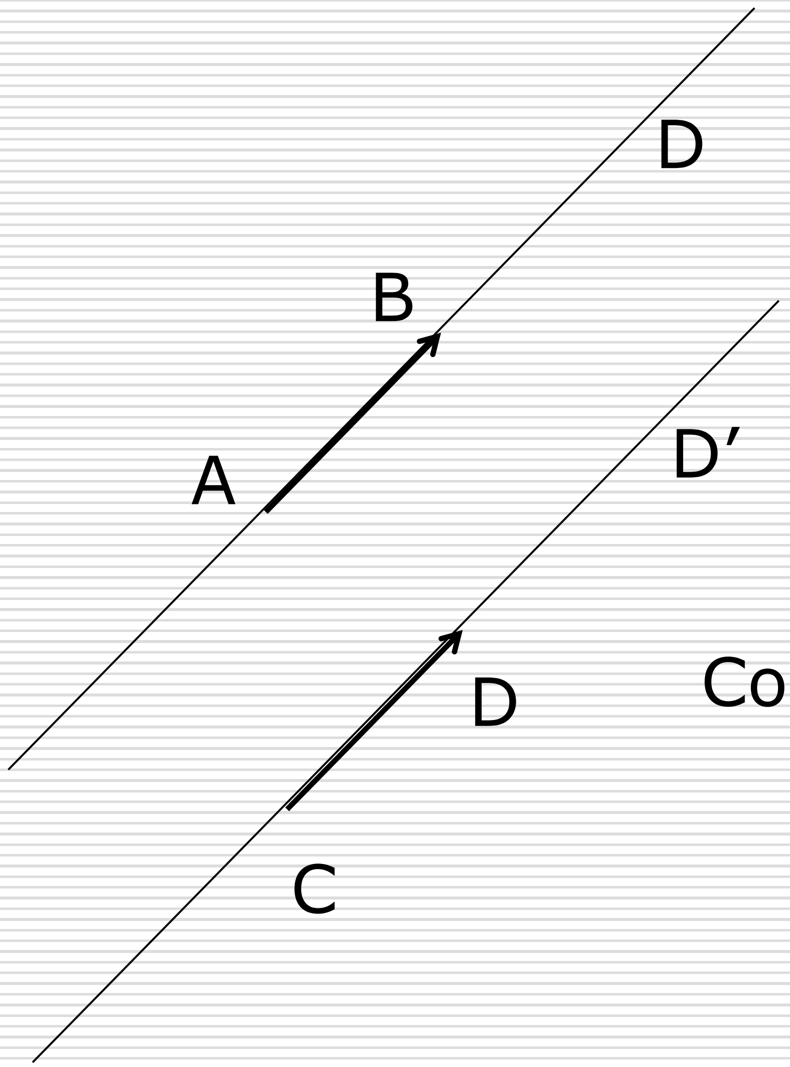
2. Vectors

Geometric definition



1 - Modulus (length) > 0 : AB = $\left\|\overrightarrow{\mathbf{AB}}\right\|$

2 - Support (straight line): D,
or every straight line parallel to D

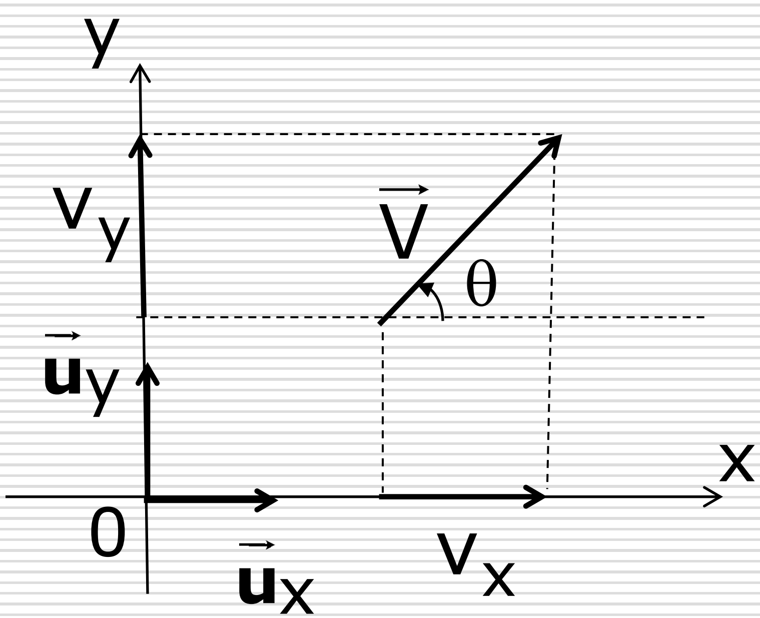
3 - Direction (arrow)

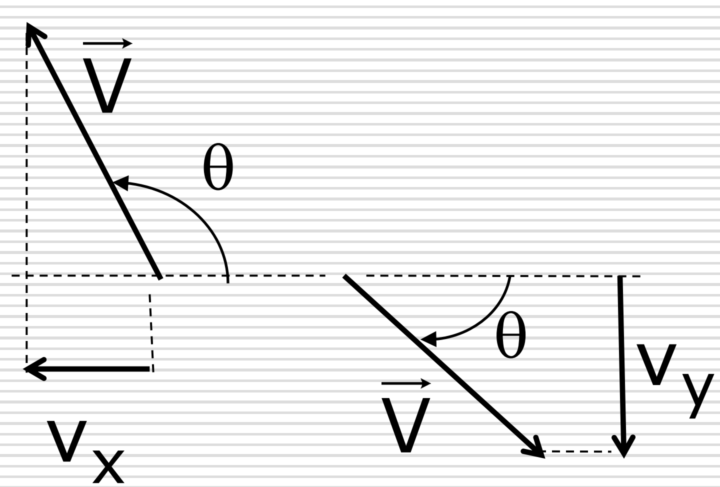
Consequence: if CD = AB
if D' // D
and if the orientation is the same
then:

$\overrightarrow{\mathbf{CD}} = \overrightarrow{\mathbf{AB}}$

2. Vectors

Algebric expression





$$\vec{\mathbf{V}} = v_x \vec{\mathbf{u}}_x + v_y \vec{\mathbf{u}}_y$$

v_x, v_y : components

$$\theta = \left(\overrightarrow{0x}, \vec{V}\right), \text{mod.} 2\pi$$

$$v_x = \left\| \vec{\mathbf{V}} \right\| \cos\theta$$

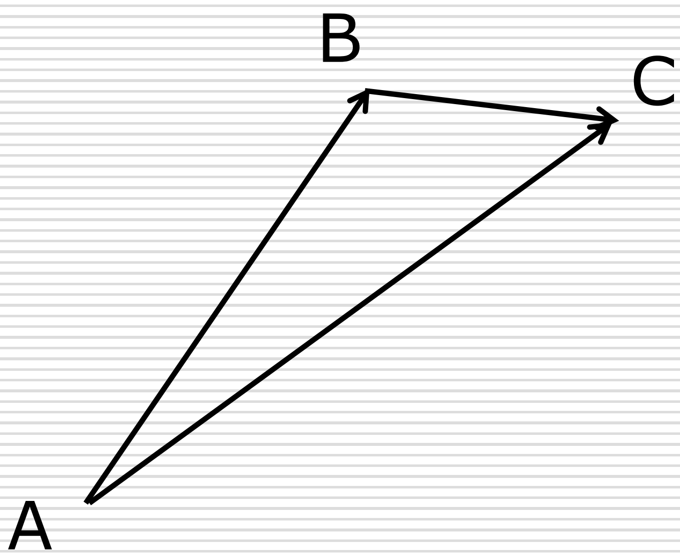
$$v_y = \left\| \vec{\mathbf{V}} \right\| \sin\theta$$

$0 \le \theta \le 2\pi$: if $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ then $v_x \le 0$

if $\pi \le \theta \le 2\pi$ then $v_y \le 0$

2. Vectors

Definitions of operations on vectors



1 - Addition (Chasles relationship)

$$\overrightarrow{\mathbf{AB}} + \overrightarrow{\mathbf{BC}} = \overrightarrow{\mathbf{AC}}$$

The addition confers to the set of vectors a structure of commutative group ($\vec{0}$ is the neutral element $-\vec{\mathbf{V}}$ the opposite element)

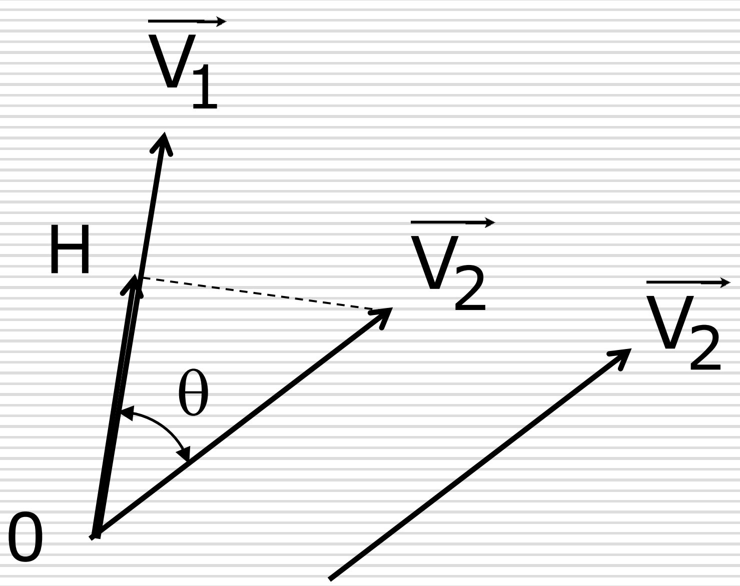
2 – Multiplication by a real number k

Distributivity/addition: $\forall k \in \mathbb{R},\ k(\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}2) = k\vec{\mathbf{V}}_1 + k\vec{\mathbf{V}}_2$

These 2 operations confer to the set of vectors a structure of commutative ring (k=1 is the neutral element)

2. Vectors

- **Dot product**



1 – Geometric definition

$$\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 = \left\|\vec{\mathbf{v}}_1\right\| . \left\|\vec{\mathbf{v}}_2\right\| \cos\left(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\right) = \left\|\vec{\mathbf{v}}_1\right\| \overline{\mathrm{OH}}$$

$$\theta = \pm\left(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\right) \text{ (commutativity)}$$

N. B.: $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \left\|\vec{\mathbf{v}}\right\|^2 \geq 0$ square of the norm

2 - Orthonormality relationship

$$\vec{\mathbf{u}}_m \cdot \vec{\mathbf{u}}_n = \delta_{mn} = \begin{cases} 0 \text{ if } m \neq n \\ 1 \text{ if } m = n \end{cases}$$

3 – Algebric expression

$$\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 = v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z}$$

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \left\|\vec{\mathbf{v}}\right\|^2 = v_x^2 + v_y^2 + v_z^2 \text{ (Pythagore Theorem)}$$

2. Vectors

Properties of the dot product

1 – Commutativity: $\vec{\mathbf{V}}_1 \cdot \vec{\mathbf{V}}_2 = \vec{\mathbf{V}}_2 \cdot \vec{\mathbf{V}}_1$

2 – Bilinearity: $\forall \alpha, \beta \in \square,\ (\alpha \vec{\mathbf{V}}_1 + \beta \vec{\mathbf{V}}_2) \cdot \vec{\mathbf{U}} = \alpha \vec{\mathbf{V}}_1 \cdot \vec{\mathbf{U}} + \beta \vec{\mathbf{V}}_2 \cdot \vec{\mathbf{U}}$

$$\forall \alpha, \beta \in \square,\ \vec{\mathbf{V}} \cdot (\alpha \vec{\mathbf{U}}_1 + \beta \vec{\mathbf{U}}_2) = \alpha \vec{\mathbf{V}} \cdot \vec{\mathbf{U}}_1 + \beta \vec{\mathbf{V}} \cdot \vec{\mathbf{U}}_2$$

Generalization : any scalar product $\langle A,B \rangle$ is a bilinear form defined on □

Properties of the norm

1 – $\|\vec{\mathbf{V}}\|^2 \geq 0$

2 – Schwartz inequality: $\|\vec{\mathbf{A}} + \vec{\mathbf{B}}\| \leq \|\vec{\mathbf{A}}\| + \|\vec{\mathbf{B}}\|$

Generalization: any norm is defined from these properties

2. Vectors

Dot product: other notations

1 - Matrix: $\vec{\mathbf{A}}$ is a matrix with one column and 3 rows: $\left[A\right] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \left[A\right]^T \left[B\right] = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \sum_{i=1}^{3} a_i b_i = a_u b_u$$

2 - Einstein convention: implicit sum on repeated indices

$$\vec{\mathbf{A}} = \sum_{i=1}^{3} a_i \vec{\mathbf{u}}_i = a_v \vec{\mathbf{u}}_v,\ v=(1,2,3),\ \overrightarrow{\mathbf{u}_1}, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3 : \text{coordinate system}$$

2. Vectors

- **Using Einstein convention**

Total differential

$$d\phi = \partial_\nu \phi \, du_\nu \quad \left(= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right)$$

Product of matrices

$$c_{ij} = \left[\mathbf{AB}\right]_{ij} = a_{ik} b_{kj} \quad \left(= \sum_{k=1}^{n} a_{ik} b_{kj} \right)$$

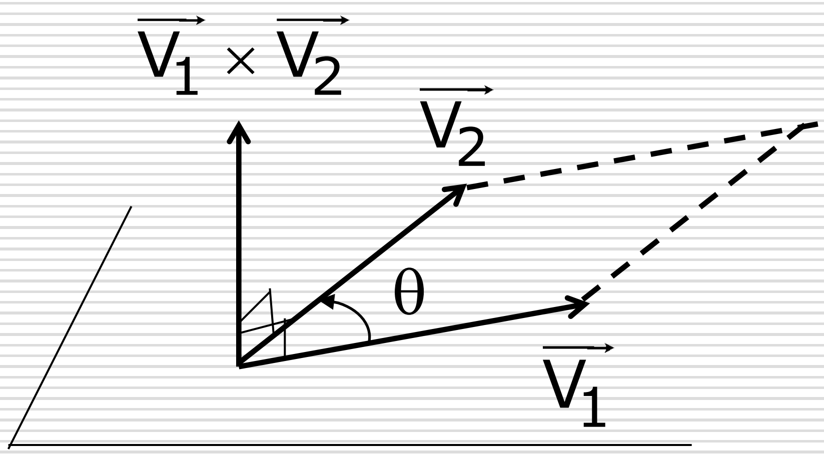
Trace of a matrix

$$\mathrm{Tr}\ \mathbf{A} = a_{ii} \quad \left(= \sum_{i=1}^{n} a_{ii} \right)$$

$$\delta_{ii} = 3$$

2. Vectors

Cross or vector product



1 – Geometric definition

$$\begin{cases} \left|\vec{\mathbf{V}}_1 \times \vec{\mathbf{V}}_2\right| = \left\|\vec{\mathbf{V}}_1\right\| . \left\|\vec{\mathbf{V}}_2\right\| \sin\left(\vec{\mathbf{V}}_1, \vec{\mathbf{V}}_2\right) \\ \text{Direction: screw or right hand rules} \end{cases}$$

$$\left|\vec{\mathbf{V}}_1 \times \vec{\mathbf{V}}_2\right| = \text{area}\left(\vec{\mathbf{V}}_1 \times \vec{\mathbf{V}}_2\right)$$

N.B.: $\vec{\mathbf{u}}_u \times \vec{\mathbf{u}}_v = \vec{\mathbf{u}}_w$, u, v, w : permutation of x, y, z

$$\left|\vec{\mathbf{u}}_x \times \vec{\mathbf{u}}_x\right| = \left|\vec{\mathbf{u}}_y \times \vec{\mathbf{u}}_y\right| = \left|\vec{\mathbf{u}}_z \times \vec{\mathbf{u}}_z\right| = 0$$

2 – Properties: - anticommutativity: $\left|\vec{\mathbf{V}}_1 \times \vec{\mathbf{V}}_2\right| = -\left|\vec{\mathbf{V}}_2 \times \vec{\mathbf{V}}_1\right|$

- bilinearity

2. Vectors

Cross product: other notations

3 – Algebric expression:

$$\vec{\mathbf{V}}_1 \times \vec{\mathbf{V}}_2 = \vec{\mathbf{u}}_x \left(v_{1y}v_{2z} - v_{1z}v_{2y}\right) + \vec{\mathbf{u}}_y \left(v_{1z}v_{2x} - v_{1x}v_{2z}\right) + \vec{\mathbf{u}}_z \left(v_{1x}v_{2y} - v_{1y}v_{2x}\right)$$

4 – Einstein convention: Levi-Civita symbol

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \epsilon_{ijk}\, \vec{\mathbf{u}}_i a_j b_k \quad \left(= \sum_i \sum_j \sum_k \vec{\mathbf{u}}_i a_j b_k \right)$$

$$\epsilon_{ijk} \begin{cases} 0, \text{ unless i, j, k are distinct (21 cases of zero)} \\ +1 \text{ if (i,j,k) are (1,2,3) in cyclic order, or even permutation (3 cases)} \\ -1 \text{ if (i,j,k) are (2,1,3) in cyclic order, or odd permutation (3 cases)} \end{cases}$$

2. Vectors

On Levi-Civita symbol

1 – $\epsilon_{ijk} = 1$ if $(i,j,k) = (1,2,3),(2,3,1),(3,1,2)$

$\epsilon_{ijk} = -1$ if $(i,j,k) = (1,3,2),(2,1,3),(3,2,1)$

2 - Property: rotating indices doesn't change sign: $\epsilon_{ijk} = \epsilon_{kij}$

3 - Component # i of the dot product

$$\left[\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right]_i = \epsilon_{ijk}\, a_j b_k \quad \left(= \sum_j \sum_k \epsilon_{ijk}\, a_j b_k \right)$$

4 - Relationship between Levi Civita and Kronecker symbols

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{lk}$$

Proof: examine the 81 cases and group symetric ones.

2. Vectors

- Cross or vector product: computation

$$\vec{\mathbf{v}}_1 \times \vec{\mathbf{v}}_2 = \det \begin{bmatrix} \vec{\mathbf{u}}_x & \vec{\mathbf{u}}_y & \vec{\mathbf{u}}_z \\ v_{1x} & v_{1y} & v_{1z} \\ v_{2x} & v_{2y} & v_{2z} \end{bmatrix}$$

Sarrus rule:

$$\det = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{bmatrix}$$

or $\vec{\mathbf{v}}_1 \times \vec{\mathbf{v}}_2 = \vec{\mathbf{u}}_x \det \begin{bmatrix} v_{1y} & v_{1z} \\ v_{2y} & v_{2z} \end{bmatrix} - \vec{\mathbf{u}}_y \det \begin{bmatrix} v_{1x} & v_{1z} \\ v_{2x} & v_{2z} \end{bmatrix} + \vec{\mathbf{u}}_z \det \begin{bmatrix} v_{1x} & v_{1y} \\ v_{2x} & v_{2y} \end{bmatrix}$

2. Vectors

Double cross product

$$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) - \vec{\mathbf{c}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}) \quad \text{(bac – cab or abacab rule)}$$

$$\text{Proof. } \left[\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})\right]_i = \epsilon_{ijk}\ a_j(\vec{\mathbf{b}} \times \vec{\mathbf{c}})_k = \epsilon_{ijk}\ a_j\ \epsilon_{klm}\ b_l\ c_m$$

In order to apply the relationship between Levi-Civita and Kronecker symbols, both Levi-Civita symbols have to begin with the same indice k. Then we use the invariance by rotating indices.

$$\left[\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})\right]_i = \epsilon_{ijk}\epsilon_{klm}\ a_j\ b_l\ c_m = \epsilon_{kij}\epsilon_{klm}\ a_j\ b_l\ c_m$$

$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)a_j\ b_l\ c_m = b_i\left(a_m c_m\right) - c_i\left(a_l b_l\right)$$

2. Vectors

Vector triple product

1 – Definition: $\vec{\mathbf{A}} \cdot \left(\vec{\mathbf{B}} \times \vec{\mathbf{C}}\right) = a_i \in_{ijk} b_j c_k$

2 – Expression: $\vec{\mathbf{A}} \cdot \left(\vec{\mathbf{B}} \times \vec{\mathbf{C}}\right) = \det \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{bmatrix}$

$$\vec{\mathbf{A}} \cdot \left(\vec{\mathbf{B}} \times \vec{\mathbf{C}}\right) = a_x \det \begin{bmatrix} b_y & b_z \\ c_y & c_z \end{bmatrix} - a_y \det \begin{bmatrix} b_x & b_z \\ c_x & c_z \end{bmatrix} + a_z \det \begin{bmatrix} b_x & b_y \\ c_x & c_y \end{bmatrix}$$

$$\vec{\mathbf{A}} \cdot \left(\vec{\mathbf{B}} \times \vec{\mathbf{C}}\right) = a_x \left(b_y c_z - b_z c_y\right) + a_y \left(b_z c_x - b_x c_z\right) + a_z \left(b_x c_y - b_y c_x\right)$$

2. Vectors

Properties of triple product

$$\vec{\mathbf{a}} \cdot \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right) = \vec{\mathbf{b}} \cdot \left(\vec{\mathbf{c}} \times \vec{\mathbf{a}}\right) = \vec{\mathbf{c}} \cdot \left(\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right)$$

Proof. Consider, for example, the first equation:

$$\vec{\mathbf{a}} \cdot \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right) = a_x \left(b_y c_z - b_z c_y\right) + a_y \left(b_z c_x - b_x c_z\right) + a_z \left(b_x c_y - b_y c_x\right)$$

$$\vec{\mathbf{b}} \cdot \left(\vec{\mathbf{c}} \times \vec{\mathbf{a}}\right) = b_x \left(c_y a_z - c_z a_y\right) + b_y \left(c_z a_x - c_x a_z\right) + b_z \left(c_x a_y - c_y a_x\right)$$

Other proof of the first equation using Levi-Civita symbols:

$$\vec{\mathbf{a}} \cdot \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right) = a_i \left(\epsilon_{ijk}\, b_j c_k\right) = \epsilon_{ijk}\, a_i b_j c_k = \epsilon_{jki}\, a_i b_j c_k = b_j\, \epsilon_{jki}\, a_i c_k$$

We can permute a and c (but not indices) in Levi-Civita symbol

$$\vec{\mathbf{a}} \cdot \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right) = b_j\, \epsilon_{jki}\, c_k a_i$$