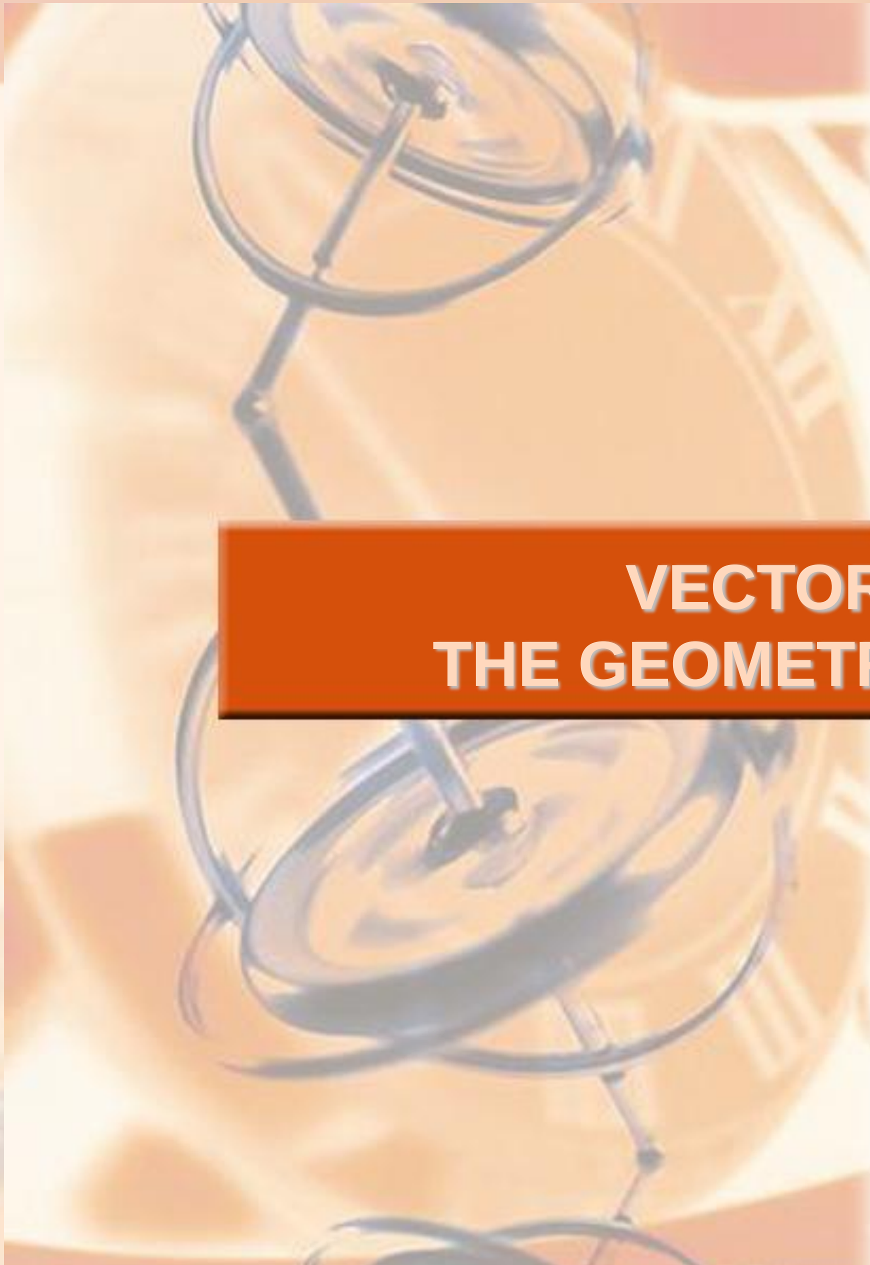


12

VECTORS AND THE GEOMETRY OF SPACE

12.4
The Cross Product

In this section, we will learn about:

Cross products of vectors

and their applications.

THE CROSS PRODUCT

The cross product **a** x **b** of two vectors **a** and **b**, unlike the dot product, is a vector.

- For this reason, it is also called the vector product.
- Note that **a** x **b** is defined only when **a** and **b** are three-dimensional (3-D) vectors.

Definition 1

If **a** = $\langle a_1, a_2, a_3 \rangle$ and **b** = $\langle b_1, b_2, b_3 \rangle$, then the cross product of **a** and **b** is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

CROSS PRODUCT

This may seem like

a strange way of defining

a product.

CROSS PRODUCT

The reason for the particular form of Definition 1 is that the cross product defined in this way has many useful properties, as we will soon see.

- In particular, we will show that the vector **a** x **b** is perpendicular to both **a** and **b**.

CROSS PRODUCT

In order to make Definition 1 easier to remember, we use the notation of determinants.

DETERMINANT OF ORDER 2

A determinant of order 2 is defined by:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

DETERMINANT OF ORDER 3

Equation 2

A determinant of order 3 can be defined in terms of second-order determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

DETERMINANT OF ORDER 3

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Observe that:

- Each term on the right side of Equation 2 involves a number a_i in the first row of the determinant.
- This is multiplied by the second-order determinant obtained from the left side by deleting the row and column in which it appears.

DETERMINANT OF ORDER 3

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Notice also the minus sign in the second term.

DETERMINANT OF ORDER 3

For example,

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= 1(0-4) - 2(6+5) + (-1)(12-0)$$

$$= -38$$

Now, let's rewrite Definition 1 using second-order determinants and the standard basis vectors **i**, **j**, and **k**.

CROSS PRODUCT

Equation 3

We see that the cross product of the vectors

$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

is:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

Equation 4

In view of the similarity between Equations 2 and 3, we often write:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

CROSS PRODUCT

The first row of the symbolic determinant in Equation 4 consists of vectors.

- However, if we expand it as if it were an ordinary determinant using the rule in Equation 2, we obtain Equation 3.

CROSS PRODUCT

The symbolic formula in Equation 4 is probably the easiest way of remembering and computing cross products.

CROSS PRODUCT

Example 1

If **a** = ⟨1, 3, 4⟩ and **b** = ⟨2, 7, –5⟩, then

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}\mathbf{k}$$

$$= (-15-28)\mathbf{i} - (-5-8)\mathbf{j} + (7-6)\mathbf{k}$$

$$= -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$

Show that **a** x **a** = **0** for any vector **a** in V_3.

- If **a** = $\langle a_1, a_2, a_3 \rangle$, then

$$\mathbf{a} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= (a_2a_3 - a_3a_2)\mathbf{i} - (a_1a_3 - a_3a_1)\mathbf{j} + (a_1a_2 - a_2a_1)\mathbf{k}$$

$$= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

CROSS PRODUCT

One of the most important properties of the cross product is given by the following theorem.

Theorem

The vector **a** x **b** is orthogonal to both **a** and **b**.

Proof

In order to show that **a** x **b** is orthogonal to **a**, we compute their dot product as follows

CROSS PRODUCT

Proof

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$

$$= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1)$$

$$= a_1a_2b_3 - a_1b_2a_3 - a_1a_2b_3 + b_1a_2a_3 + a_1b_2a_3 - b_1a_2a_3$$

$$= 0$$

Proof

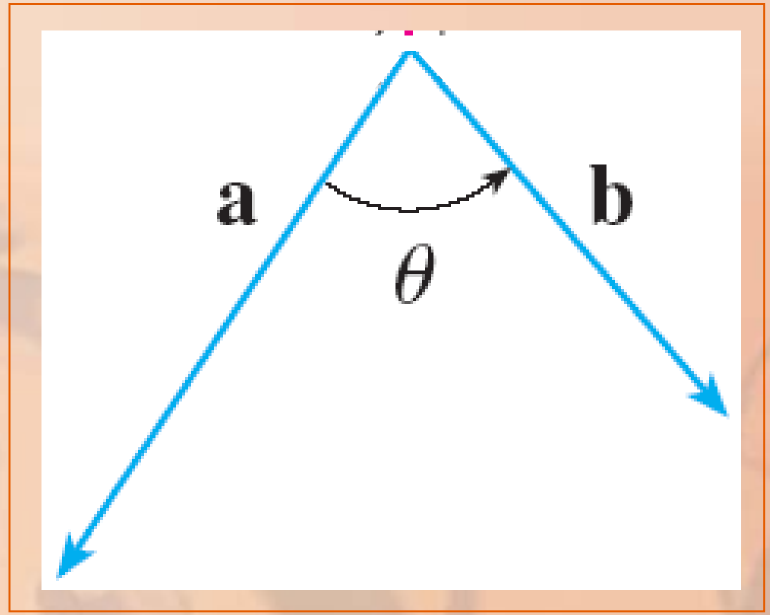
A similar computation shows that

$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$

- Therefore, **a** x **b** is orthogonal to both **a** and **b.**

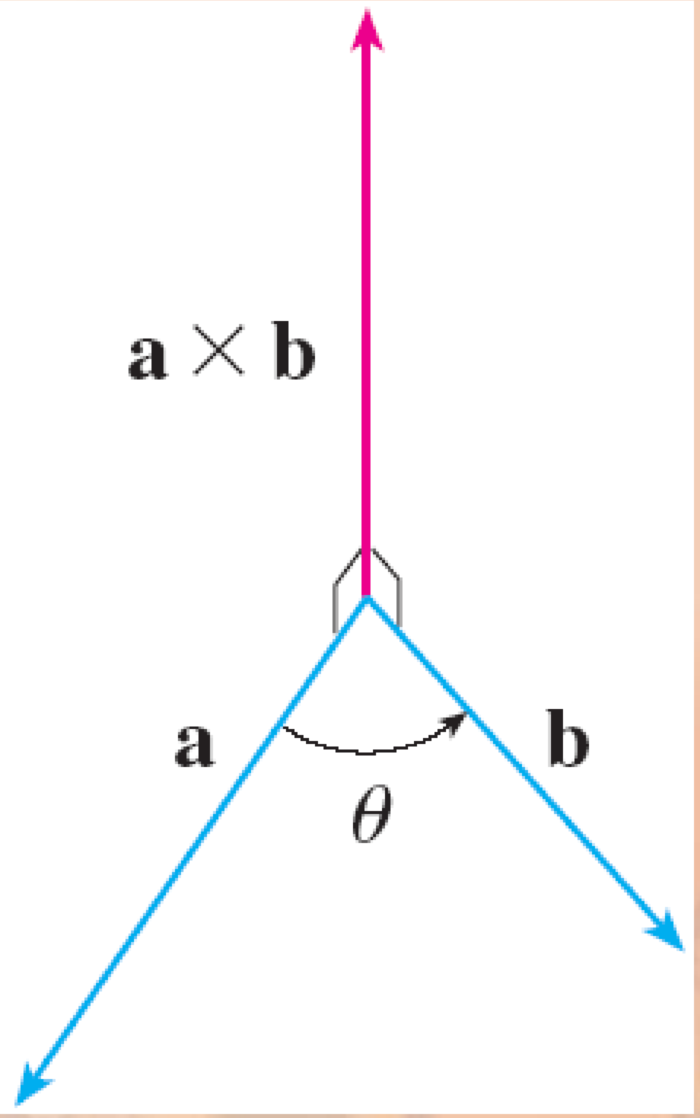
CROSS PRODUCT

Let **a** and **b** be represented by directed line segments with the same initial point, as shown.



CROSS PRODUCT

Then, Theorem 5 states that the cross product **a** x **b** points in a direction perpendicular to the plane through **a** and **b**.

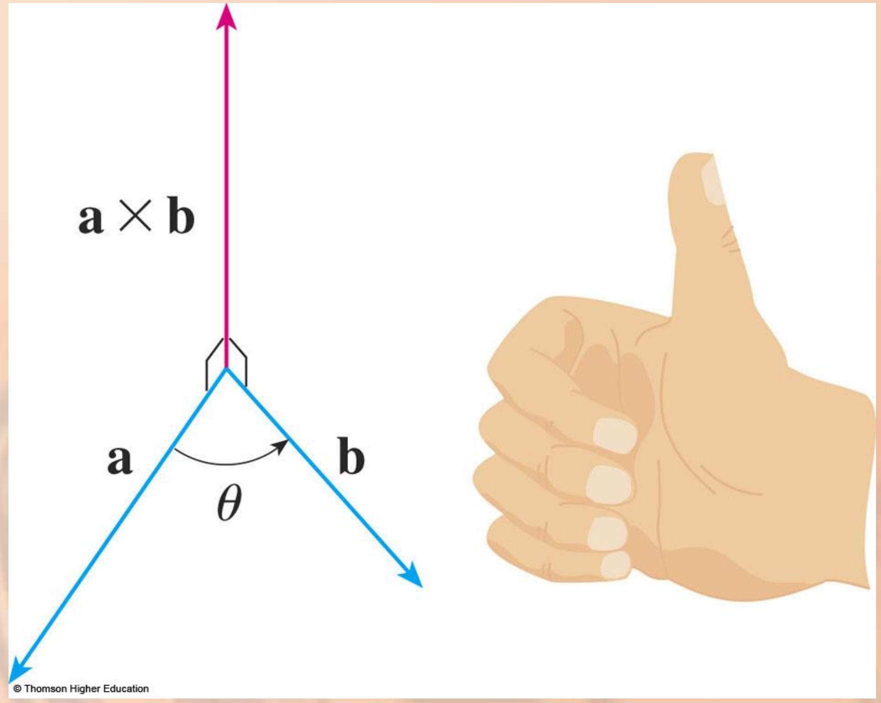


CROSS PRODUCT

It turns out that the direction of **a** x **b** is given by the right-hand rule, as follows.

RIGHT-HAND RULE

If the fingers of your right hand curl in the direction of a rotation (through an angle less than 180°) from **a** to **b**, then your thumb points in the direction of **a** x **b**.



CROSS PRODUCT

We know the direction of the vector **a** x **b**.

The remaining thing we need to complete its geometric description is its length |**a** x **b**|.

- This is given by the following theorem.

Theorem 6

If θ is the angle between **a** and **b**

(so $0 \leq \theta \leq \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

Proof

From the definitions of the cross product and length of a vector, we have:

$$|\mathbf{a} \times \mathbf{b}|^2$$

$$= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$

$$= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2$$
$$- 2a_1a_3b_1b_3 + a_1^2b_3^2 + a_1^2b_2^2$$
$$- 2a_1a_2b_1b_2 + a_2^2b_1^2$$

Proof

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$$= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$$

$$= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2 \cos^2\theta \qquad \text{[Th. 3 in Sec. 12.3]}$$

$$= |\mathbf{a}|^2|\mathbf{b}|^2 (1 - \cos^2\theta)$$

$$= |\mathbf{a}|^2|\mathbf{b}|^2 \sin^2\theta$$

Proof

Taking square roots and observing that

$\sqrt{\sin^2\theta} = \sin\theta$ because $\sin\theta \geq 0$ when

$0 \leq \theta \leq \pi$, we have:

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

CROSS PRODUCT

A vector is completely determined by its magnitude and direction.

Thus, we can now say that **a** x **b** is the vector that is perpendicular to both **a** and **b**, whose:

- Orientation is determined by the right-hand rule
- Length is |**a**||**b**| sin θ

CROSS PRODUCT

In fact, that is exactly how physicists define **a** x **b.**

Corollary 7

Two nonzero vectors **a** and **b** are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

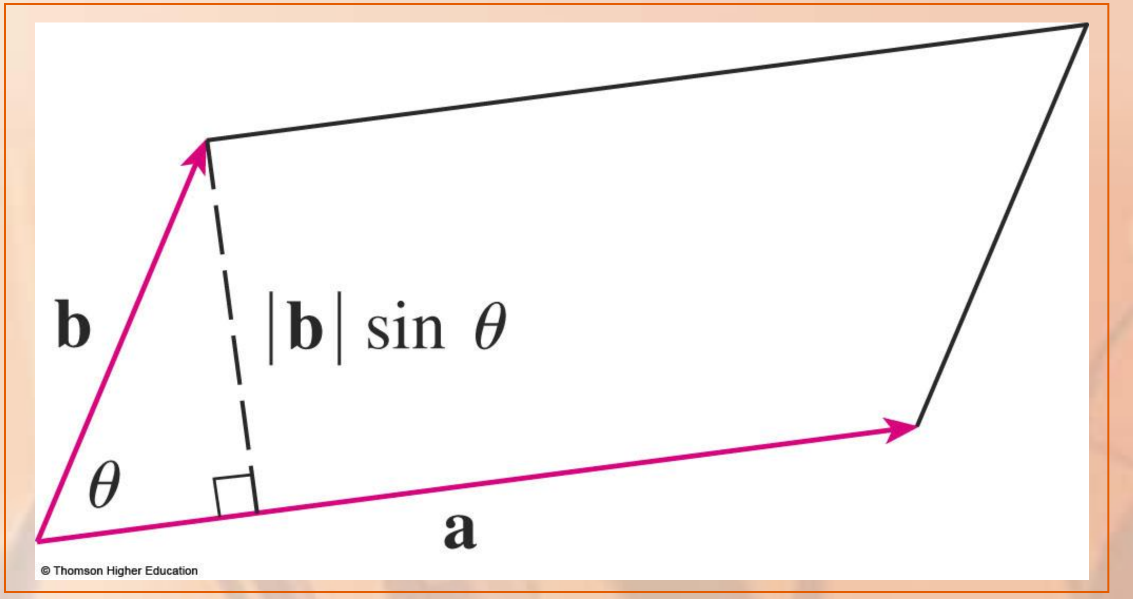
Proof

Two nonzero vectors **a** and **b** are parallel if and only if $\theta = 0$ or π.

- In either case, $\sin \theta = 0$.
- So, $|\mathbf{a} \times \mathbf{b}| = 0$ and, therefore, $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

CROSS PRODUCT

The geometric interpretation of Theorem 6 can be seen from this figure.

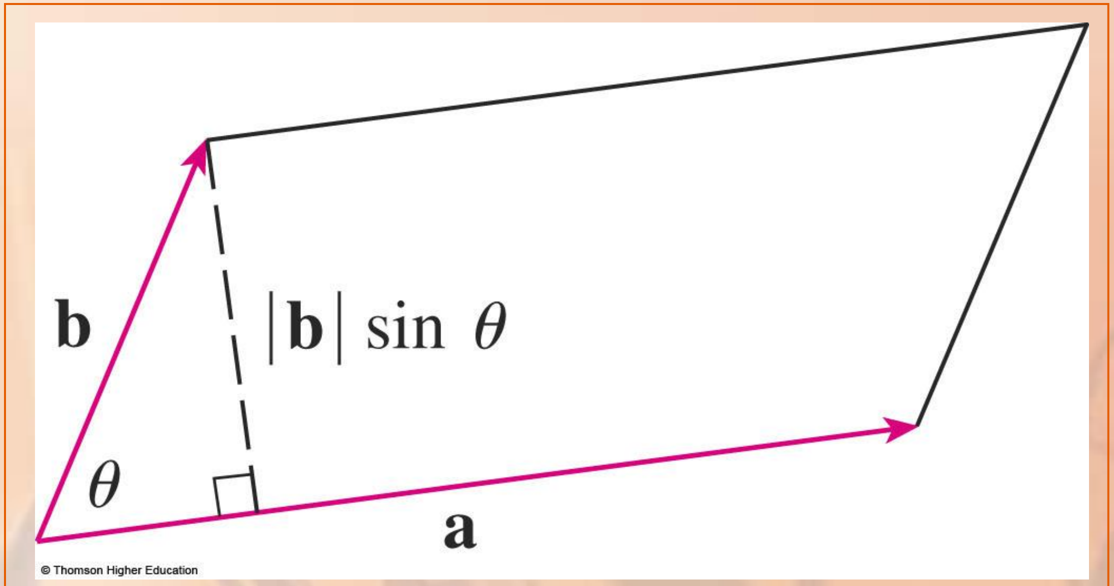


CROSS PRODUCT

If **a** and **b** are represented by directed line segments with the same initial point, then they determine a parallelogram with base |**a**|, altitude |**b**| sin *θ*, and area

$A = |\mathbf{a}|(|\mathbf{b}| \sin \theta)$

$= |\mathbf{a} \times \mathbf{b}|$



CROSS PRODUCT

Thus, we have the following way of interpreting the magnitude of a cross product.

CROSS PRODUCT MAGNITUDE

The length of the cross product **a** x **b** is equal to the area of the parallelogram determined by **a** and **b**.

Example 3

Find a vector perpendicular to the plane that passes through the points

$P(1, 4, 6)$, $Q(-2, 5, -1)$, $R(1, -1, 1)$

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

- Therefore, it is perpendicular to the plane through *P*, *Q*, and *R*.

From Equation 1 in Section 12.2, we know that:

$$\overrightarrow{PQ} = (-2-1)\,\mathbf{i} + (5-4)\,\mathbf{j} + (-1-6)\,\mathbf{k}$$
$$= -3\mathbf{i} + \mathbf{j} - 7\mathbf{k}$$

$$\overrightarrow{PR} = (1-1)\,\mathbf{i} + (-1-4)\,\mathbf{j} + (1-6)\,\mathbf{k}$$
$$= -5\mathbf{j} - 5\mathbf{k}$$

We compute the cross product of these vectors:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= (-5 - 35)\,\mathbf{i} - (15 - 0)\,\mathbf{j} + (15 - 0)\,\mathbf{k}$$

$$= -40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}$$

Therefore, the vector $\langle -40, -15, 15 \rangle$ is perpendicular to the given plane.

- Any nonzero scalar multiple of this vector, such as $\langle -8, -3, 3 \rangle$, is also perpendicular to the plane.

Example 4

Find the area of the triangle with vertices

$P(1, 4, 6)$, $Q(-2, 5, -1)$, $R(1, -1, 1)$

Example 4

In Example 3, we computed that

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -40, -15, 15 \rangle$$

- The area of the parallelogram with adjacent sides *PQ* and *PR* is the length of this cross product:

$$\left| \overrightarrow{PQ} \times \overrightarrow{PR} \right| = \sqrt{(-40)^2 + (-15)^2 + 15^2} = 5\sqrt{82}$$

The area *A* of the triangle *PQR* is half the area of this parallelogram, that is:

$$\frac{5}{2}\sqrt{82}$$

CROSS PRODUCT

If we apply Theorems 5 and 6 to the standard basis vectors **i**, **j**, and **k** using $\theta = \pi/2$, we obtain:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

Observe that:

$$\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$$

- Thus, the cross product is not commutative.

CROSS PRODUCT

Also,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

However,

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$$

- So, the associative law for multiplication does not usually hold.
- That is, in general, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

CROSS PRODUCT

However, some of the usual laws of algebra do hold for cross products.

CROSS PRODUCT

The following theorem summarizes the properties of vector products.

CROSS PRODUCT PROPERTIES **Theorem 8**

If **a**, **b**, and **c** are vectors and *c* is a scalar, then

1. **a** x **b** = –**b** x **a**

2. (*c***a**) x **b** = *c*(**a** x **b**) = **a** x (*c***b**)

3. **a** x (**b** + **c**) = **a** x **b** + **a** x **c**

CROSS PRODUCT PROPERTIES

Theorem 8

4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

CROSS PRODUCT PROPERTIES

These properties can be proved by writing the vectors in terms of their components and using the definition of a cross product.

- We give the proof of Property 5 and leave the remaining proofs as exercises.

CROSS PRODUCT PROPERTY 5 **Proof**

Let

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

$$\mathbf{b} = \langle b_1, b_2, b_3 \rangle$$

$$\mathbf{c} = \langle c_1, c_2, c_3 \rangle$$

CROSS PRODUCT PROPERTY 5 — Proof—Equation 9

Then,

$$
\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) \\
&\quad + a_3(b_1c_2 - b_2c_1) \\
&= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 \\
&\quad + a_3b_1c_2 - a_3b_2c_1 \\
&= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 \\
&\quad + (a_1b_2 - a_2b_1)c_3 \\
&= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}
\end{aligned}
$$

SCALAR TRIPLE PRODUCT

The product **a** · (**b** x **c**) that occurs in Property 5 is called the scalar triple product of the vectors **a**, **b**, and **c**.

SCALAR TRIPLE PRODUCTS

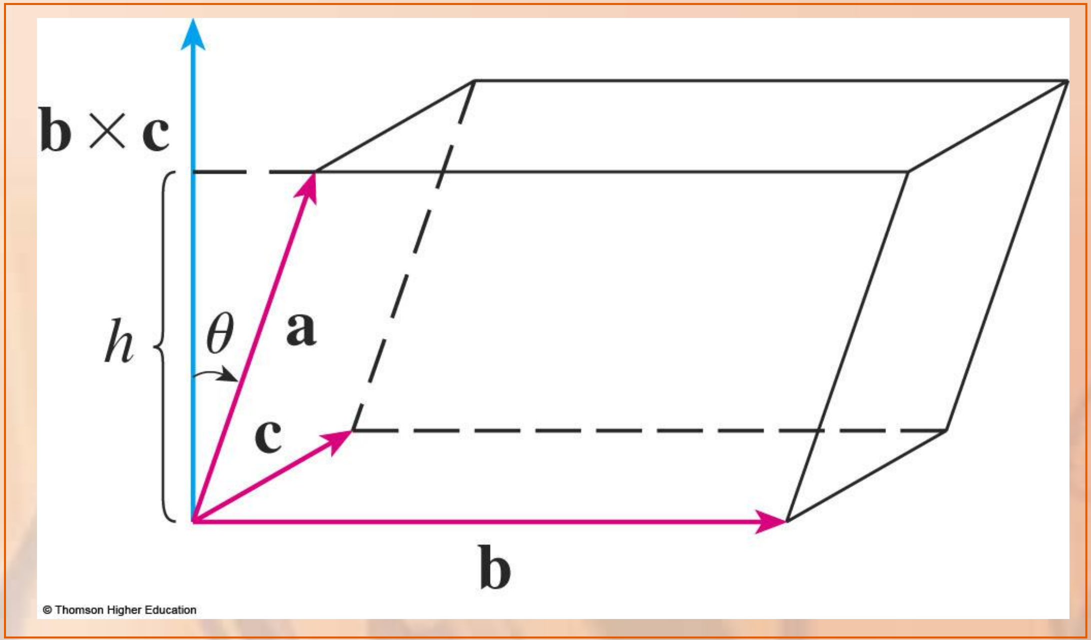
Equation 10

Notice from Equation 9 that we can write the scalar triple product as a determinant:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

SCALAR TRIPLE PRODUCTS

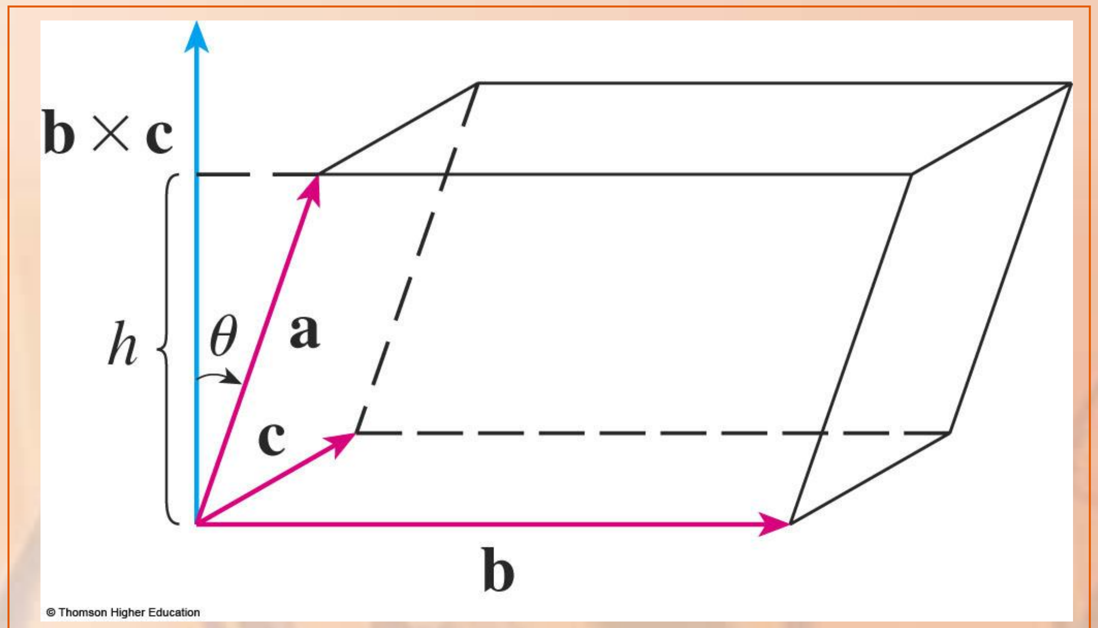
The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors **a**, **b**, and **c**.



SCALAR TRIPLE PRODUCTS

The area of the base parallelogram is:

$$A = |\mathbf{b} \times \mathbf{c}|$$

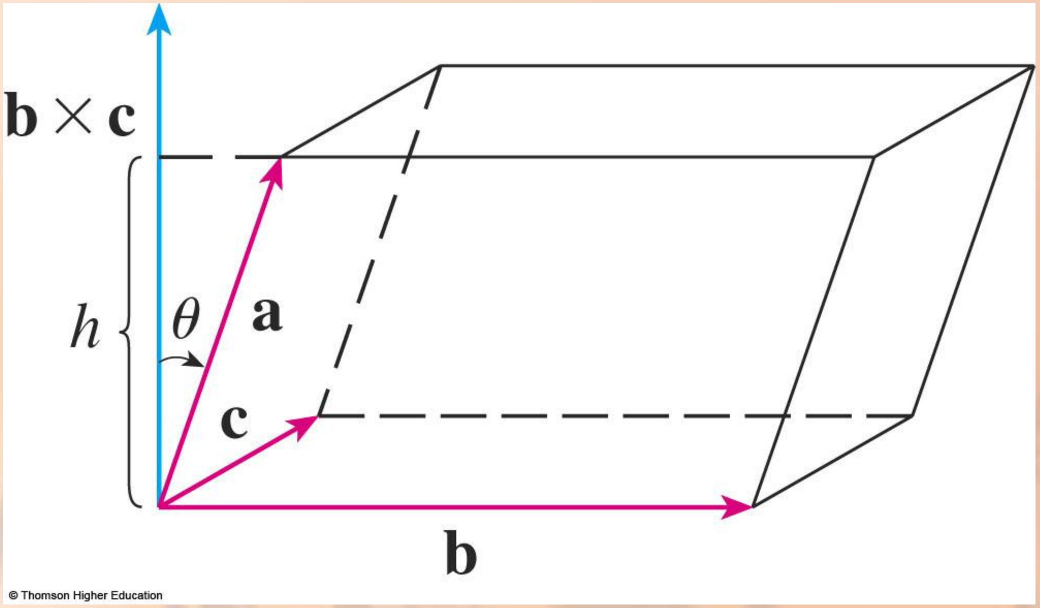


SCALAR TRIPLE PRODUCTS

If θ is the angle between **a** and **b** x **c**, then the height h of the parallelepiped is:

$$h = |\mathbf{a}||\cos \theta|$$

- We must use $|\cos \theta|$ instead of $\cos \theta$ in case $\theta > \pi/2$.



SCALAR TRIPLE PRODUCTS

Hence, the volume of the parallelepiped is:

$$
\begin{aligned}
V &= Ah \\
&= |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos \theta| \\
&= |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|
\end{aligned}
$$

- Thus, we have proved the following formula.

SCALAR TRIPLE PRODUCTS

Formula 11

The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

COPLANAR VECTORS

If we use Formula 11 and discover that the volume of the parallelepiped determined by **a**, **b**, and **c** is 0, then the vectors must lie in the same plane.

- That is, they are coplanar.

Example 5

Use the scalar triple product to show that the vectors

$\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, $\mathbf{c} = \langle 0, -9, 18 \rangle$

are coplanar.

We use Equation 10 to compute their scalar triple product:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7\begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

$$= 1(18) - 4(36) - 7(-18) = 0$$

Example 5

Hence, by Formula 11, the volume of the parallelepiped determined by **a**, **b**, and **c** is 0.

- This means that **a**, **b**, and **c** are coplanar.

VECTOR TRIPLE PRODUCT

The product **a** x (**b** x **c**) that occurs in Property 6 is called the vector triple product of **a**, **b**, and **c**.

- Property 6 will be used to derive Kepler's First Law of planetary motion in Chapter 13.
- Its proof is left as Exercise 46

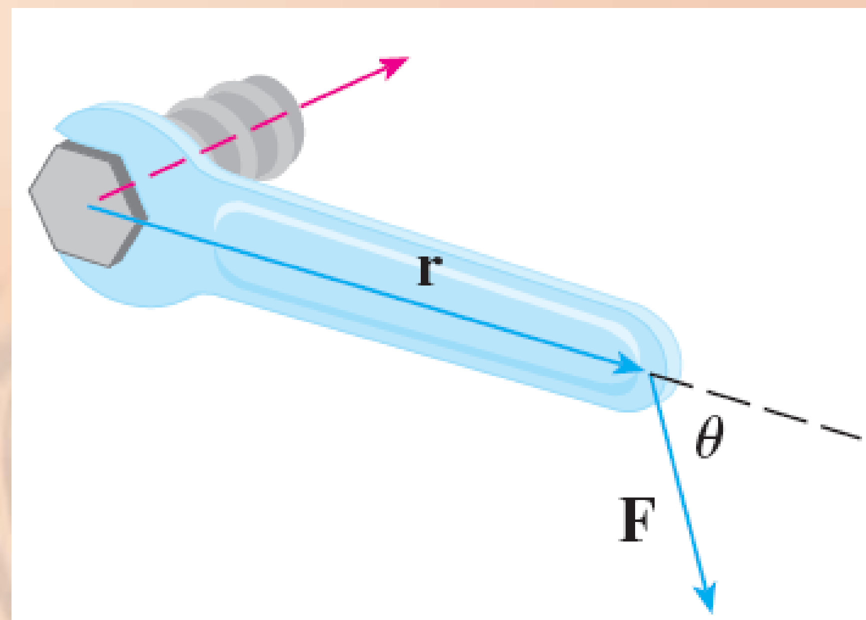
CROSS PRODUCT IN PHYSICS

The idea of a cross product occurs often in physics.

In particular, we consider a force **F** acting on a rigid body at a point given by a position vector **r**.

CROSS PRODUCT IN PHYSICS

For instance, if we tighten a bolt by applying a force to a wrench, we produce a turning effect.

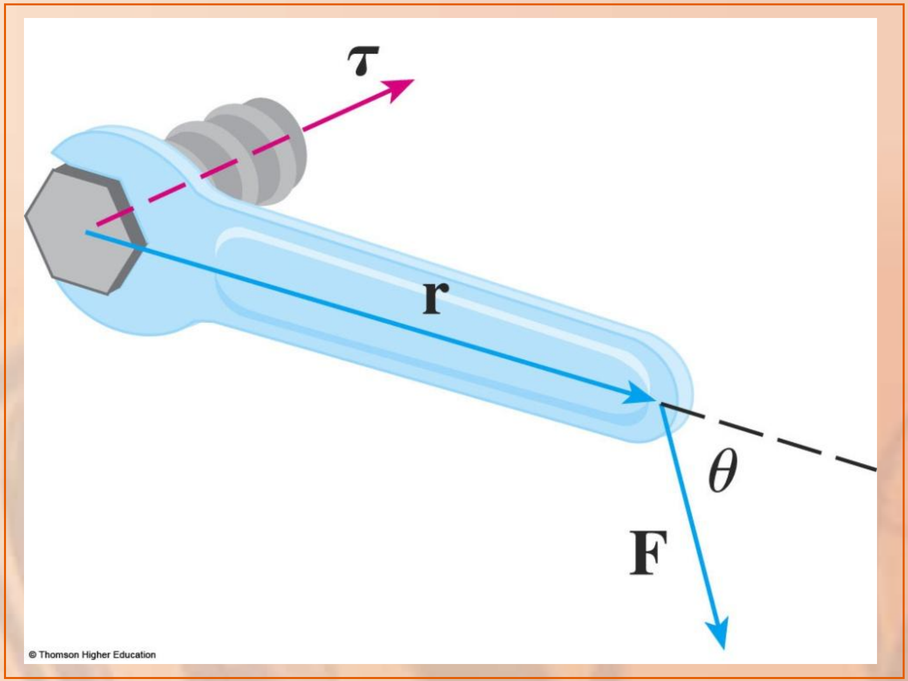


TORQUE

The torque τ (relative to the origin) is defined to be the cross product of the position and force vectors

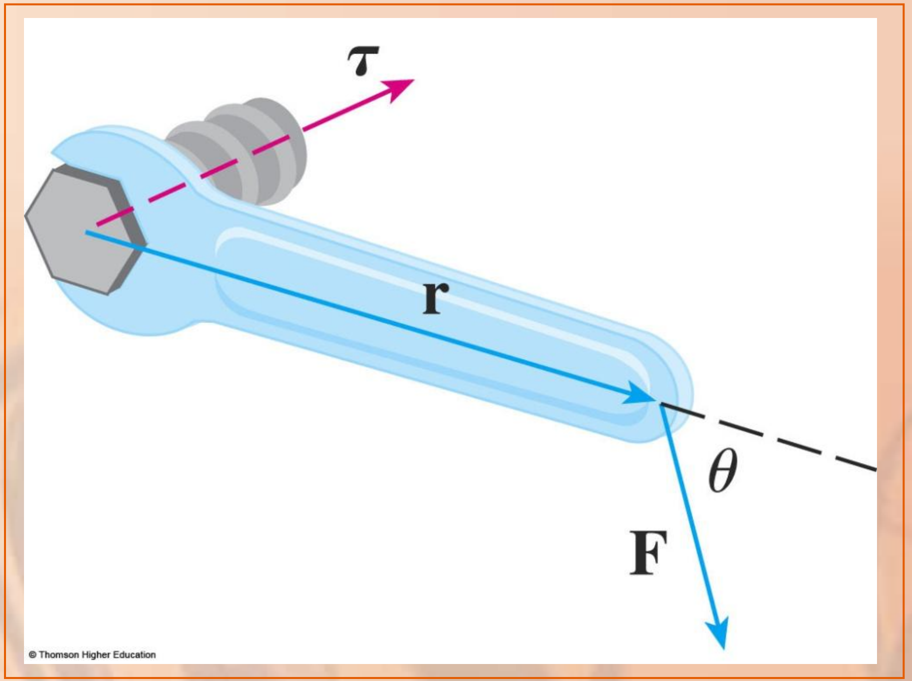
$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

- It measures the tendency of the body to rotate about the origin.



TORQUE

The direction of the torque vector indicates the axis of rotation.



TORQUE

According to Theorem 6, the magnitude of the torque vector is

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin \theta$$

where θ is the angle between the position and force vectors.

TORQUE

Observe that the only component of **F** that can cause a rotation is the one perpendicular to **r**—that is, |**F**| sin θ.

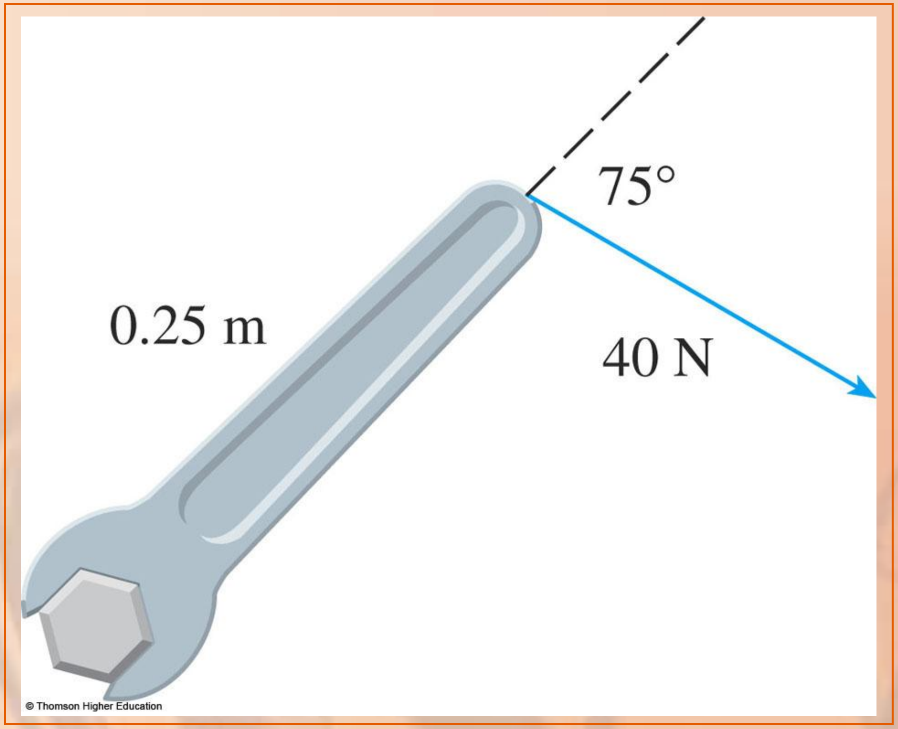
- The magnitude of the torque is equal to the area of the parallelogram determined by **r** and **F**.

TORQUE

Example 6

A bolt is tightened by applying a 40-N force to a 0.25-m wrench, as shown.

- Find the magnitude of the torque about the center of the bolt.



Example 6

The magnitude of the torque vector is:

$$
\begin{aligned}
|\boldsymbol{\tau}| &= |\mathbf{r} \times \mathbf{F}| \\
&= |\mathbf{r}||\mathbf{F}| \sin 75° \\
&= (0.25)(40) \sin 75° \\
&= 10 \sin 75° \\
&\approx 9.66 \text{ N}\cdot\text{m}
\end{aligned}
$$

If the bolt is right-threaded, then the torque vector itself is

$$\boldsymbol{\tau} = |\boldsymbol{\tau}|\ \mathbf{n} \approx 9.66\ \mathbf{n}$$

where **n** is a unit vector directed down into the slide.