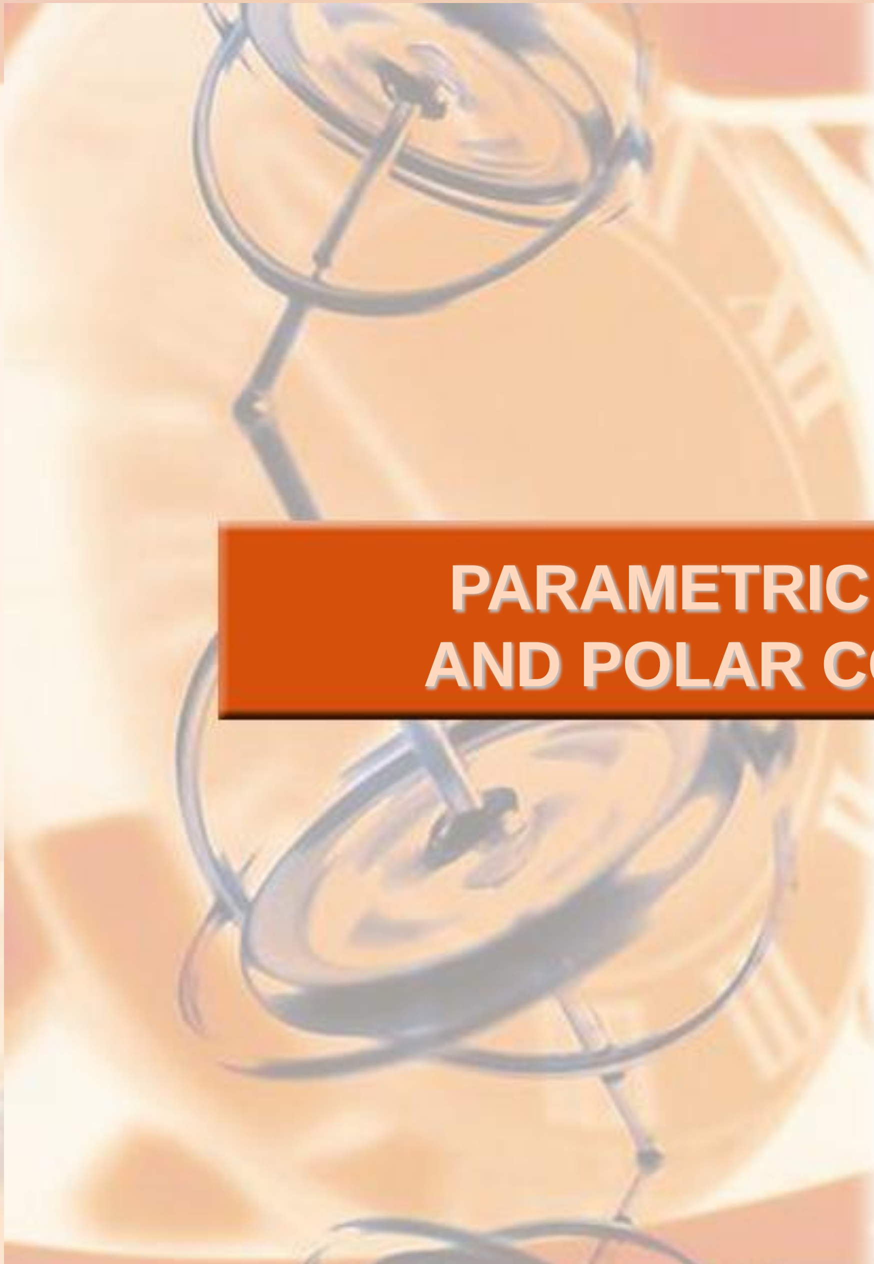


10

PARAMETRIC EQUATIONS AND POLAR COORDINATES

10.4

Areas and Lengths in Polar Coordinates

In this section, we will:

Develop the formula for the area of a region whose boundary is given by a polar equation.

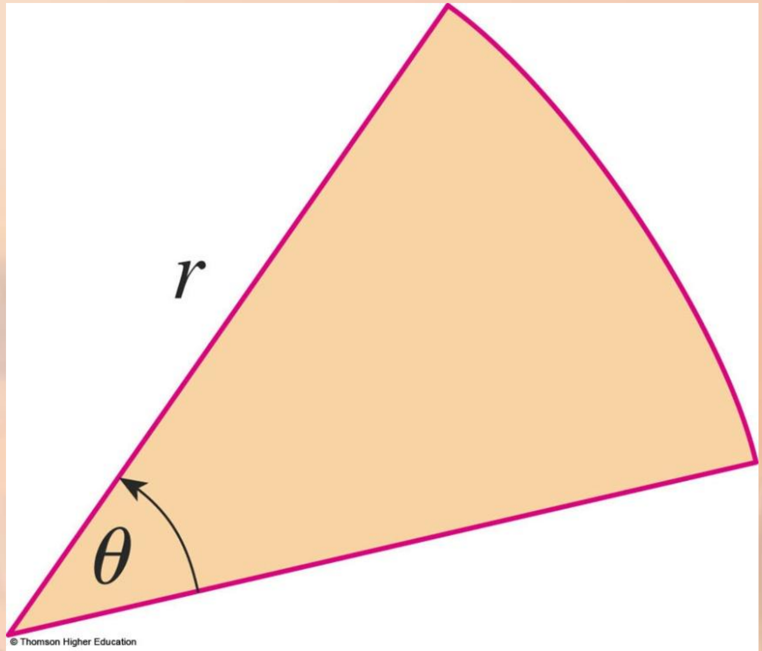
AREAS IN POLAR COORDINATES **Formula 1**

We need to use the formula for the area of a sector of a circle

$$A = \tfrac{1}{2} r^2 \theta$$

where:

- r is the radius.
- θ is the radian measure of the central angle.



AREAS IN POLAR COORDINATES

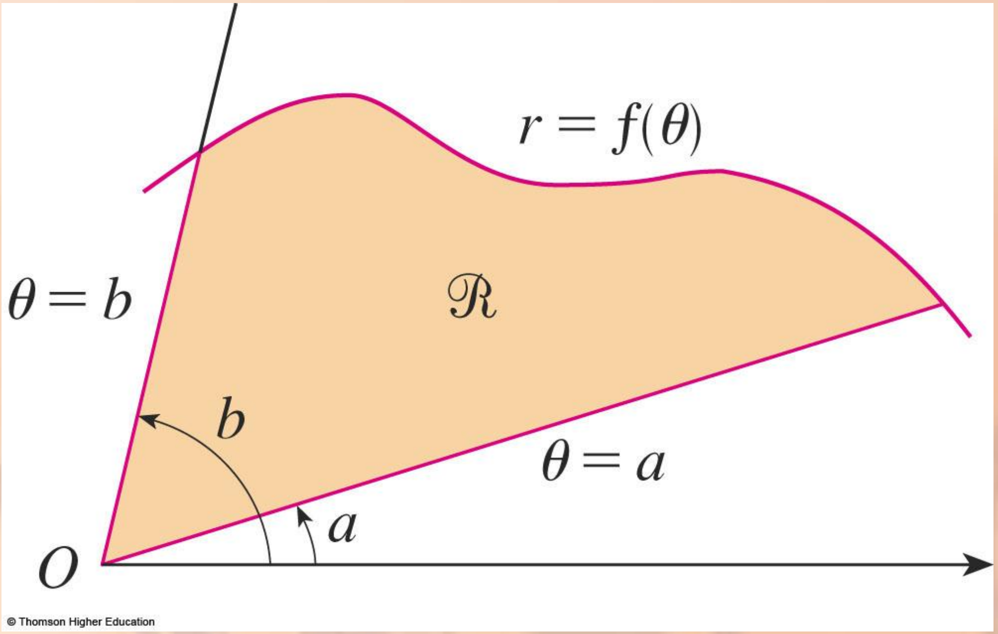
Formula 1 follows from the fact that the area of a sector is proportional to its central angle:

$$A = (\theta/2\pi)\pi r^2 = \tfrac{1}{2}r^2\theta$$

AREAS IN POLAR COORDINATES

Let $\mathcal{R}$ be the region bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where:

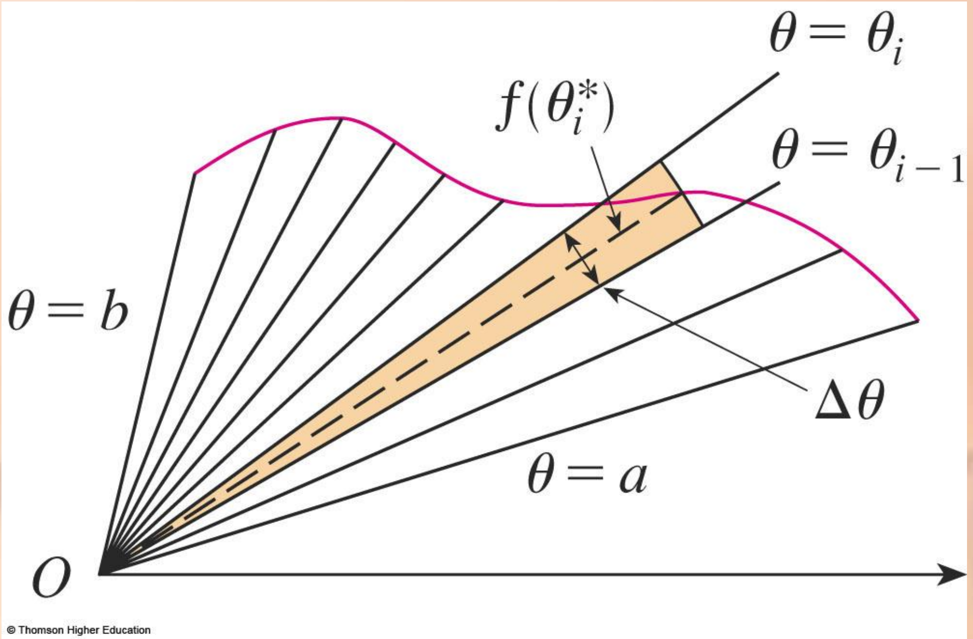
- f is a positive continuous function.
- $0 < b - a \leq 2\pi$



AREAS IN POLAR COORDINATES

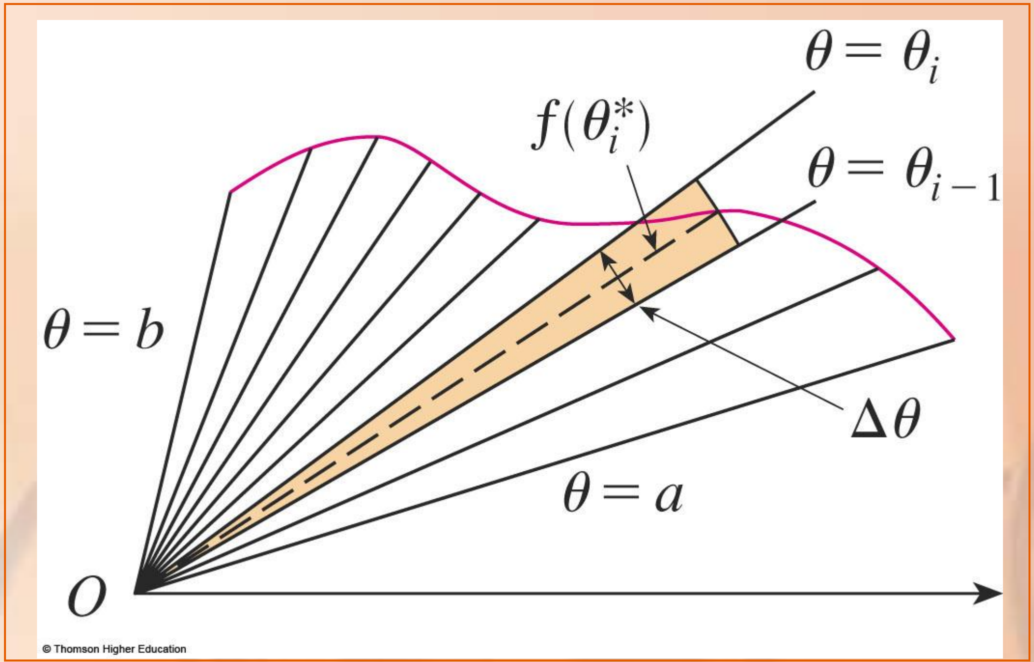
We divide the interval [*a, b*] into subintervals with endpoints θ_0, θ_1, θ_2, …, θ_n, and equal width $\Delta\theta$.

- Then, the rays $\theta = \theta_i$ divide $\mathcal{R}$ into smaller regions with central angle $\Delta\theta = \theta_i - \theta_{i-1}$.



AREAS IN POLAR COORDINATES

If we choose θ_i^* in the i th subinterval $[\theta_{i-1}, \theta_i]$ then the area ΔA_i of the i th region is the area of the sector of a circle with central angle $\Delta\theta$ and radius $f(\theta^*)$.



Thus, from Formula 1, we have:

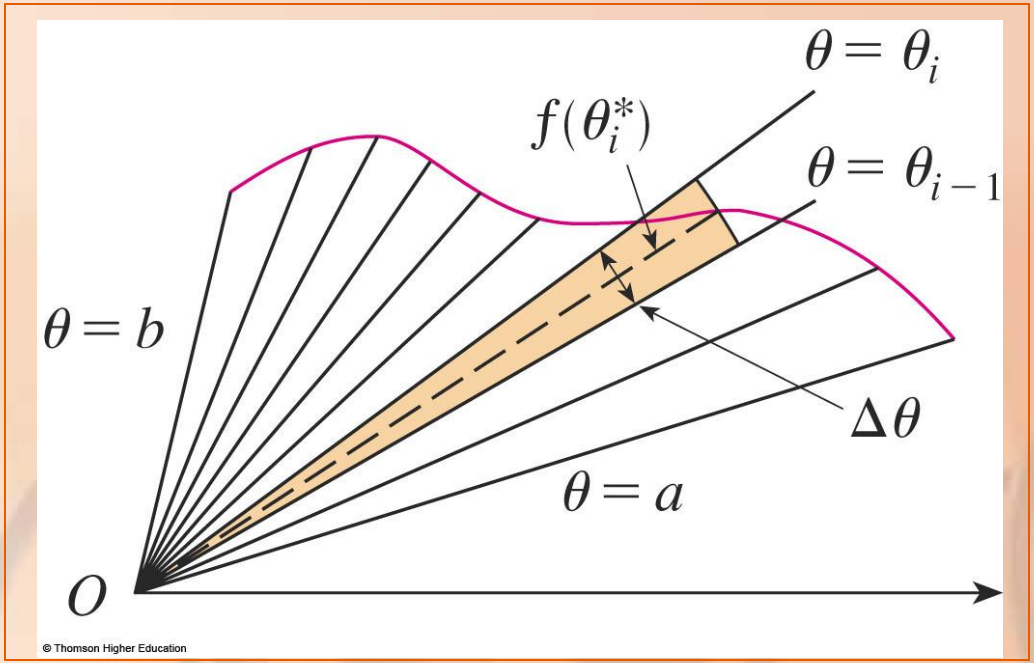
$$\Delta A_i \approx \tfrac{1}{2}[f(\theta_i^*)]^2\,\Delta\theta$$

So, an approximation to the total area A of $\mathcal{R}$ is:

$$A \approx \sum_{i=1}^{n} \tfrac{1}{2}[f(\theta_i^*)]^2\,\Delta\theta$$

AREAS IN POLAR COORDINATES

It appears that the approximation in Formula 2 improves as $n \to \infty$.



However, the sums in Formula 2 are Riemann sums for the function $g(\theta) = \frac{1}{2}[f(\theta)]^2$.

So,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\tfrac{1}{2}[f(\theta_i^*)]^2\,\Delta\theta = \int_a^b \tfrac{1}{2}[f(\theta)]^2\,d\theta$$

AREAS IN POLAR COORDINATES **Formula 3**

Therefore, it appears plausible—and can, in fact, be proved—that the formula for the area *A* of the polar region $\mathcal{R}$ is:

$$A = \int_a^b \tfrac{1}{2}[f(\theta)]^2 \, d\theta$$

 Formula 4

Formula 3 is often written as

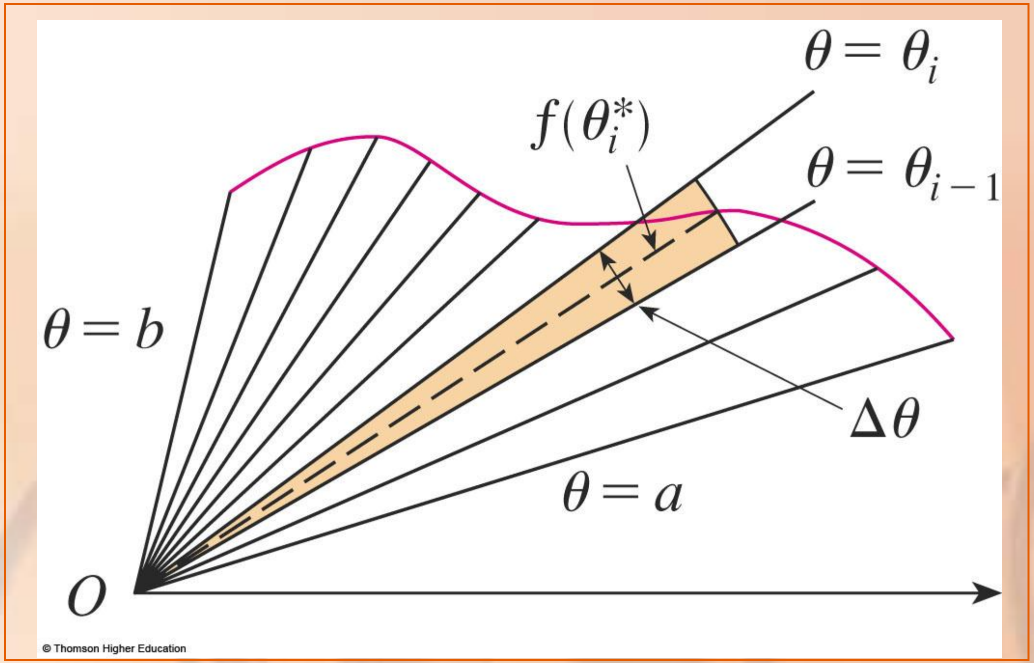
$$A = \int_a^b \tfrac{1}{2} r^2 \, d\theta$$

with the understanding that $r = f(\theta)$.

- Note the similarity between Formulas 1 and 4.

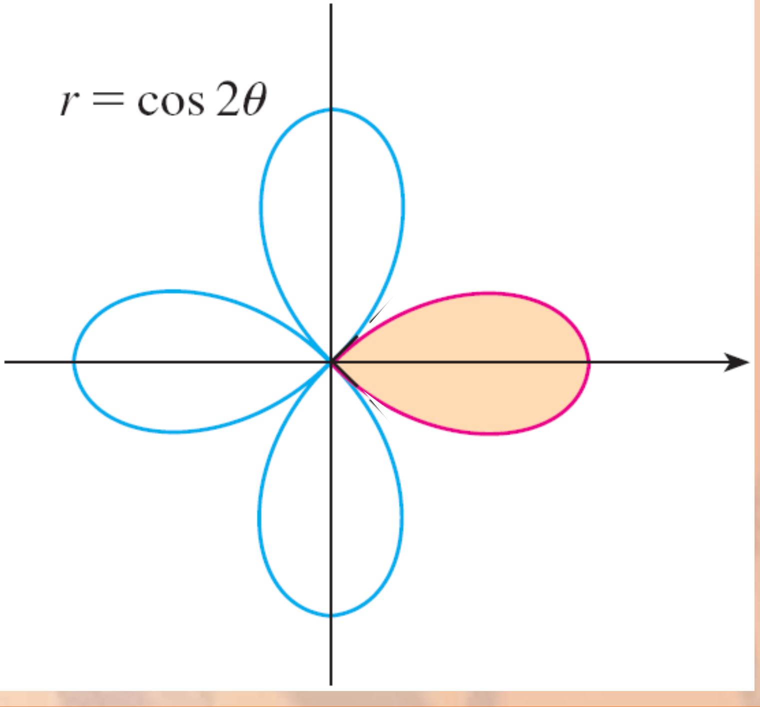
AREAS IN POLAR COORDINATES **Formula 4**

When we apply Formula 3 or 4, it is helpful to think of the area as being swept out by a rotating ray through *O* that starts with angle *a* and ends with angle *b*.



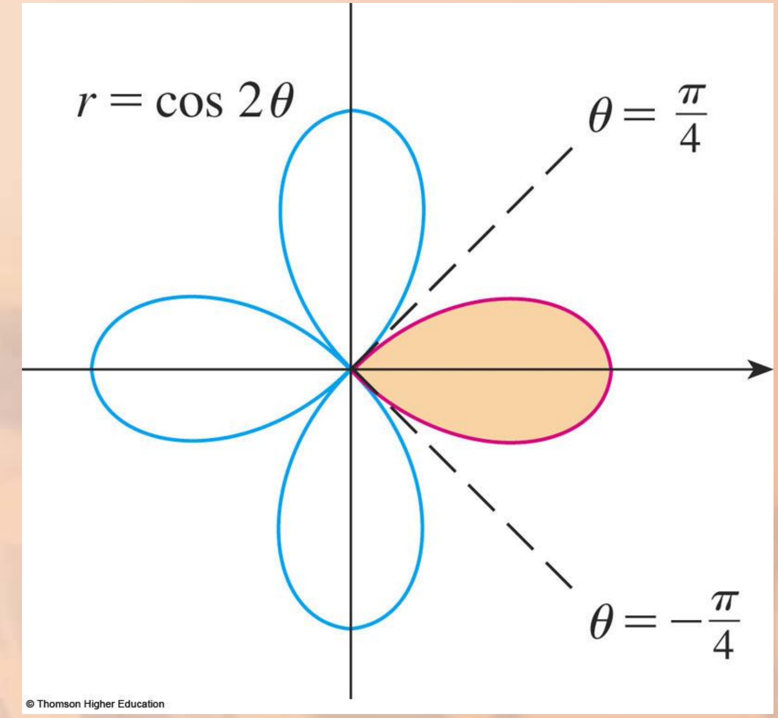
Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

- The curve $r = \cos 2\theta$ was sketched in Example 8 in Section 10.3



AREAS IN POLAR COORDINATES Example 1

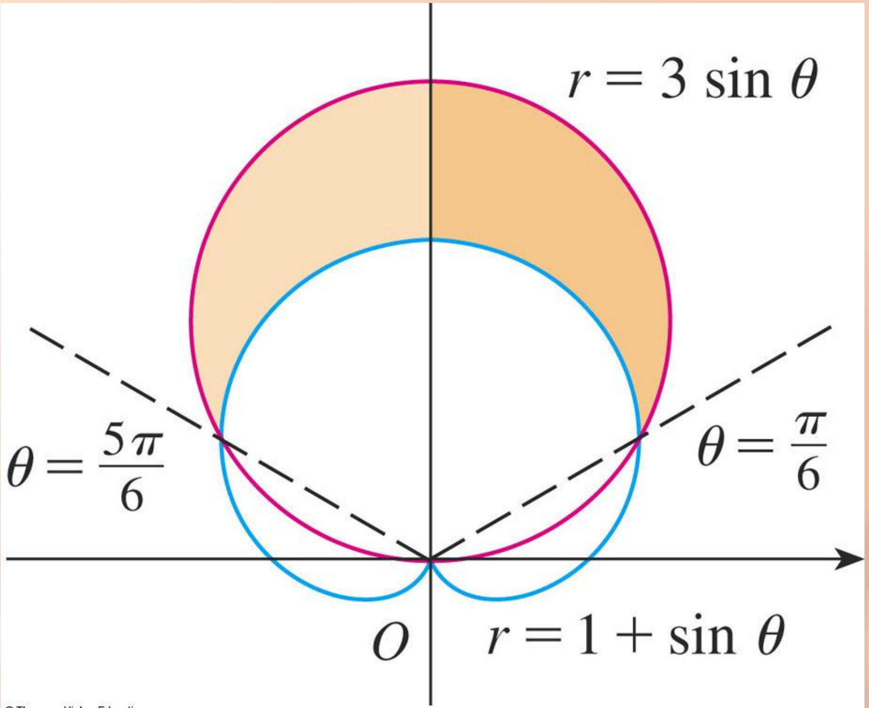
Notice that the region enclosed by the right loop is swept out by a ray that rotates from $\theta = -\pi/4$ to $\theta = \pi/4$.



Hence, Formula 4 gives:

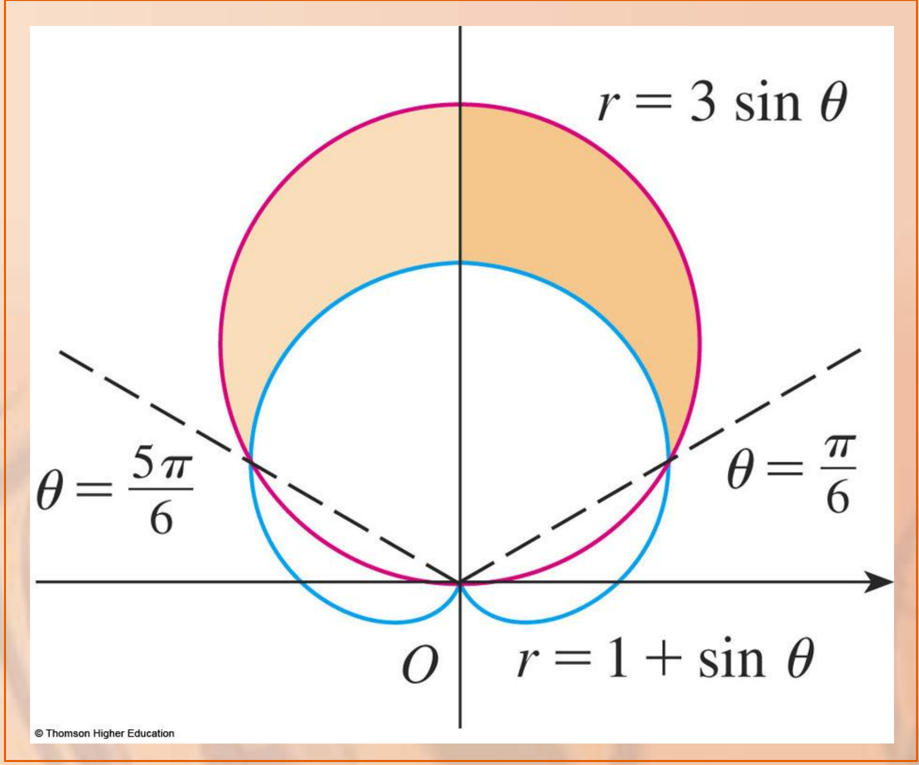
$$
\begin{aligned}
A &= \int_{-\pi/4}^{\pi/4} \tfrac{1}{2} r^2 \, d\theta \\
&= \tfrac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta \, d\theta \\
&= \int_{0}^{\pi/4} \cos^2 2\theta \, d\theta \\
&= \int_{0}^{\pi/4} \tfrac{1}{2} (1 + \cos 4\theta) \, d\theta \\
&= \tfrac{1}{2} \Big[\theta + \tfrac{1}{4} \sin 4\theta \Big]_0^{\pi/4} = \frac{\pi}{8}
\end{aligned}
$$

Find the area of the region that lies inside the circle $r = 3 \sin \theta$ and outside the cardioid $r = 1 + \sin \theta$.



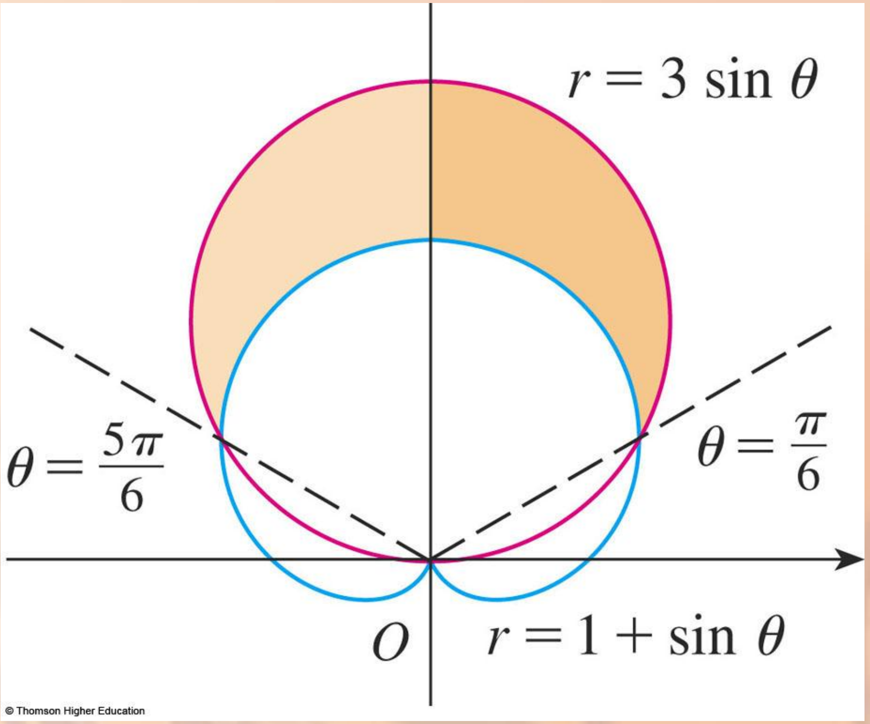
 Example 2

The values of *a* and *b* in Formula 4 are determined by finding the points of intersection of the two curves.



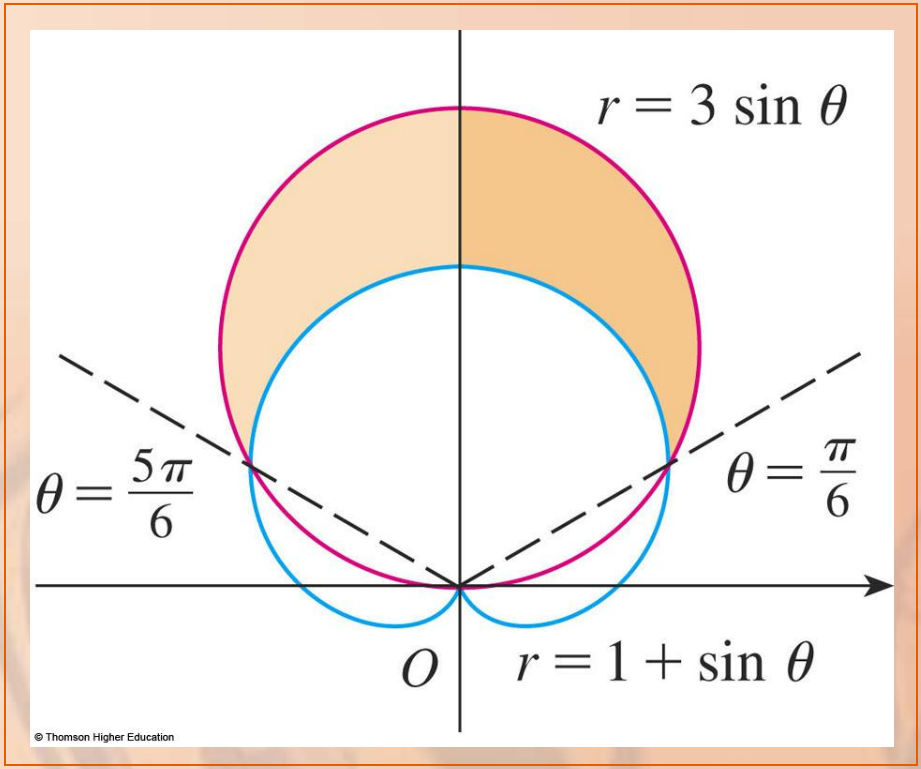
They intersect when 3 sin $\theta = 1 + \sin\theta$, which gives $\sin\theta = ½$.

- So, $\theta = \pi/6$ and $5\pi/6$.



AREAS IN POLAR COORDINATES **Example 2**

The desired area can be found by subtracting the area inside the cardioid between $\theta = \pi/6$ and $\theta = 5\pi/6$ from the area inside the circle from $\pi/6$ to $5\pi/6$.



Thus,

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2\, d\theta$$

$$- \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1+\sin\theta)^2\, d\theta$$

As the region is symmetric about the vertical axis $\theta = \pi/2$, we can write:

$$A = 2\left[\tfrac{1}{2}\int_{\pi/6}^{\pi/2} 9\sin^2\theta\,d\theta - \tfrac{1}{2}\int_{\pi/6}^{\pi/2}(1+2\sin\theta+\sin^2\theta)\,d\theta\right]$$

$$= \int_{\pi/6}^{\pi/2}(8\sin^2\theta - 1 - 2\sin\theta)\,d\theta$$

$$= \int_{\pi/6}^{\pi/2}(3 - 4\cos 2\theta - 2\sin\theta)\,d\theta \quad \left[\text{as}\sin^2 = \tfrac{1}{2}(1-\cos 2\theta)\right]$$

$$= 3\theta - 2\sin 2\theta + 2\cos\theta\Big]_{\pi/6}^{\pi/2}$$

$$= \pi$$

Example 2 illustrates the procedure for finding the area of the region bounded by two polar curves.

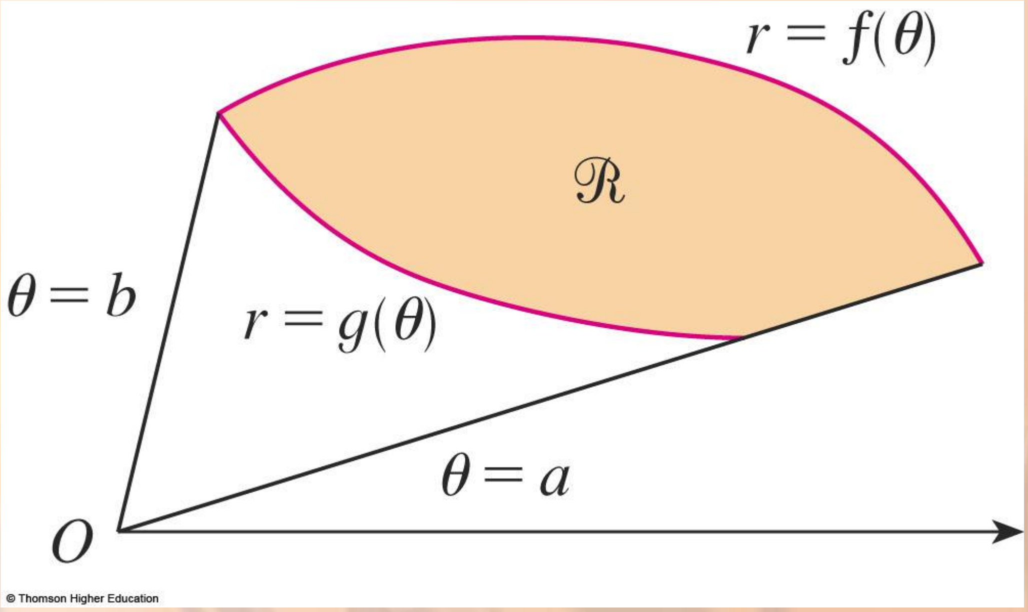
AREAS IN POLAR COORDINATES

In general, let $\mathcal{R}$ be a region that is bounded by curves with polar equations

$$r = f(\theta),\ r = g(\theta),\ \theta = a,\ \theta = b,$$

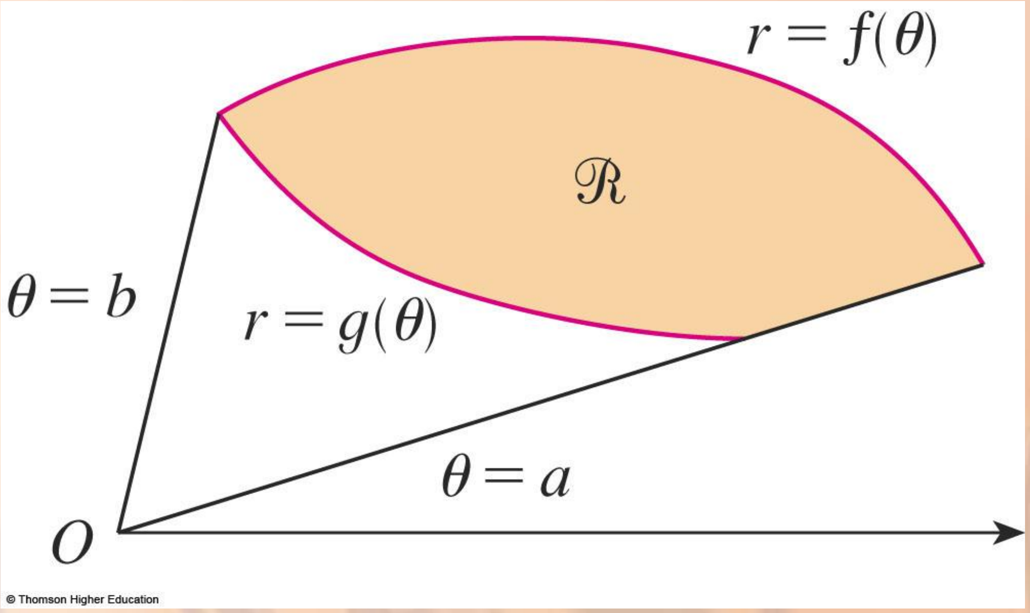
where:

- $f(\theta) \geq g(\theta) \geq 0$
- $0 < b - a < 2\pi$



AREAS IN POLAR COORDINATES

The area A of $\mathcal{R}$ is found by subtracting the area inside $r = g(\theta)$ from the area inside $r = f(\theta)$.



AREAS IN POLAR COORDINATES

So, using Formula 3, we have:

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 \, d\theta - \int_a^b \frac{1}{2}[g(\theta)]^2 \, d\theta$$

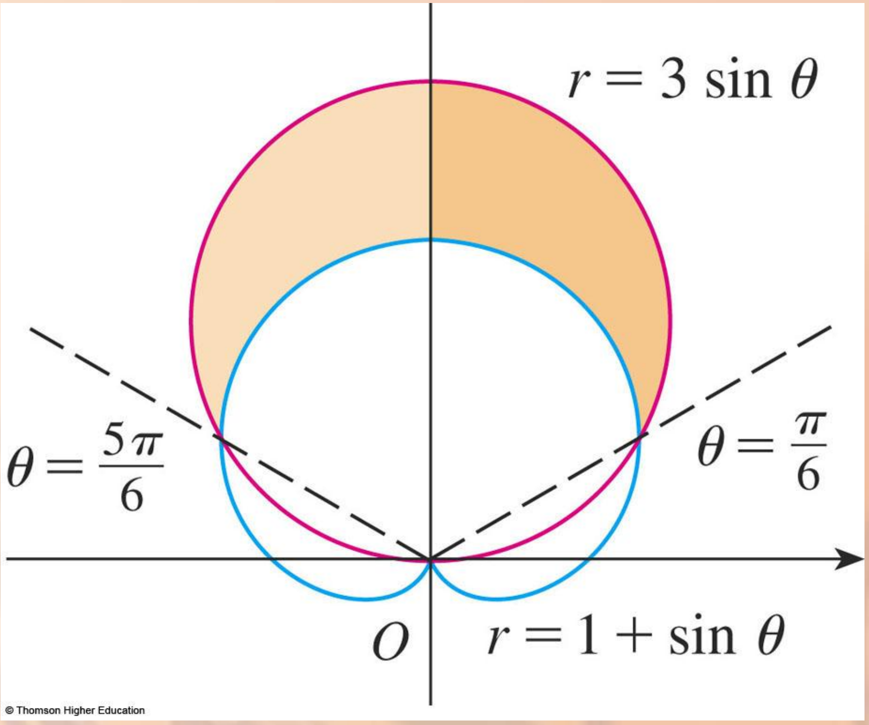
$$= \frac{1}{2}\int_a^b \left([f(\theta)]^2 - [g(\theta)]^2\right) d\theta$$

CAUTION

The fact that a single point has many representations in polar coordinates sometimes makes it difficult to find all the points of intersection of two polar curves.

CAUTION

For instance, it is obvious from this figure that the circle and the cardioid have three points of intersection.



CAUTION

However, in Example 2, we solved the equations $r = 3 \sin \theta$ and $r = 1 + \sin \theta$ and found only two such points:

$$(3/2, \pi/6) \text{ and } (3/2, 5\pi/6)$$

CAUTION

The origin is also a point of intersection.

However, we can't find it by solving the equations of the curves.

- The origin has no single representation in polar coordinates that satisfies both equations.

CAUTION

Notice that, when represented as $(0, 0)$ or $(0, \pi)$, the origin satisfies $r = 3 \sin \theta$.

- So, it lies on the circle.

CAUTION

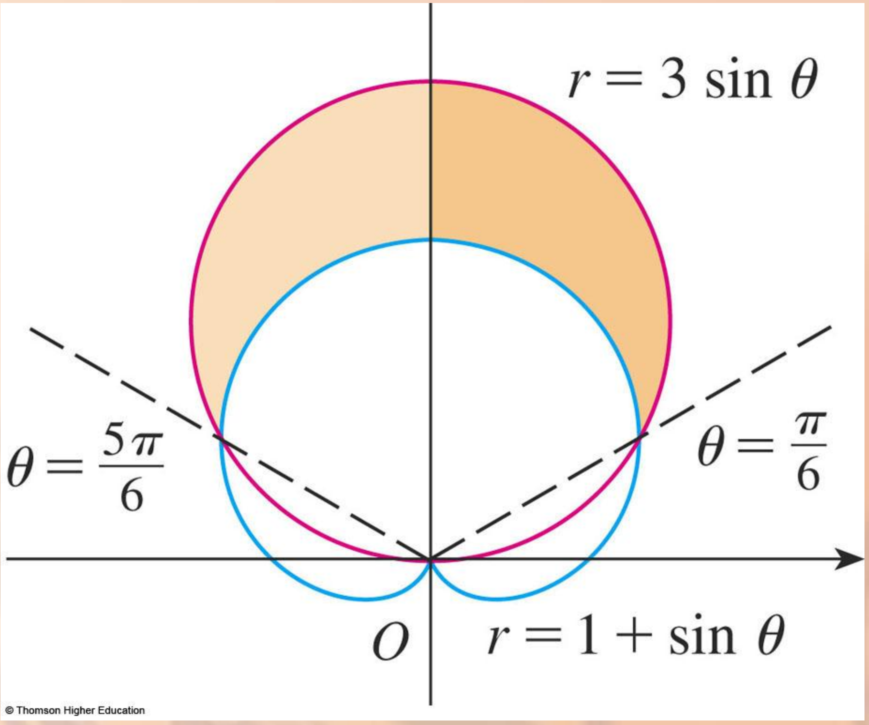
When represented as $(0, 3\pi/2)$, it satisfies $r = 1 + \sin\theta$.

- So, it lies on the cardioid.

CAUTION

Think of two points moving along the curves as the parameter value θ increases from 0 to 2π.

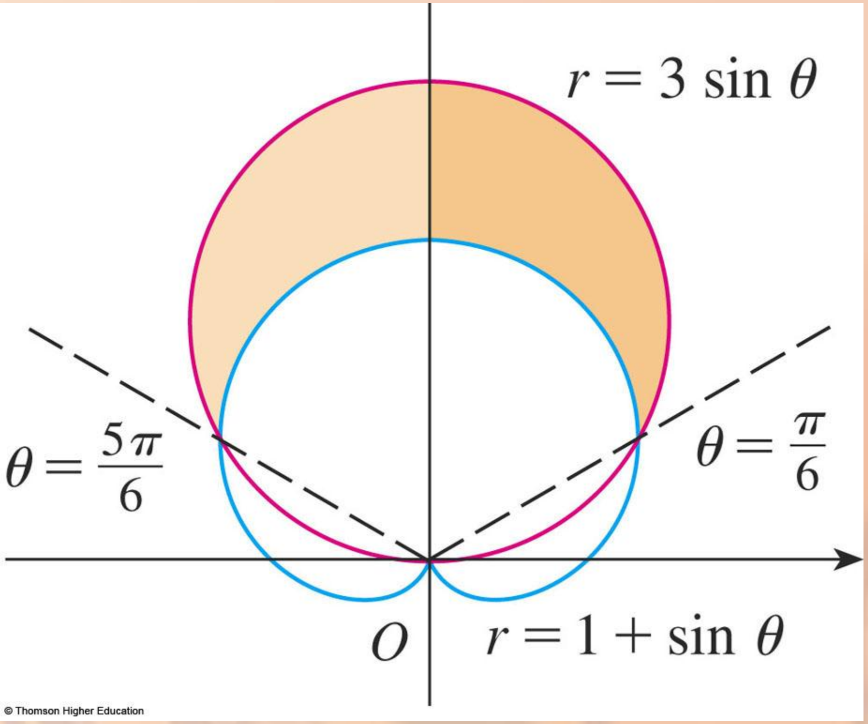
- On one curve, the origin is reached at $\theta = 0$ and $\theta = \pi$.
- On the other, it is reached at $\theta = 3\pi/2$.



CAUTION

- The points don't collide at the origin since they reach the origin at different times.

- However, the curves intersect there nonetheless.



CAUTION

Thus, to find all points of intersection of two polar curves, it is recommended that you draw the graphs of both curves.

- It is especially convenient to use a graphing calculator or computer to help with this task.

Find all points of intersection of the curves $r = \cos 2\theta$ and $r = ½$.

- If we solve the equations $r = \cos 2\theta$ and $r = ½$, we get $\cos 2\theta = ½$.
- Therefore, $2\theta = \pi/3, 5\pi/3, 7\pi/3, 11\pi/3$.

- Thus, the values of θ between 0 and 2π that satisfy both equations are:

$$\theta = \pi/6,\ 5\pi/6,\ 7\pi/6,\ 11\pi/6$$

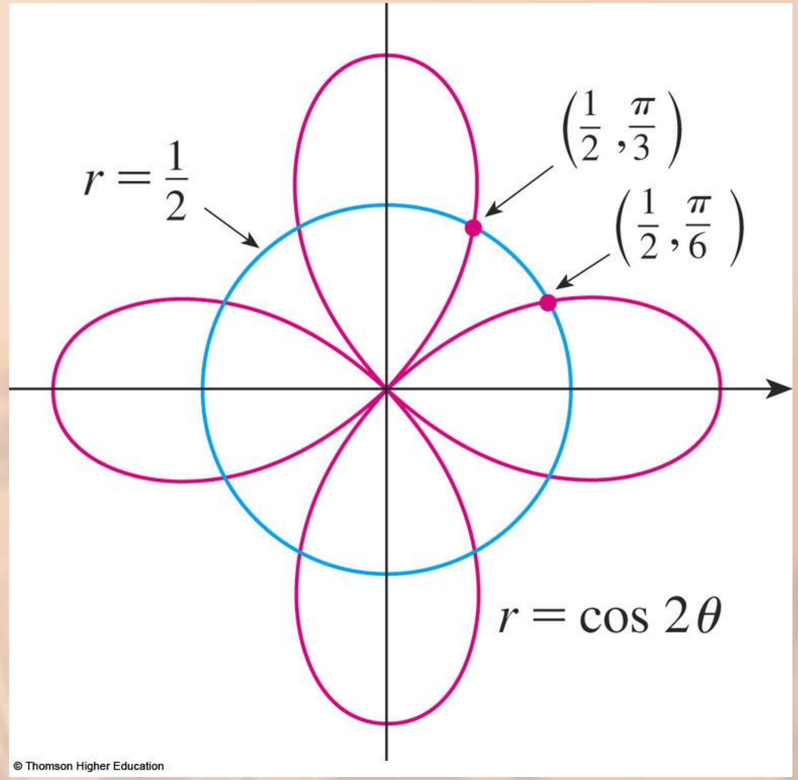
We have found four points of intersection:

$(\frac{1}{2}, \pi/6)$, $(\frac{1}{2}, 5\pi/6)$, $(\frac{1}{2}, 7\pi/6)$, $(\frac{1}{2}, 11\pi/6)$

POINTS OF INTERSECTION **Example 3**

However, you can see that the curves have four other points of intersection:

$(\frac{1}{2}, \pi/3), (\frac{1}{2}, 2\pi/3), (\frac{1}{2}, 4\pi/3), (\frac{1}{2}, 5\pi/3)$

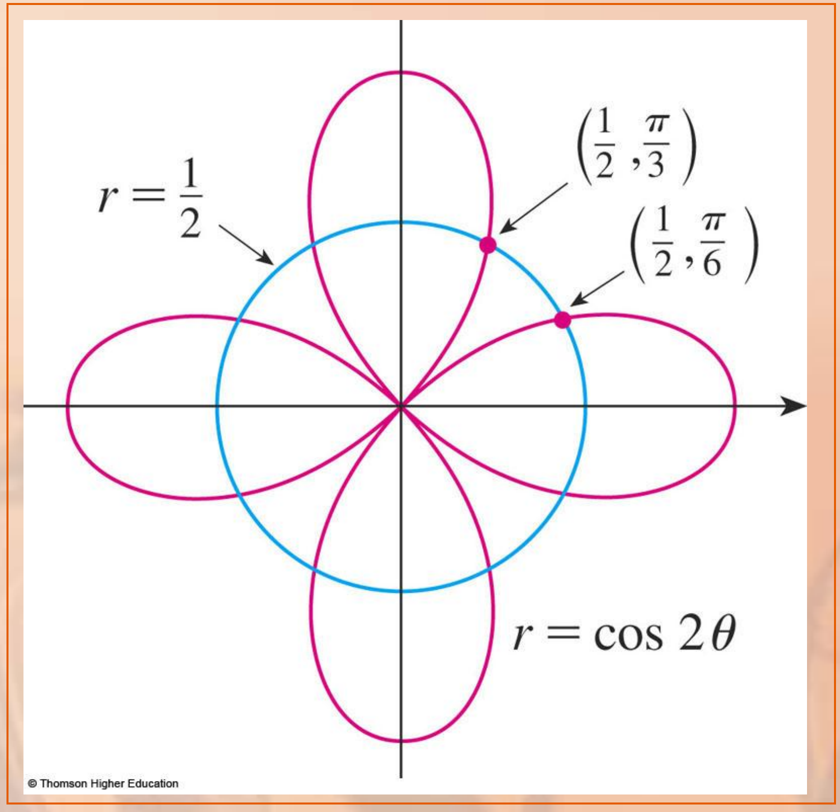


POINTS OF INTERSECTION Example 3

These can be found using symmetry or by noticing that another equation of the circle is $r = -\frac{1}{2}$.

Then, we solve

$r = \cos 2\theta$ and $r = -\frac{1}{2}$.



ARC LENGTH

To find the length of a polar curve $r = f(\theta)$, a ≤ θ ≤ b, we regard θ as a parameter and write the parametric equations of the curve as:

$$x = r \cos \theta = f(\theta)\cos \theta$$

$$y = r \sin \theta = f(\theta)\sin \theta$$

Using the Product Rule and differentiating with respect to θ, we obtain:

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

So, using $\cos^2\theta + \sin^2\theta = 1$,we have:

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta$$

$$+ \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

Formula 5

Assuming that *f'* is continuous, we can use Theorem 6 in Section 10.2 to write the arc length as:

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

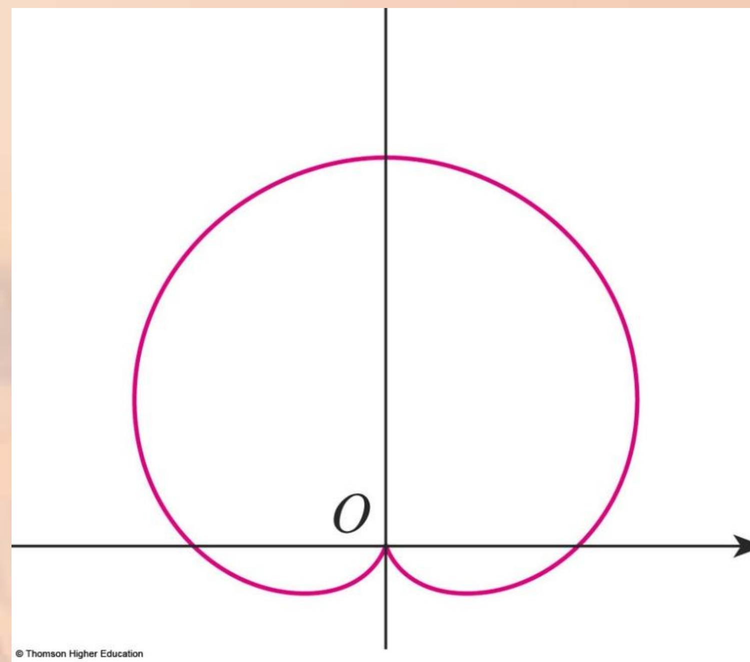
Therefore, the length of a curve with polar equation $r = f(\theta)$, $a \le \theta \le b$, is:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Find the length of the cardioid

$$r = 1 + \sin\theta$$

- We sketched it in Example 7 in Section 10.3



Its full length is given by the parameter interval

$0 \le \theta \le 2\pi$.

So, Formula 5 gives:

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2+2\sin\theta}\, d\theta$$

ARC LENGTH

We could evaluate this integral by multiplying and dividing the integrand by $\sqrt{2-2\sin\theta}$ or we could use a computer algebra system.

- In any event, we find that the length of the cardioid is $L = 8$.