15

MULTIPLE INTEGRALS

15.4

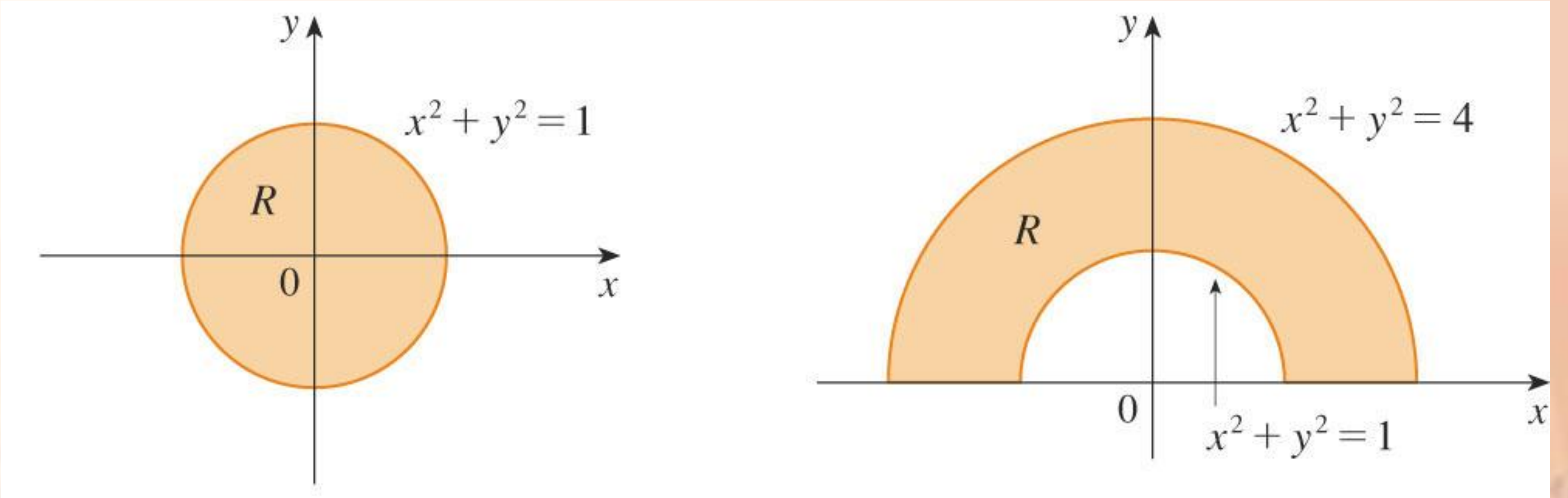
Double Integrals in Polar Coordinates

In this section, we will learn:

How to express double integrals in polar coordinates.

DOUBLE INTEGRALS IN POLAR COORDINATES

Suppose that we want to evaluate a double integral $\iint_R f(x,y)\,dA$, where *R* is one of the regions shown here.

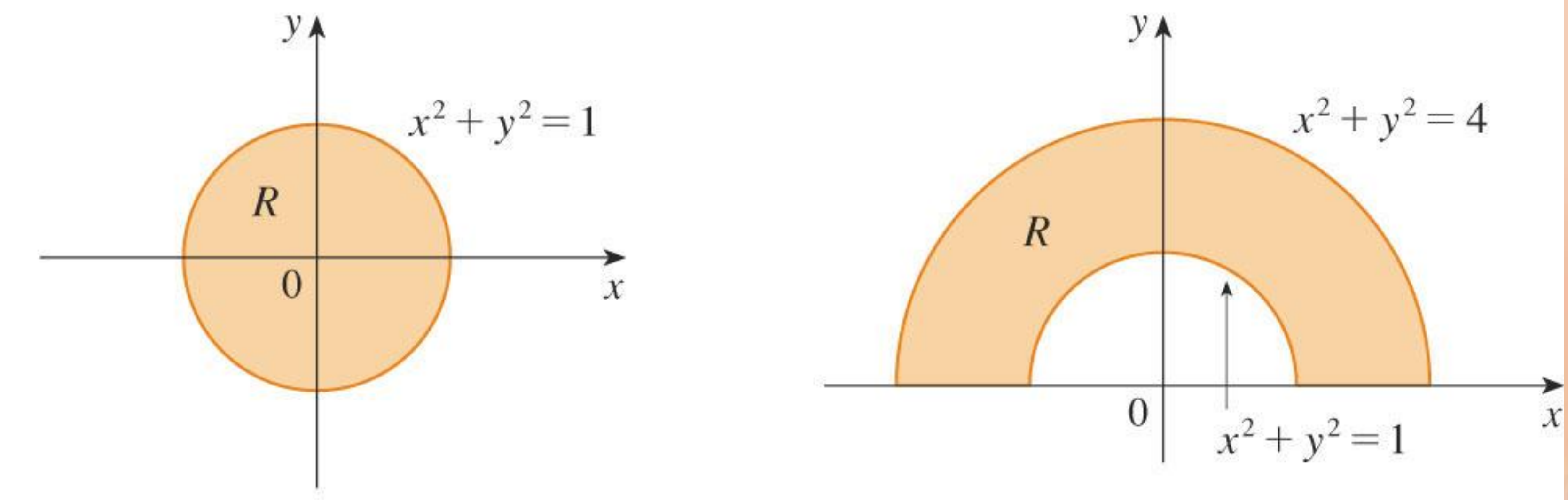


(a) $R=\{(r,\theta)\mid 0\leq r\leq 1, 0\leq\theta\leq 2\pi\}$

(b) $R=\{(r,\theta)\mid 1\leq r\leq 2, 0\leq\theta\leq \pi\}$

DOUBLE INTEGRALS IN POLAR COORDINATES

In either case, the description of *R* in terms of rectangular coordinates is rather complicated but *R* is easily described by polar coordinates.



(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$

(b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

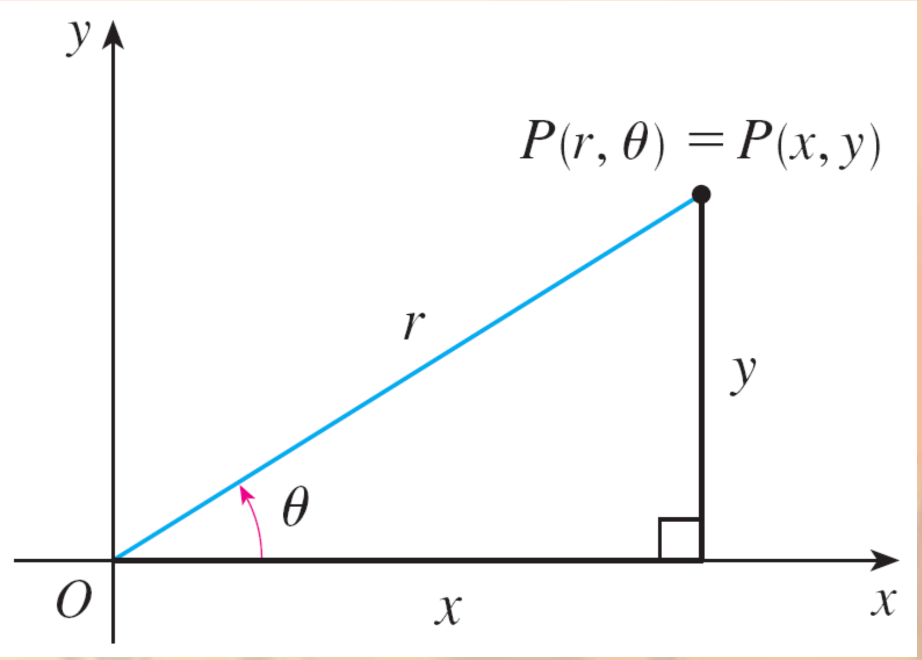
DOUBLE INTEGRALS IN POLAR COORDINATES

Recall from this figure that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$r^2 = x^2 + y^2$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

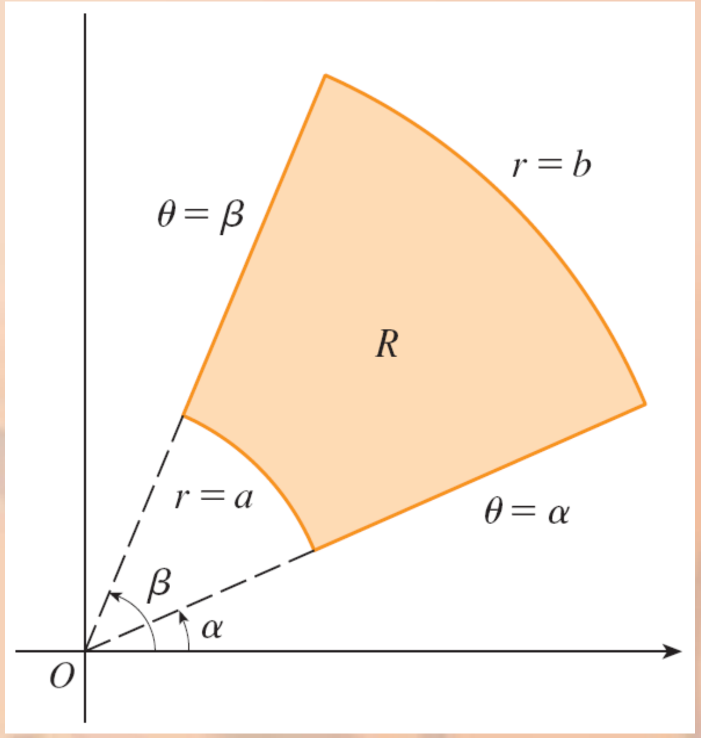


POLAR RECTANGLE

The regions in the first figure are special cases of a polar rectangle

$$R = \{(r, \theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$$

shown here.



POLAR RECTANGLE

To compute the double integral

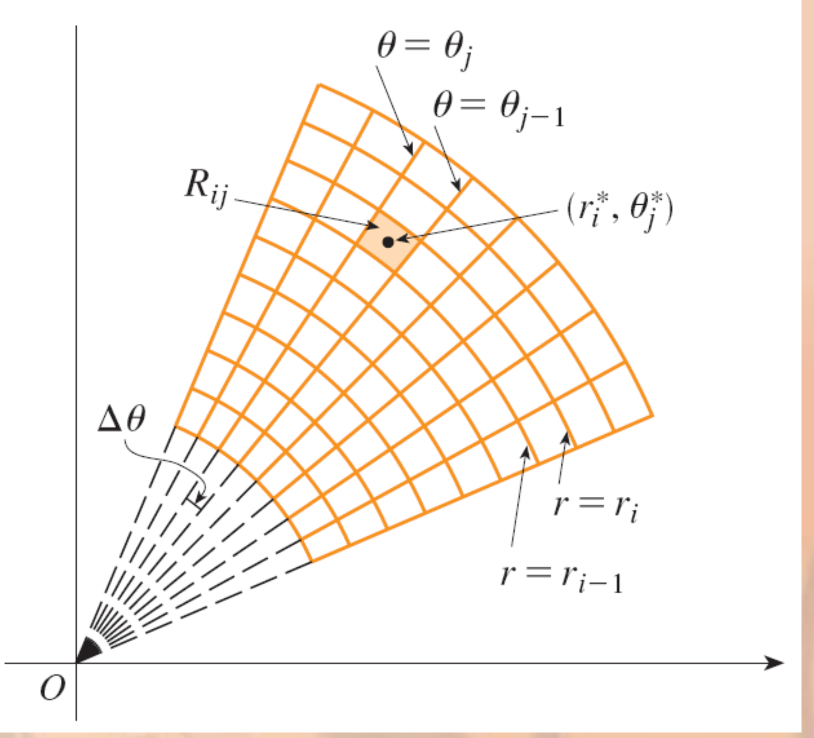
$$\iint_R f(x, y)\, dA$$

where *R* is a polar rectangle, we divide:

- The interval [a, b] into m subintervals [r_{i-1}, r_i] of equal width $\Delta r = (b - a)/m$.
- The interval [α, β] into n subintervals [θ_{j-1}, θ_j] of equal width $\Delta\theta = (\beta - \alpha)/n$.

POLAR RECTANGLES

Then, the circles $r = r_i$ and the rays $\theta = \theta_i$ divide the polar rectangle R into the small polar rectangles shown here.



POLAR SUBRECTANGLE

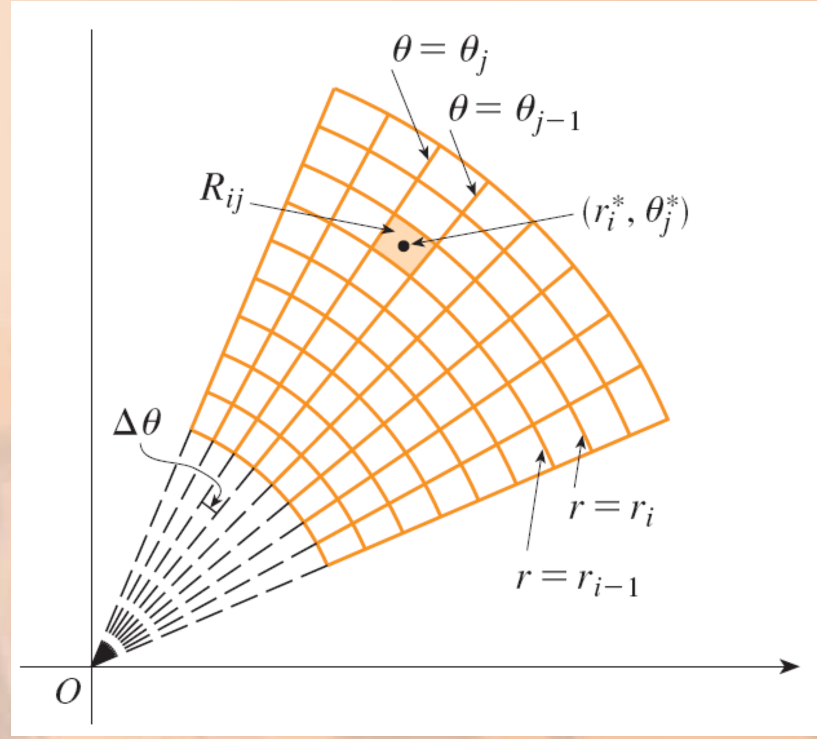
The "center" of the polar subrectangle

$$R_{ij} = \{(r, \theta) \mid r_{i-1} \le r \le r_i, \theta_{j-1} \le \theta \le \theta_j\}$$

has polar coordinates

$$r_i^* = ½ (r_{i-1} + r_i)$$

$$\theta_j^* = ½ (\theta_{j-1} + \theta_j)$$



POLAR SUBRECTANGLE

We compute the area of R_{ij} using the fact that the area of a sector of a circle with radius r and central angle θ is $\frac{1}{2}r^2\theta$.

POLAR SUBRECTANGLE

Subtracting the areas of two such sectors, each of which has central angle $\Delta\theta = \theta_j - \theta_{j-1}$, we find that the area of R_{ij} is:

$$
\begin{aligned}
\Delta A_i &= \tfrac{1}{2} r_i^2\, \theta - \tfrac{1}{2} r_{i-1}^2\, \Delta\theta \\
&= \tfrac{1}{2} (r_i^2 - r_{i-1}^2)\, \Delta\theta \\
&= \tfrac{1}{2} (r_i + r_{i-1})(r_i - r_{i-1})\, \Delta\theta \\
&= r_i^*\, \Delta r\, \Delta\theta
\end{aligned}
$$

POLAR RECTANGLES

We have defined the double integral

$$\iint_R f(x, y)\, dA$$

in terms of ordinary rectangles.

However, it can be shown that, for continuous functions *f*, we always obtain the same answer using polar rectangles.

The rectangular coordinates of the center of R_{ij} are $(r_i^* \cos \theta_j^*, r_i^* \sin \theta_j^*)$.

So, a typical Riemann sum is:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\,\Delta A_i$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, r_i^*\,\Delta r\,\Delta\theta$$

POLAR RECTANGLES

If we write $g(r, \theta) = r\, f(r \cos \theta, r \sin \theta)$, the Riemann sum in Equation 1 can be written as:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} g(r_i^*, \theta_j^*)\, \Delta r\, \Delta \theta$$

- This is a Riemann sum for the double integral

$$\int_{\alpha}^{\beta} \int_{a}^{b} g(r, \theta)\, dr\, d\theta$$

POLAR RECTANGLES

Thus, we have:

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\,\Delta A_i$$

$$= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*,\theta_j^*)\,\Delta r\,\Delta\theta$$

$$= \int_\alpha^\beta \int_a^b g(r,\theta)\,dr\,d\theta$$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

CHANGE TO POLAR COORDS. **Formula 2**

If *f* is continuous on a polar rectangle *R* given by

$$0 \le a \le r \le b, \ \alpha \le \theta \le \beta$$

where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

CHANGE TO POLAR COORDS.

Formula 2 says that we convert from rectangular to polar coordinates in a double integral by:

- Writing $x = r \cos \theta$ and $y = r \sin \theta$
- Using the appropriate limits of integration for r and θ
- Replacing dA by $dr\, d\theta$

CHANGE TO POLAR COORDS.

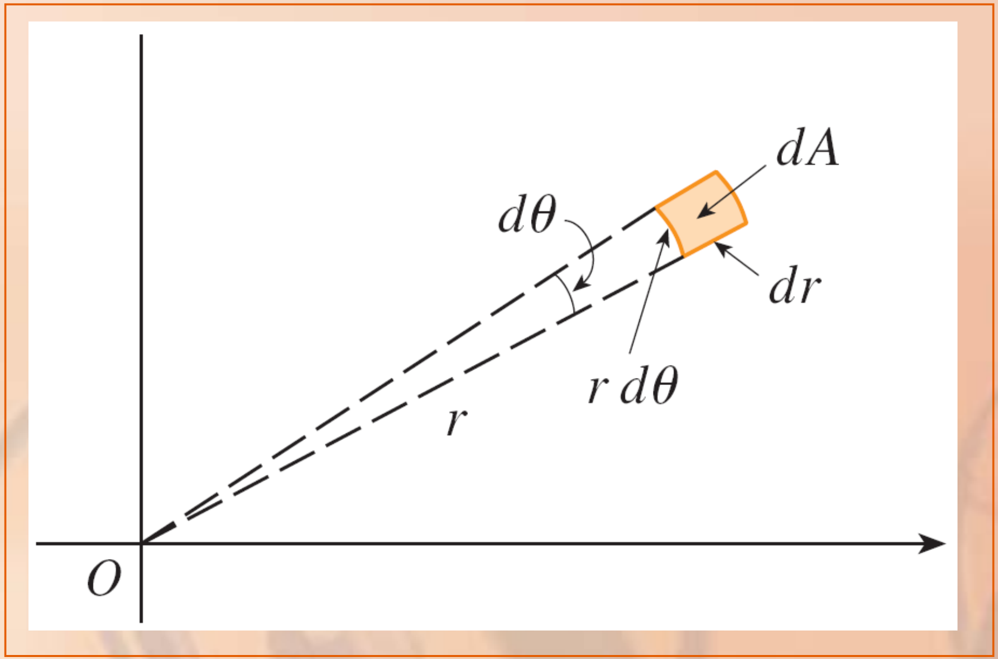
Be careful not to forget the additional factor r on the right side of Formula 2.

CHANGE TO POLAR COORDS.

A classical method for remembering the formula is shown here.

- The "infinitesimal" polar rectangle can be thought of as an ordinary rectangle with dimensions $r\,d\theta$ and dr.

- So, it has "area" $dA = r\,dr\,d\theta$.



CHANGE TO POLAR COORDS. **Example 1**

Evaluate

$$\iint_R (3x + 4y^2)\, dA$$

where *R* is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

The region R can be described as:

$$R = \{(x, y) \mid y \geq 0,\ 1 \leq x^2 + y^2 \leq 4\}$$

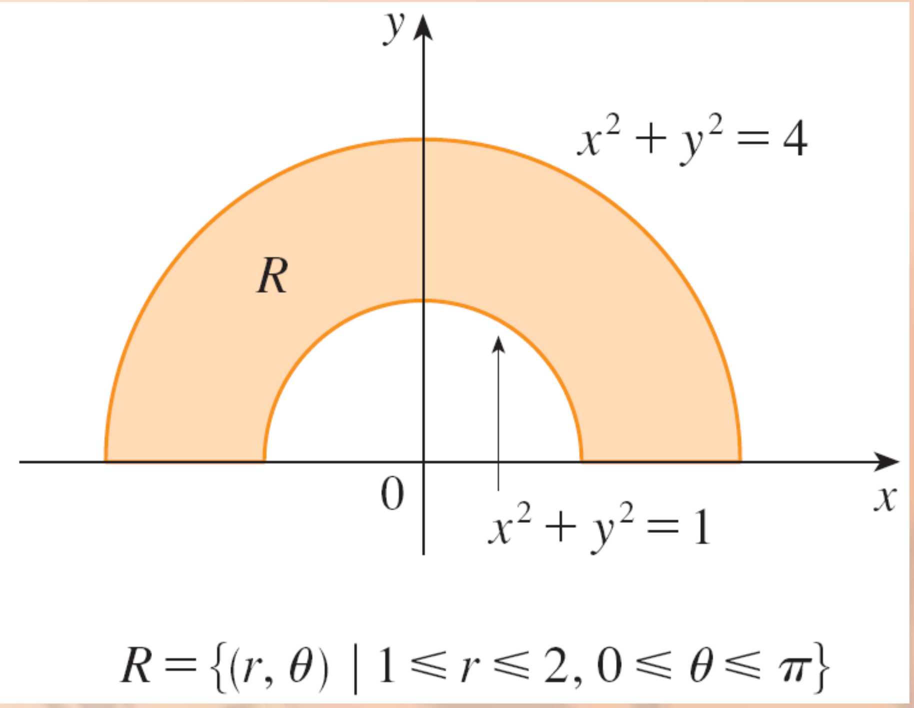
CHANGE TO POLAR COORDS. **Example 1**

It is the half-ring shown here.

In polar coordinates,

it is given by:

$1 \leq r \leq 2,\ 0 \leq \theta \leq \pi$



Hence, by Formula 2,

$$\iint_R (3x + 4y^2)\, dA$$

$$= \int_0^{\pi} \int_1^2 (3r \cos\theta + 4r^2 \sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi} \int_1^2 (3r^2 \cos\theta + 4r^3 \sin^2\theta)\, dr\, d\theta$$

$$= \int_0^{\pi} [r^3 \cos\theta + r^4 \sin^2\theta]_{r=1}^{r=2}\, d\theta$$

CHANGE TO POLAR COORDS. **Example 1**

$$=\int_0^{\pi}(7\cos\theta+15\sin^2\theta]\,d\theta$$

$$=\int_0^{\pi}[7\cos\theta+\tfrac{15}{2}(1-\cos 2\theta)]\,d\theta$$

$$=7\sin\theta+\frac{15\theta}{2}-\frac{15}{4}\sin 2\theta\Big]_0^{\pi}$$

$$=\frac{15\pi}{2}$$

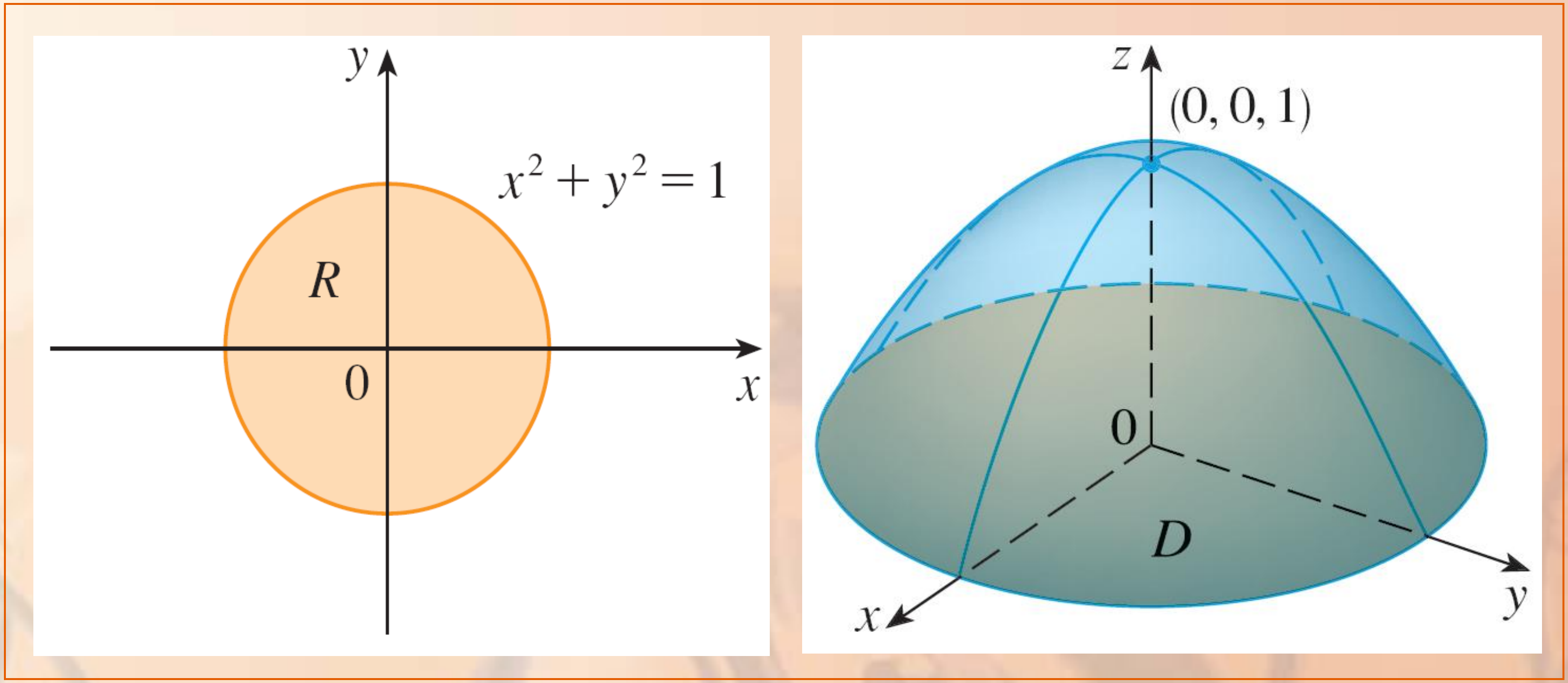
Find the volume of the solid bounded by:

- The plane $z = 0$
- The paraboloid $z = 1 - x^2 - y^2$

If we put $z = 0$ in the equation of the paraboloid, we get $x^2 + y^2 = 1$.

- This means that the plane intersects the paraboloid in the circle $x^2 + y^2 = 1$.

So, the solid lies under the paraboloid and above the circular disk *D* given by $x^2 + y^2 \le 1$.



In polar coordinates, *D* is given by

$0 \le r \le 1,\ 0 \le \theta \le 2\pi$.

- As $1 - x^2 - y^2 = 1 - r^2$, the volume is:

$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (r - r^3)\, dr$$

$$= 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = \frac{\pi}{2}$$

CHANGE TO POLAR COORDS. Example 2

Had we used rectangular coordinates instead, we would have obtained:

$$V = \iint_D (1 - x^2 - y^2)\, dA$$

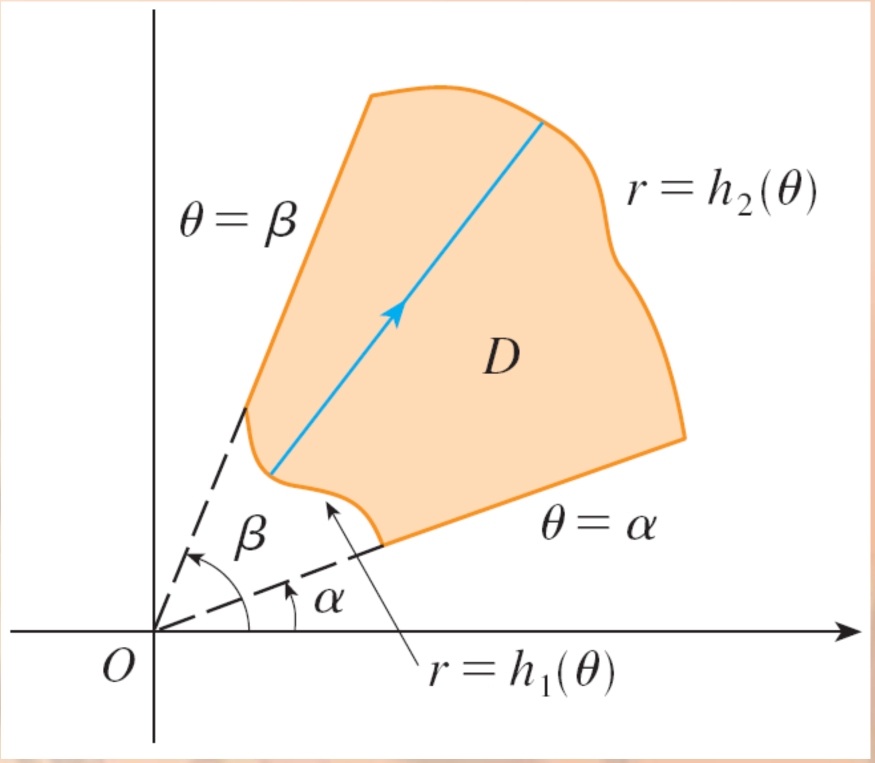
$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx$$

- This is not easy to evaluate because it involves finding $\int (1 - x^2)^{3/2}\, dx$

CHANGE TO POLAR COORDS.

What we have done so far can be extended to the more complicated type of region shown here.

- It's similar to the type II rectangular regions considered in Section 15.3



CHANGE TO POLAR COORDS.

In fact, by combining Formula 2 in this section with Formula 5 in Section 15.3, we obtain the following formula.

CHANGE TO POLAR COORDS. **Formula 3**

If *f* is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

then

$$\iint_D f(x, y)\, dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

CHANGE TO POLAR COORDS.

In particular, taking $f(x, y) = 1$, $h_1(\theta) = 0$, and $h_2(\theta) = h(\theta)$ in the formula, we see that the area of the region D bounded by $\theta = \alpha$, $\theta = \beta$, and $r = h(\theta)$ is:

$$A(D) = \iint_D 1\,dA = \int_\alpha^\beta \int_0^{h(\theta)} r\,dr\,d\theta = \int_\alpha^\beta \left[\frac{r^2}{2}\right]_0^{h(\theta)} d\theta$$

$$= \int_\alpha^\beta \tfrac{1}{2}[h(\theta)]^2\,d\theta$$

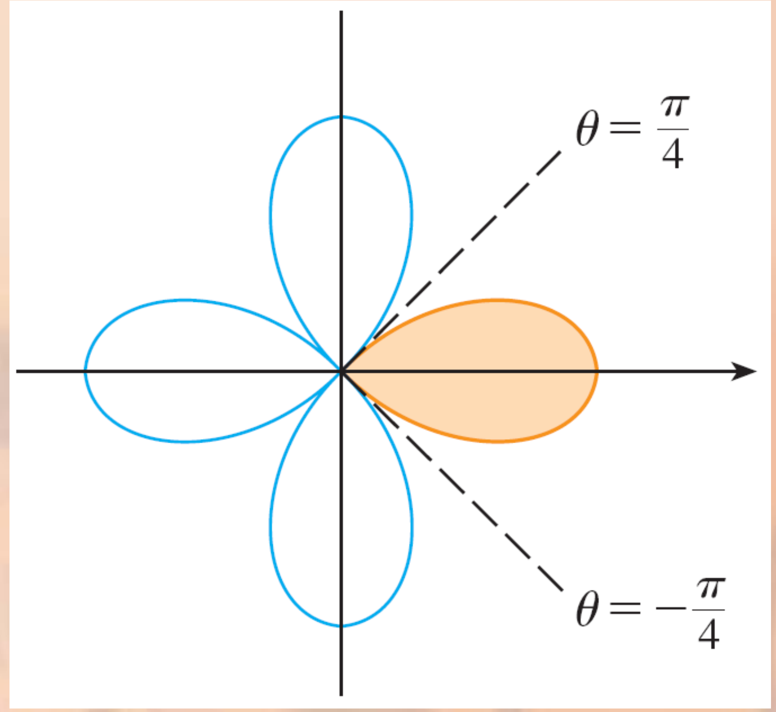
- This agrees with Formula 3 in Section 10.4

Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

CHANGE TO POLAR COORDS. **Example 3**

From this sketch of the curve, we see that a loop is given by the region

$$D = \{(r, \theta) \mid -\pi/4 \le \theta \le \pi/4,\ 0 \le r \le \cos 2\theta\}$$



So, the area is:

$$A(D)=\iint_D dA=\int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} r\,dr\,d\theta$$

$$=\int_{-\pi/4}^{\pi/4}\left[\tfrac{1}{2}r^2\right]_0^{\cos 2\theta}d\theta$$

$$=\tfrac{1}{2}\int_{-\pi/4}^{\pi/4}\cos^2 2\theta\,d\theta$$

$$=\tfrac{1}{4}\int_{-\pi/4}^{\pi/4}(1+\cos 4\theta)\,d\theta$$

$$=\tfrac{1}{4}\left[\theta+\tfrac{1}{4}\sin 4\theta\right]_{-\pi/4}^{\pi/4}=\frac{\pi}{8}$$

CHANGE TO POLAR COORDS. **Example 4**

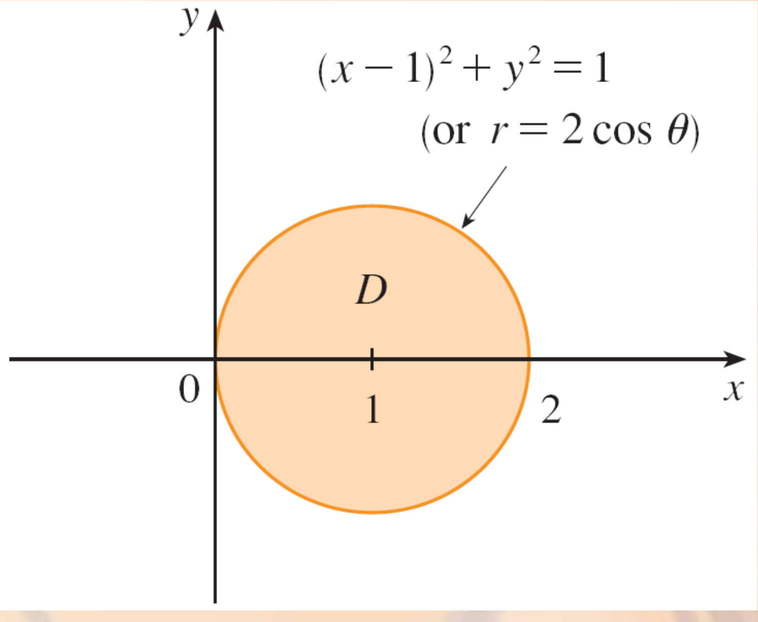
Find the volume of the solid that lies:

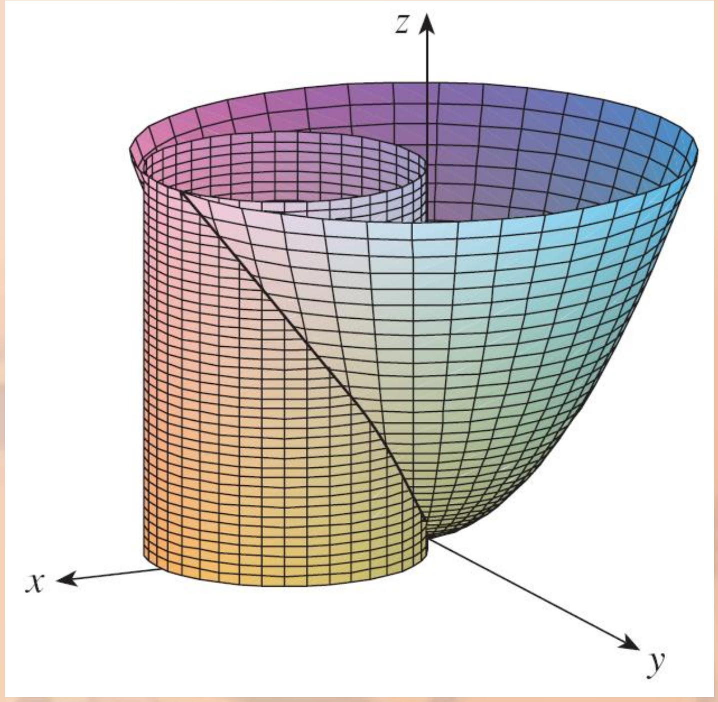
- Under the paraboloid $z = x^2 + y^2$
- Above the xy-plane
- Inside the cylinder $x^2 + y^2 = 2x$

CHANGE TO POLAR COORDS. Example 4

The solid lies above the disk *D* whose boundary circle has equation $x^2 + y^2 = 2x$.

- After completing the square, that is: $(x - 1)^2 + y^2 = 1$





CHANGE TO POLAR COORDS. **Example 4**

In polar coordinates, we have:

$$x^2 + y^2 = r^2 \text{ and } x = r \cos \theta$$

So, the boundary circle becomes:

$$r^2 = 2r \cos \theta$$

or $$r = 2 \cos \theta$$

Thus, the disk D is given by:

$D =$

$\{(r, \theta) \mid -\pi/2 \leq \theta \leq \pi/2 , 0 \leq r \leq 2 \cos \theta\}$

So, by Formula 3, we have:

$$
\begin{aligned}
V &= \iint_D (x^2 + y^2)\, dA \\
&= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 r\, dr\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4} \right]_0^{2\cos\theta} d\theta \\
&= 4 \int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta
\end{aligned}
$$

CHANGE TO POLAR COORDS. **Example 4**

$$= 8\int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$= 8\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= 2\int_0^{\pi/2} [1 + 2\cos 2\theta + \tfrac{1}{2}(1+\cos 4\theta)] \, d\theta$$

$$= 2[\tfrac{3}{2}\theta + \sin 2\theta + \tfrac{1}{8}\sin 4\theta]_0^{\pi/2}$$

$$= 2\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}$$