Chapter 3

Digital Signals and Systems

Introduction

- In this chapter we discuss the basic concepts and the mathematical tools that form the basis for the representation and analysis of *discrete-time signals* and *systems*.

- The chapter continues to study some properties of *linear systems* such as *time invariance*, *causality*, *impulse response*, *difference equations*, and *digital convolution*.

Digital Signals

- A discrete-time signal $x[n]$ is a sequence of numbers of an integer variable n, where $n \in \mathbb{Z}$.

$x(0)$: zero-th sample amplitude at the sample number $n = 0$,
$x(1)$: first sample amplitude at the sample number $n = 1$,
$x(2)$: second sample amplitude at the sample number $n = 2$,
$x(3)$: third sample amplitude at the sample number $n = 3$, and so on.

- The *duration* or *length* L_x of $x[n]$ is the number of samples from the first nonzero sample $x[n_1]$to the last nonzero sample $x[n_2]$, $L_x = n_2$-n_1+1.

- the *support* of the sequence is the range $n_1 \leq n \leq n_2$ or $[n_1, n_2]$.

- The symbol ↑ denotes the index n = 0; it is omitted when the table starts at n = 0.

Functional

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

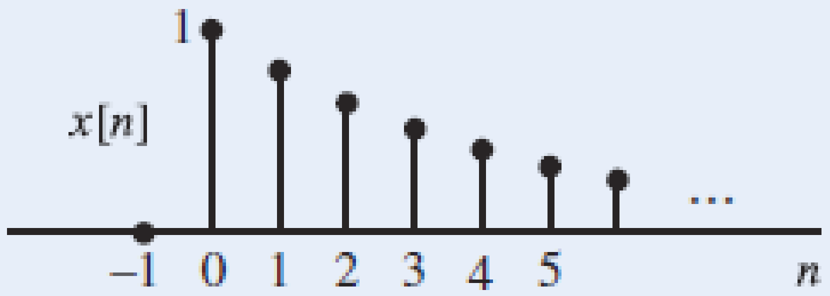
Tabular

n	...	-2	-1	0	1	2	3	...
$x[n]$	...	0	0	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	...

Sequence

$$x[n] = \{\ \ldots\ 0\ \underset{\uparrow}{1}\ \tfrac{1}{2}\ \tfrac{1}{4}\ \tfrac{1}{8}\ \ldots\ \}$$

Pictorial



Signal representation

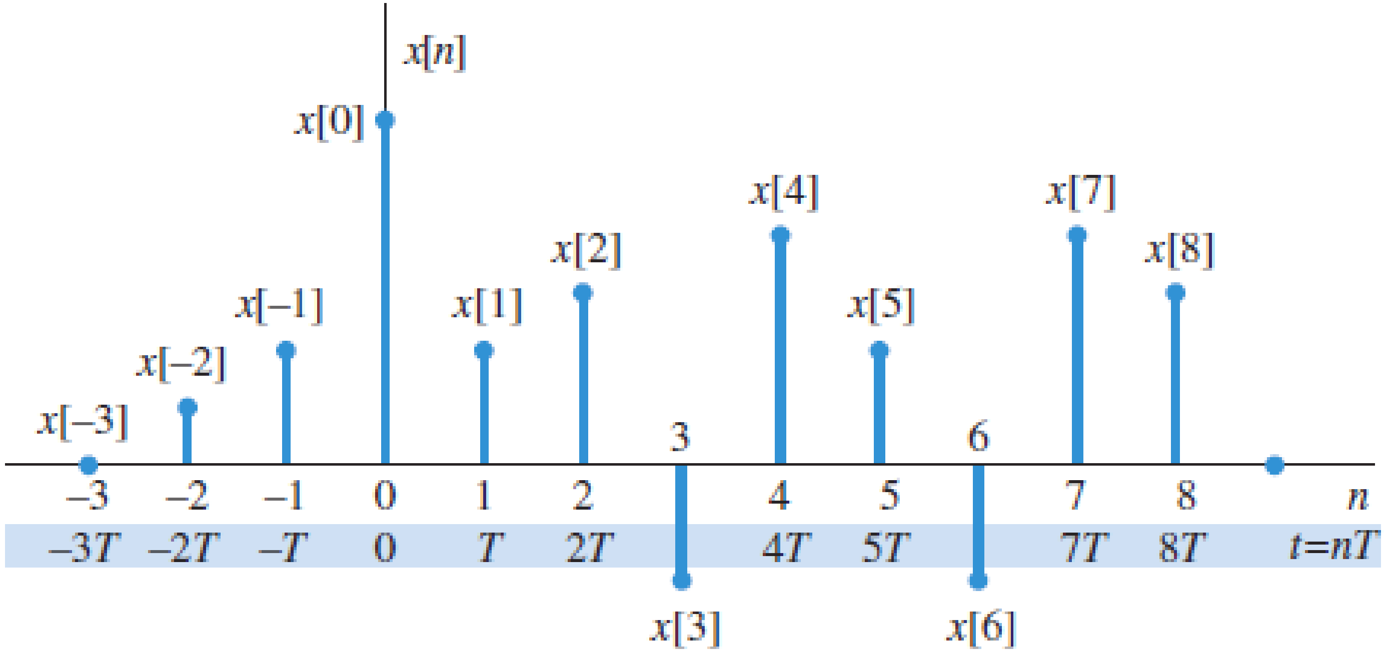
Digital Signals Contd.

Energy: The energy of a sequence x[n] is defined by the formula

$$\mathcal{E}_x \triangleq \sum_{n=-\infty}^{\infty} |x[n]|^2.$$

Power: the power of a sequence x[n] is defined as the average energy per sample.

$$\mathcal{P}_x \triangleq \lim_{L\to\infty}\left[\frac{1}{2L+1}\sum_{n=-L}^{L} |x[n]|^2\right].$$

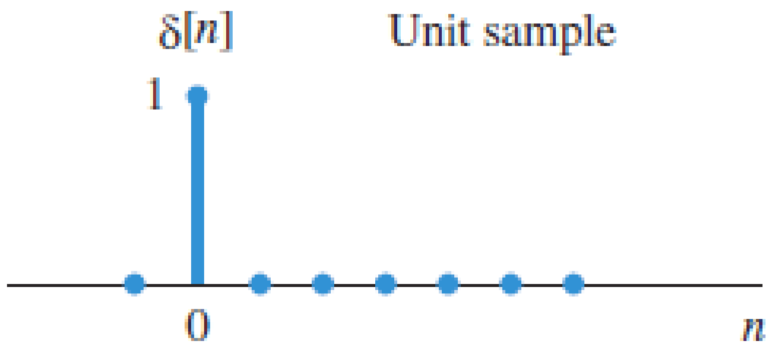


Representation of a sampled signal

Common Digital Sequences

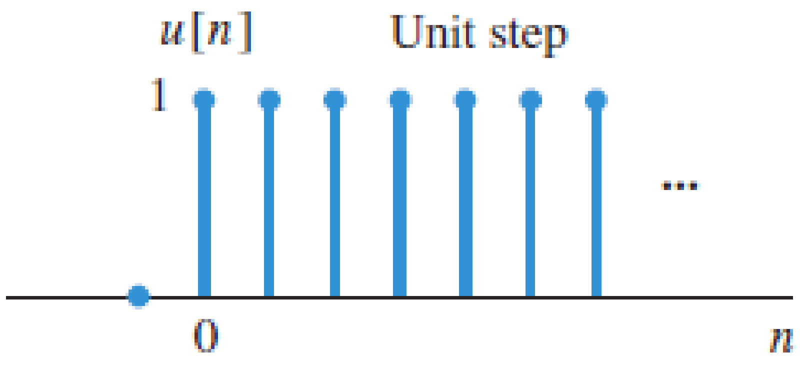
1. Unit-impulse sequence:

$$\delta[n] \triangleq \begin{cases} 1, & n = 0 \\ 0. & n \neq 0 \end{cases}$$



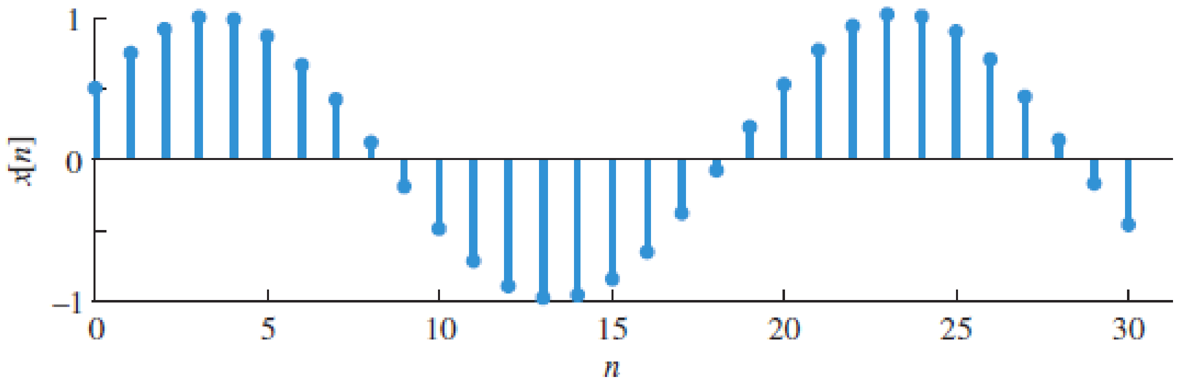
2. Unit-step sequence:

$$u[n] \triangleq \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



3. Sinusoidal sequence

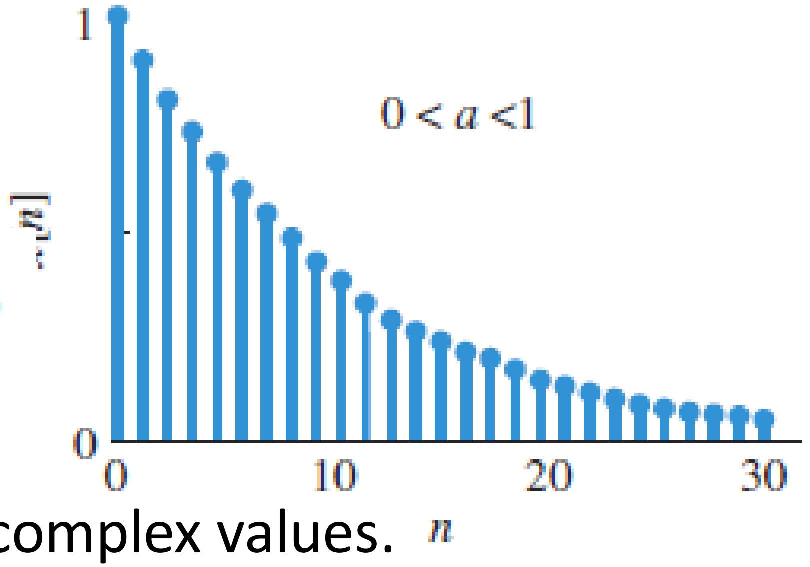
$$x[n] = A\cos(\omega_0 n + \phi), \quad -\infty < n < \infty$$

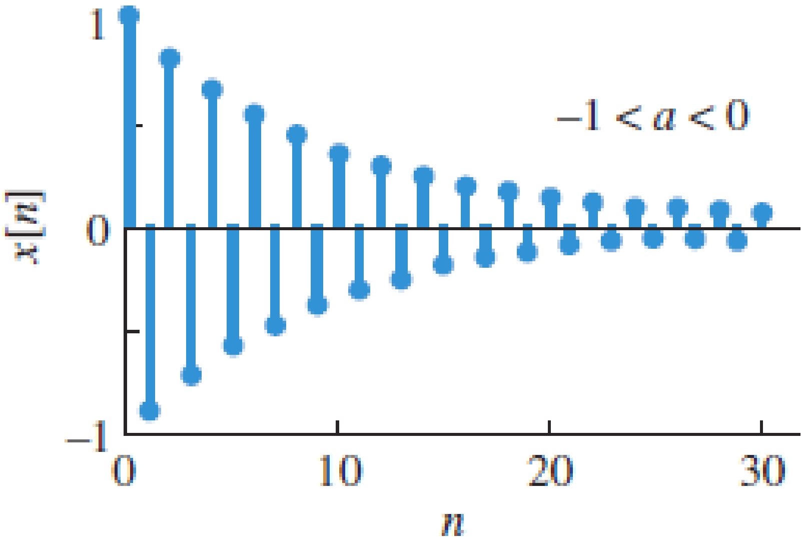


Common Digital Sequences

4. Exponential sequence

$$x[n] \triangleq Aa^n, \quad -\infty < n < \infty$$





where A and a can take real or complex values.

Signal generation and plotting in MATLAB

```
n=(-10:10); x=2*cos(2*pi*0.05*n);
```

```
[x,n] = delta(n1,n0,n2); % Unit impulse sequence
[x,n] = unitstep(n1,n0,n2); % Unit step sequence
[x,n] = unitpulse(n1,n2,n3,n4); % Unit pulse sequence
    x = persegen(xp,Np,Nps); % Periodic sequence
```

generate a unit sample and unit step sequences at n=n0 in the range n=(n1:n2).

creates a rectangular pulse of unit amplitude from n2 to n3 and zero elsewhere

generates Nps periods of a periodic signal with period Np.

```
stem(n,x,'fill'); ylabel('x[n]'); xlabel('n');
```

plot the sequence as a discrete-time signal

Operations on sequences

$y[n] = x_1[n] + x_2[n],$ (signal addition)

$y[n] = x_1[n] - x_2[n],$ (signal subtraction)

$y[n] = x_1[n] \cdot x_2[n],$ (signal multiplication)

$y[n] = x_1[n]/x_2[n],$ (signal division)

$y[n] = a \cdot x_2[n].$ (signal scaling)

- *Time-reversal or folding*: reflects the sequence $x[n]$ about the origin $n = 0$. $y[n] = x[-n],$

$x[-n] = x[n]$ *even or symmetric*

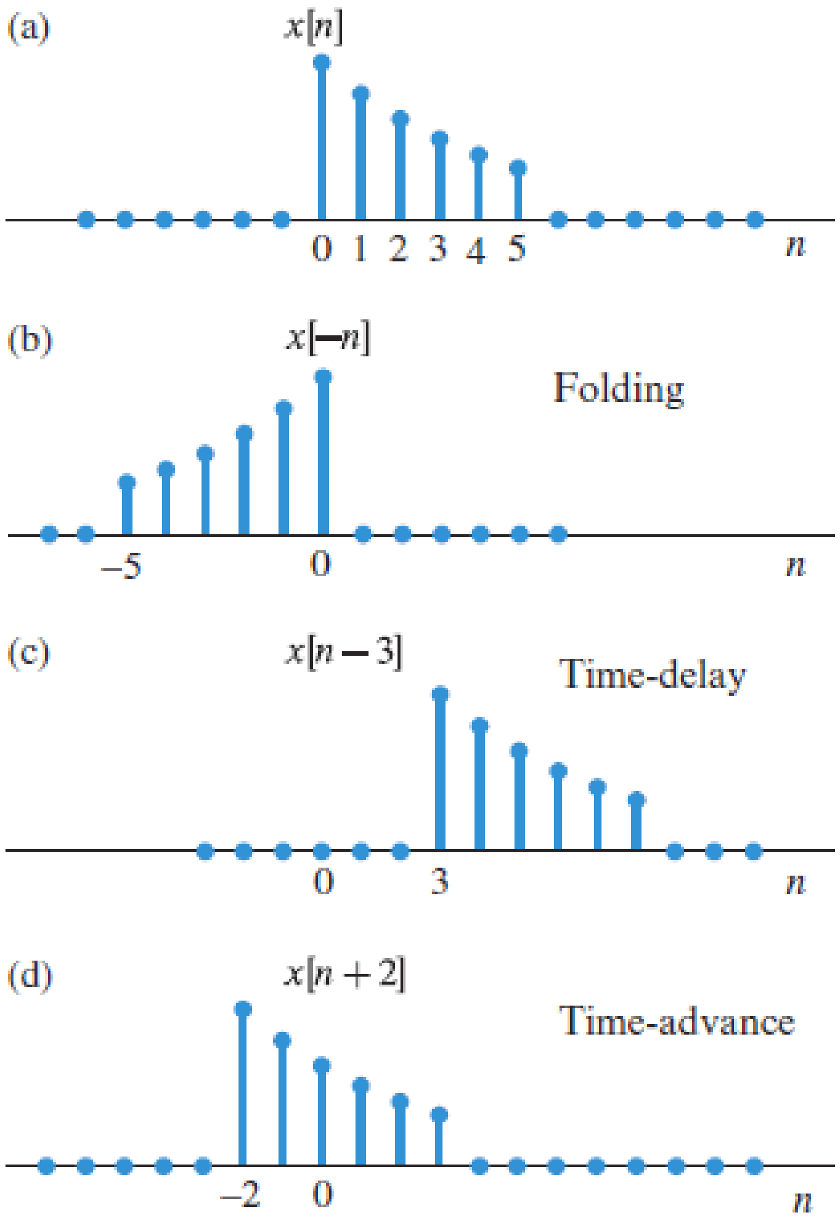
$x[-n] = -x[n]$ *odd or anti-symmetric*

- *Time-shifting:* the sequence $x[n]$ is shifted by n_0 samples

$$y[n] = x[n - n_0].$$

If $n_0 > 0$ *shift to the right (time-delay)*

If $n_0 < 0$ *shift to the left (time-advance.)*



Folding and time-shifting

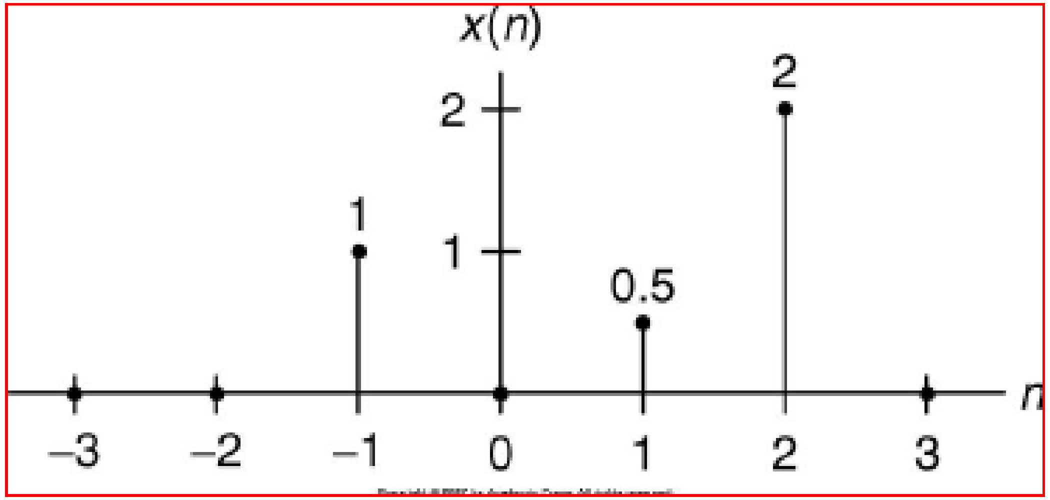
Example 1

Given the following,

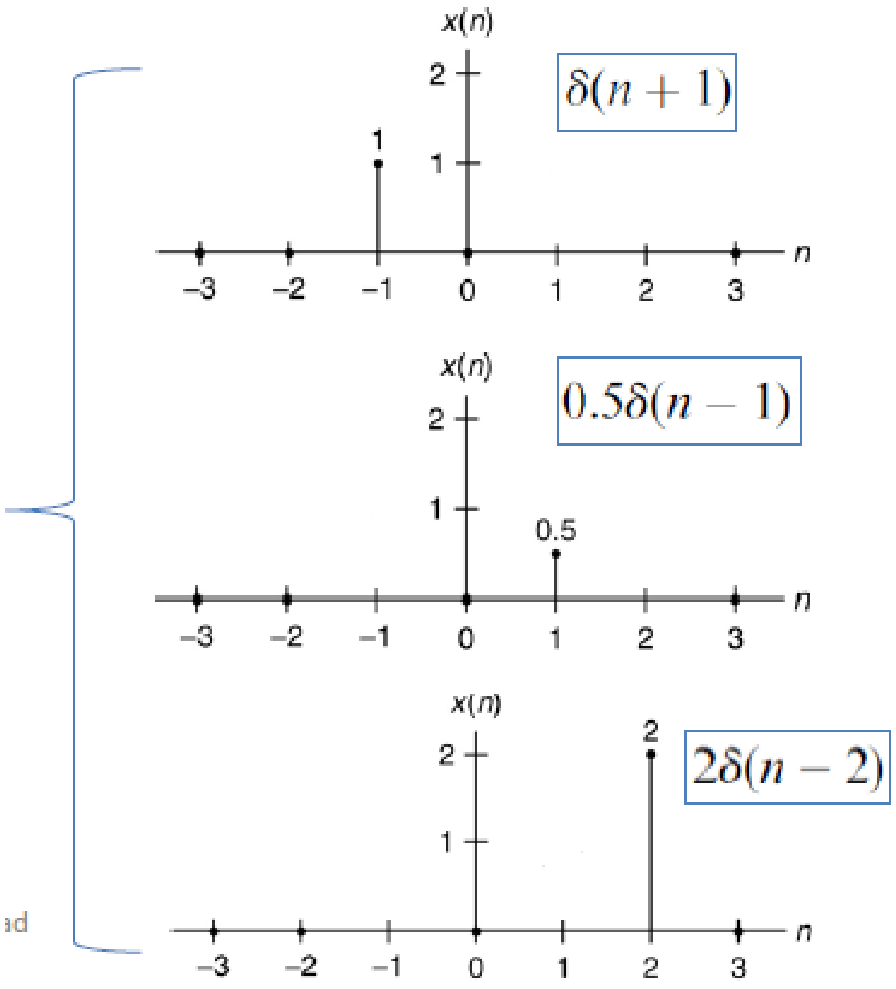
$$x(n) = \delta(n+1) + 0.5\delta(n-1) + 2\delta(n-2),$$

a. Sketch this sequence.

Solution:



The sum



Generation of Digital Signals

- To generate the digital sequence $x(n)$ from the analog signal $x(t)$:

uniformly sampling at the time interval of $\Delta t = T$

$$x(n) = x(t)\Big|_{t=nT} = x(nT)$$

Example 2

Convert analog signal $x(t)$ into digital signal $x(n)$, when sampling period is 125 microsecond, also plot sample values.

$$\boxed{x(t) = 10e^{-5000t}u(t)}$$

Solution:

$$t = nT = n \times 0.000125 = 0.000125n$$

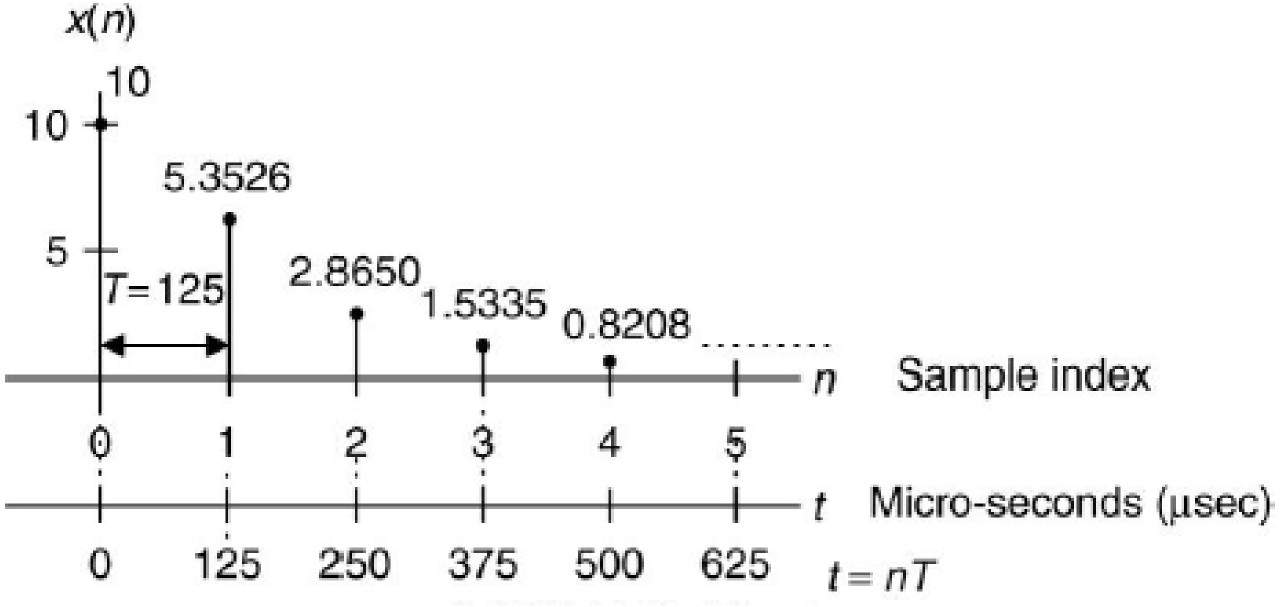
$$x(n) = x(nT) = 10e^{-5000\times 0.000125n}u(nT) = 10e^{-0.625n}u(n)$$

Example 2 (contd.)

The first five sample values:

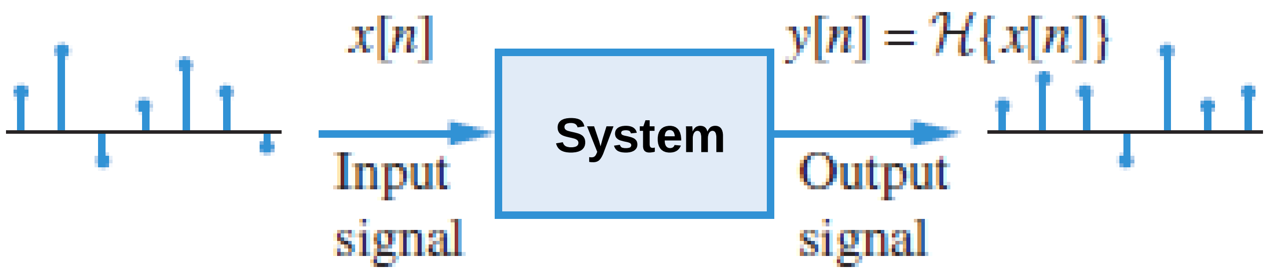
$$x(0) = 10e^{-0.625\times 0}u(0) = 10.0$$
$$x(1) = 10e^{-0.625\times 1}u(1) = 5.3526$$
$$x(2) = 10e^{-0.625\times 2}u(2) = 2.8650$$
$$x(3) = 10e^{-0.625\times 3}u(3) = 1.5335$$
$$x(4) = 10e^{-0.625\times 4}u(4) = 0.8208$$

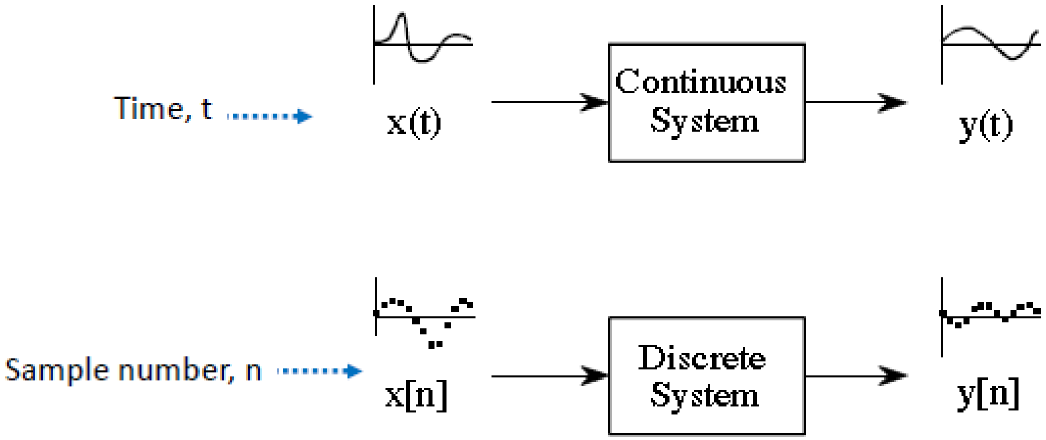
Plot of the digital sequence:



Digital Systems

A *Digital system* is a computational process or algorithm that transforms or maps a sequence $x[n]$, called the *input signal*, into another sequence $y[n]$, called the *output signal*.





Example 3

Determine the response of the following system to the input signal $x(n) = \begin{cases} |n|, & -3 \le n \le 3 \\ 0, & otherwise \end{cases}$

and the system's output $y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$

Solution:

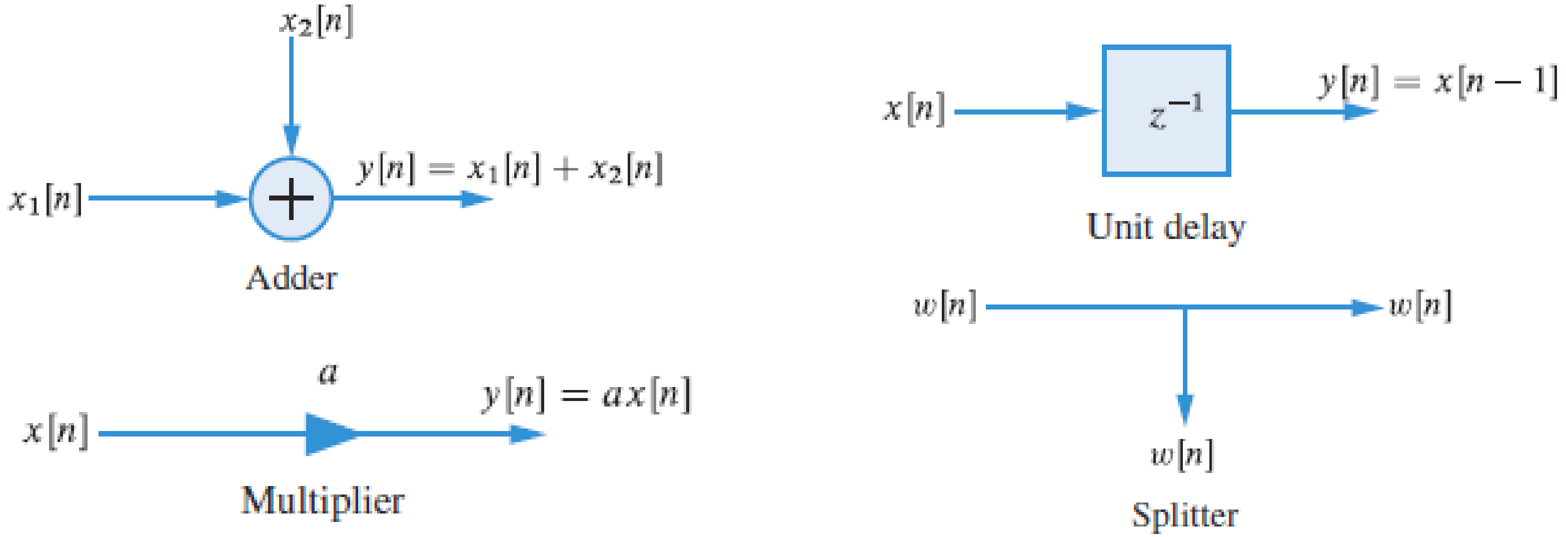
The output of this system is the mean value of the present, immediate past, and the immediate future samples.

For $n = 0$ ➡ $y(0) = \frac{1}{3}[x(-1) + x(0) + x(1)] = \frac{1}{3}[1 + 0 + 1] = \frac{2}{3}$

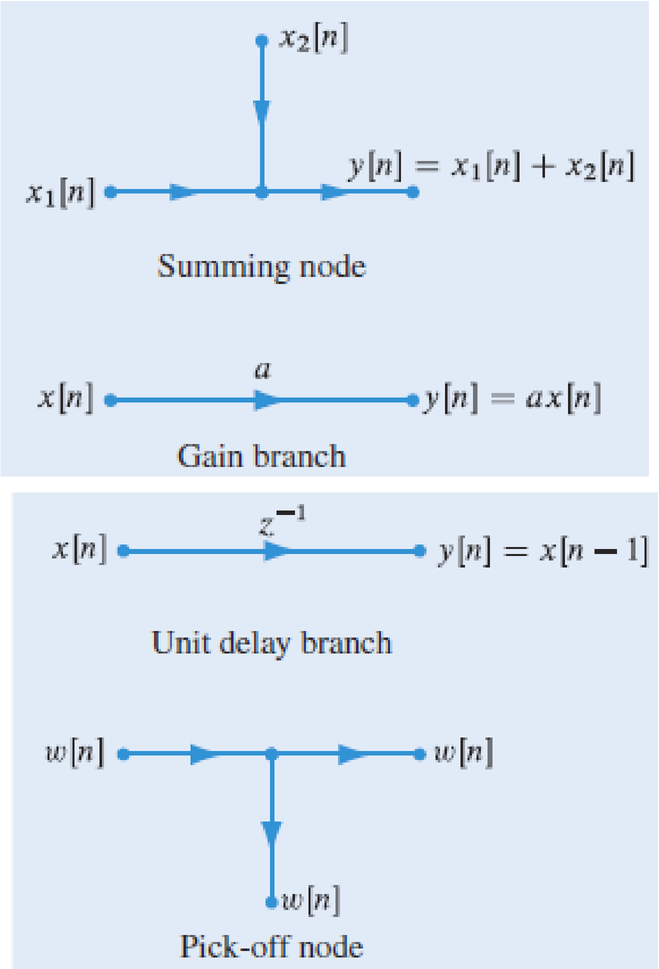
Repeating this computation for every value of n ➡ $y(n) = \left\{\ldots, 0, 1, \frac{5}{3}, 2, 1, \frac{2}{3}, 1, 2, \frac{5}{3}, 1, 0, \ldots\right\}$

Block Diagram of Discrete-Time Systems

- Operations required in the implementation of a discrete-time system can be depicted in one of two ways: a *block diagram* or a *signal flow graph*.



Block Diagram Elements



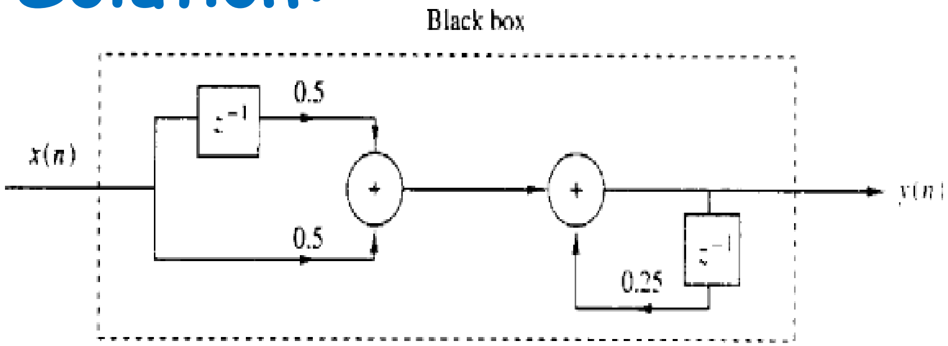
Signal Flow Graph Elements

Example 4

Sketch the block diagram representation of the discrete-time system described by the input-output relation.

$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{2}x(n) + \frac{1}{2}x(n-1)$$

Solution:



Classification of Discrete-Time Systems

Static and dynamic systems

$y(n) = x(n) + 3x(n-1)$ Dynamic (have memory) past or future samples of the input

$y(n) = n\,x(n) + b\,x^3(n)$ Static or memory-less no past or future samples of the input

Causality

A system is called *causal* if the present value of the output does not depend on future values of the input. Causality is required for systems that should operate in real-time.

Example: $y(n) = 0.5x(n) + 2.5x(n-2), \text{ for } n \geq 0$

- If the output of a system depends on future values of its input, the system is *non-causal*.

Example: $y(n) = 0.25x(n-1) + 0.5x(n+1) - 0.4y(n-1), \text{ for } n \geq 0$

Stability

A system is said to be *stable*, in the Bounded-Input Bounded-Output (BIBO) sense. Stability is a property that should be satisfied by every practical system.

$$|x[n]| \leq M_x < \infty \Rightarrow |y[n]| \leq M_y < \infty.$$

Example:

- The moving-average system is *stable*: $y[n] = \frac{1}{3}\{x[n] + x[n-1] + x[n-2]\} \leq 3M_x.$ for $|x[n]| \leq M_x$
- The accumulator system is *unstable* : $y[n] = \sum_{k=0}^{\infty} x[n-k]$ becomes unbounded as $n \rightarrow \infty$.

Example 9

Given a linear system given by: $y(n) = 0.25y(n-1) + x(n)$ for $n \geq 0$ and $y(-1) = 0$

Which is described by the unit-impulse response: $h(n) = (0.25)^n u(n)$

Determine whether the system is stable or not.

Solution: To determine whether a system is stable, we apply the following equation:

$$S = \sum_{k=-\infty}^{\infty} |h(k)| = \ldots + |h(-1)| + |h(0)| + |h(1)| + \ldots < \infty.$$

$$S = \sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=-\infty}^{\infty} \left|(0.25)^k u(k)\right|$$

Using definition of step function: $u(k) = 1$ for $k \geq 0$ $\Rightarrow$ $S = \sum_{k=0}^{\infty} (0.25)^k = 1 + 0.25 + 0.25^2 + \ldots.$

For a< 1, we know $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$ where $a = 0.25 < 1$

Therefore $S = 1 + 0.25 + 0.25^2 + \ldots = \frac{1}{1-0.25} = \frac{4}{3} < \infty$

The summation is finite, so the system is stable.

Linearity

A digital system is *linear* if and only if it satisfy the ***superposition principle,*** for every real or complex constant a_1, a_2 and every input signal $x_1[n]$and $x_2[n]$:

$$H[a_1 x_1[n] + a_2 x_2[n]] = a_1 H[x_1[n]] + a_2 H[x_2[n]]$$

Homogeneity & Additivity

Additivity

Homogeneity
(deals with amplitude)

linearity means that the output due to a sum of input signals equals the sum of outputs due to each signal alone

---

time invariance

system is called *time-invariant* or fixed if and only if for every input $x[n]$and every time shift n_0

$$y[n] = \mathcal{H}\{x[n]\} \Rightarrow y[n - n_0] = \mathcal{H}\{x[n - n_0]\},$$

a time shift in the input results in a corresponding time shift in the output

Time-invariance means that the system does not change over time.

Example 5 (Linear System)

Let a digital amplifier, $y(n) = 10x(n)$

$x(n) \rightarrow \boxed{\text{X 10}} \rightarrow y(n)$

If the inputs are: $x_1(n) = u(n)$ and $x_2(n) = \delta(n)$

Outputs will be: $y_1(n) = 10u(n)$ and $y_2(n) = 10\delta(n)$, respectively.

$x_1(n) = u(n) \rightarrow \boxed{\text{X 10}} \rightarrow y_1(n) = 10u(n)$

$x_2(n) = \delta(n) \rightarrow \boxed{\text{X 10}} \rightarrow y_2(n) = 10\delta(n)$

If we apply combined input to the system: $x(n) = 2x_1(n) + 4x_2(n) = 2u(n) + 4\delta(n)$

The output will be: $y(n) = 10x(n) = 10(2u(n) + 4\delta(n)) = \boxed{20u(n) + 40\delta(n)}$

Linear System

Individual outputs:	$2y_1(n) = 2\times 10x_1(n) = 20u(n)$ $4y_2(n) = 4\times 10x_2(n) = 40\delta(n)$

$y(n) = 2y_1(n) + 4y_2(n)$

$2x_1(n) = 2u(n) \rightarrow \boxed{\text{X 10}} \rightarrow 2y_1(n) = 20u(n)$

$4x_2(n) = 4\delta(n) \rightarrow \boxed{\text{X 10}} \rightarrow 4y_2(n) = 40\delta(n)$

Example 5 (Non-Linear System)

$x(n) \rightarrow$ System $\rightarrow y(n) = x^2(n)$

$x_1[n] \rightarrow$ System $\rightarrow y_1[n] = x_1{}^2\,[n]$

$x_2(n) \rightarrow$ System $\rightarrow y_2[n] = x_2{}^2\,[n]$

Linear ???

$a_1x_1[n] + a_2x_2[n] \rightarrow$ System $\rightarrow a_1y_1[n] + a_2y_2[n]$

If the input is: $x[n] = a_1x_1[n] + a_2x_2[n]$

Then the output is: $y[n] = (x[n])^2 = (a_1x_1[n] + a_2x_2[n]\,)^2 = a_1{}^2\,x_1{}^2[n] + a_2{}^2x_2{}^2[n] + 2a_2a_1x_1[n]x_2[n]$ **(1)**

Individual outputs:

$$a_1y_1[n] = a_1x_1{}^2\,[n]$$
$$a_2y_2[n] = a_2x_2{}^2\,[n]$$

$\Rightarrow$ $a_1y_1[n] + a_2y_2[n] = a_1x_1{}^2\,[n] + a_2x_2{}^2\,[n]$ **(2)**

(1) (2) $\Rightarrow$

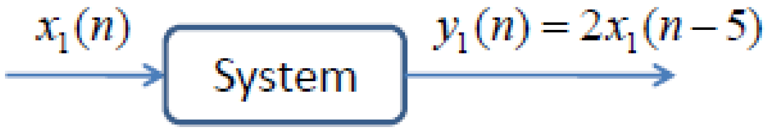
$$y[n] \neq a_1y_1[n] + a_2y_2[n]$$

Non Linear System

Example 6 (Time Invariant System)

Given the linear system $y[n] = 2x[n-5]$, find whether the system is time invariant or not.

Solution:



Let the shifted input be: $x_2[n] = x_1\,[n - n_0]$,

Therefore system output: $y_2[n] = 2x_2\,[n-5] = 2x_1\,[n - n_0 - 5]$,

Equal

Shifting $y_1[n] = 2x_1\,[n-5]$ by n_0 samples leads to $y_1[n - n_0] = 2x_1\,[n - 5 - n_0]$

Time Invariant

Example 6 (Not Time Invariant System)

Given the linear system $y(n) = n\,x(n)$, find whether the system is time invariant or not.

Solution:

$$x_1(n) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(n) = nx_1(n)$$

Let the shifted input be: $x_2(n) = x_1(n - n_0)$

Therefore system output due to the shifted input $\Rightarrow$ $y_2(n) = nx_2(n) = nx_1(n - n_0)$

NOT Equal

Shifting $y_1(n) = nx_1(n)$ by n_0 samples (replace n by $n - n_0$) leads to

$$y_1(n - n_0) = (n - n_0)x_1(n - n_0)$$

NOT Time Invariant

Difference Equation

A causal, linear, time-invariant system (LTI) can be described by a ***difference equation*** as follow:

$$\boxed{\underbrace{y(n) + a_1 y(n-1) + \cdots + a_N y(n-N)}_{\text{Outputs}} = \underbrace{b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M)}_{\text{Inputs}}}$$

After rearranging: $y(n) = -a_1 y(n-1) - \cdots - a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M)$

Finally:

$$\boxed{y(n) = -\sum_{i=1}^{N} a_i y(n-i) + \sum_{j=0}^{M} b_j x(n-j)}$$

Example 7 Identify non zero system coefficients of the following difference equations.

Solution:

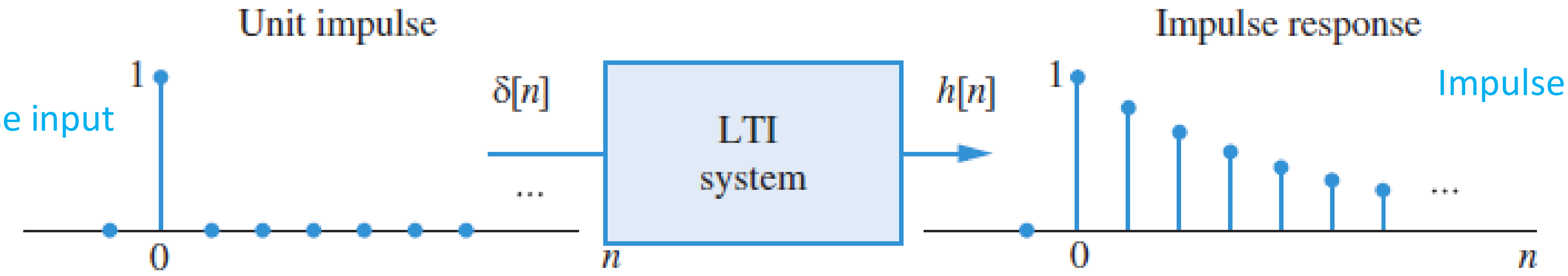
$y(n) = 0.25y(n-1) + x(n)$ ⟶ $b_0 = 1, \quad a_1 = -0.25$

$y(n) = x(n) + 0.5x(n-1)$ ⟶ $b_0 = 1, \quad b_1 = 0.5$

System Representation Using Impulse Response

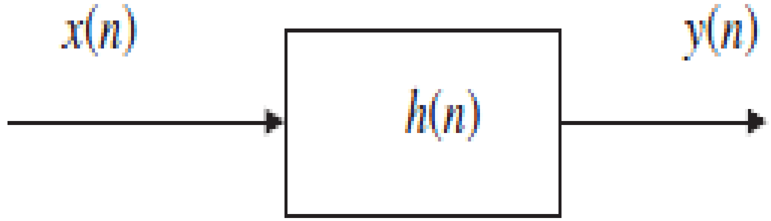
- The *linearity* and *time invariance* properties greatly simplify the analysis of linear systems (output of a decomposed, scaled and shifted input signal = sum of outputs of individual inputs)
- A *linear time-invariant* system (LTI system) can be completely described by its *impulse response* $h[n]$ due to the impulse input $\delta(n)$ with zero initial conditions.

Impulse input



Impulse Response

Any input



$y(n) = h(n) \otimes x(n)$

Convolution

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$y(n) = \cdots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \cdots$$

Example 8 (a)

Given the linear time-invariant system:

$$y(n) = 0.5x(n) + 0.25x(n-1) \text{ with an initial condition } x(-1) = 0,$$

a. Determine the unit-impulse response $h(n)$.

b. Draw the system block diagram.

c. Write the output using the obtained impulse response.

Solution:

a. let $x(n) = \delta(n)$, then $h(n) = y(n) = 0.5x(n) + 0.25x(n-1) = 0.5\delta(n) + 0.25\delta(n-1)$

Therefore, $$h(n) = \begin{cases} 0.5 & n = 0 \\ 0.25 & n = 1 \\ 0 & elsewhere \end{cases}$$

b. The block diagram of the LTI system

$$x(n) \longrightarrow \boxed{h(n) = 0.5\delta(n) + 0.25\delta(n-1)} \longrightarrow y(n)$$

c. The system output

$$y(n) = h(0)x(n) + h(1)x(n-1)$$

Example 8 (b)

Given the difference equation

$$y(n) = 0.25y(n-1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0,$$

a. Determine the unit-impulse response $h(n)$.

b. Draw the system block diagram.

c. Write the output using the obtained impulse response.

Solution:

a. Let $x(n) = \delta(n)$, then $h(n) = 0.25h(n-1) + \delta(n)$

To solve for $h(n)$, we evaluate

$$h(0) = 0.25h(-1) + \delta(0) = 0.25 \times 0 + 1 = 1$$

$$h(1) = 0.25h(0) + \delta(1) = 0.25 \times 1 + 0 = 0.25$$

$$h(2) = 0.25h(1) + \delta(2) = 0.25 \times 0.5 + 0 = 0.0625$$

....

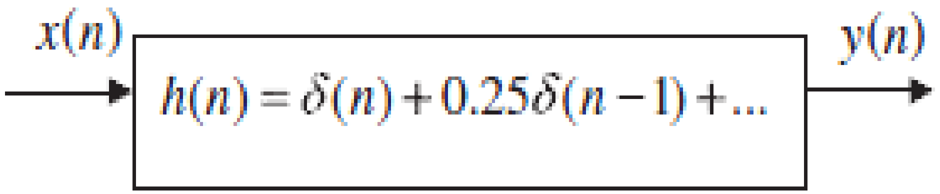
With the calculated results, we can predict the impulse response as

$$h(n) = (0.25)^n u(n) = \delta(n) + 0.25\delta(n-1) + 0.0625\delta(n-2) + \cdots$$

Infinite!

Example 8 (b) - contd.

b. The system block diagram



c. The output sequence

$$
\begin{aligned}
y(n) &= h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \cdots \\
&= x(n) + 0.25x(n-1) + 0.0625x(n-2) + \cdots
\end{aligned}
$$

Finite Impulse Response (FIR) system:

When the difference equation contains no previous outputs, i.e. '*a*' coefficients are zero. (See example 8 (a))

Infinite Impulse Response (IIR) system:

When the difference equation contains previous outputs, i.e. 'a' coefficients are not all zero. (See example 8 (b))

Digital Convolution

- A LTI system can be represented using a digital convolution

$$y(n) = h(n) * x(n) \Rightarrow y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$
$$= \cdots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \cdots$$

- The unit-impulse response $h(n)$ relates the system input and output.

- The sequences are interchangeable.

Commutative

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)}$$

$$x[n] * h[n] = h[n] * x[n]$$

- Convolution sum requires $h(n)$ to be reversed and shifted.

- If $h(n)$ is the given sequence, $h(-n)$ is the ***reversed sequence***.

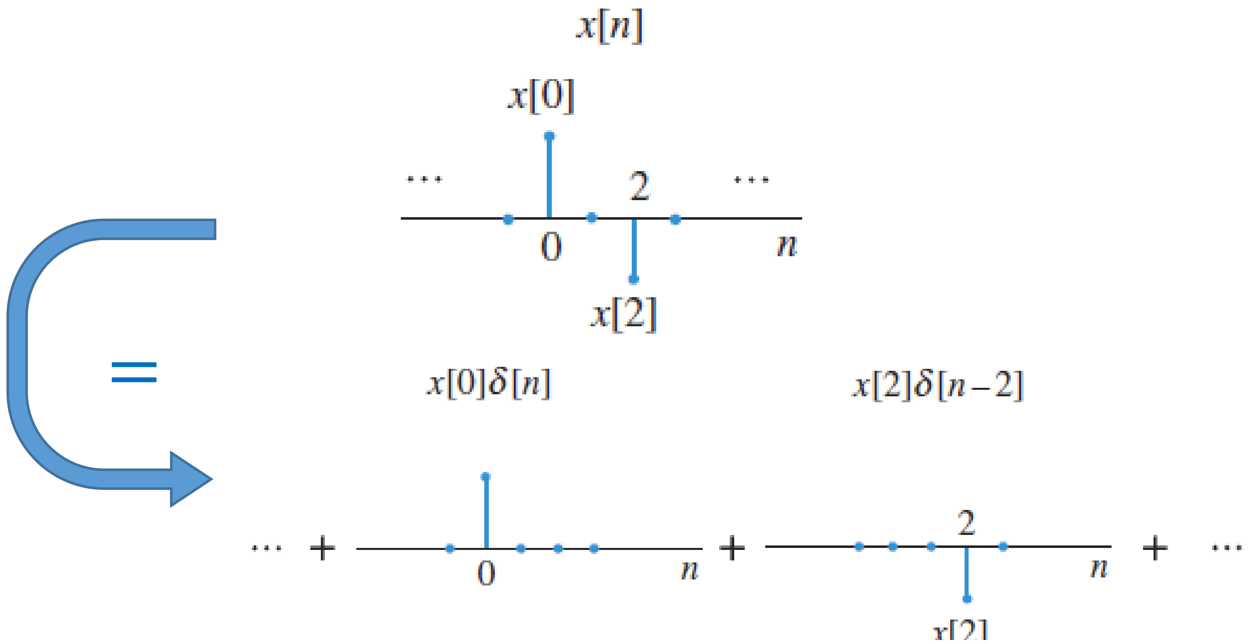
Signal decomposition into impulses

$$x_k[n] = \begin{cases} x[k], & n = k \\ 0, & n \neq k \end{cases}$$ ← Sample k of $x[n]$

$$x_k[n] = \delta[n-k];$$ ← Impulse at $\mathrm{n} = k$

$$\delta[n-k] = \begin{cases} 1, & n = k \\ 0. & n \neq k \end{cases}$$ ← Shifted Impulse

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]. \quad -\infty < n < \infty$$

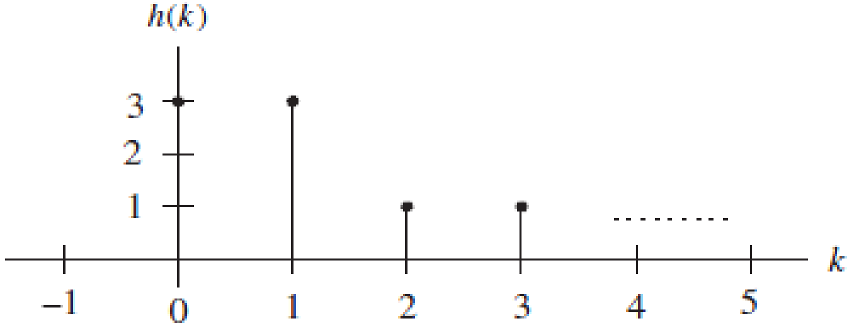


Reversed Sequence

Example: Given a sequence

$$h(k) = \begin{cases} 3, & k = 0,1 \\ 1, & k = 2,3 \\ 0 & \textit{elsewhere} \end{cases}$$

Sketch the sequence $h(k)$ and reversed sequence $h(-k)$.



Solution:

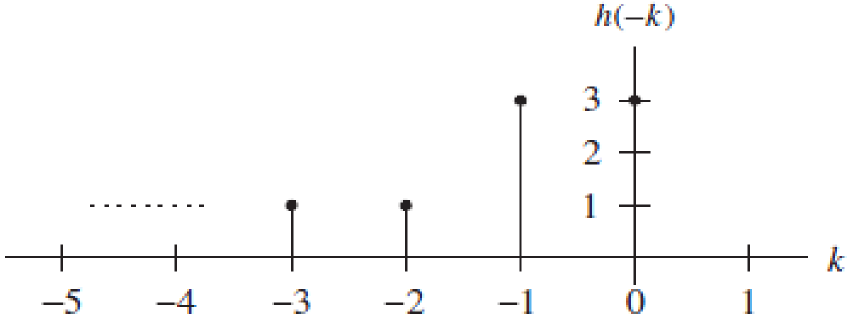
$k > 0,\ h(-k) = 0$

$k = 0,\ h(-0) = h(0) = 3$

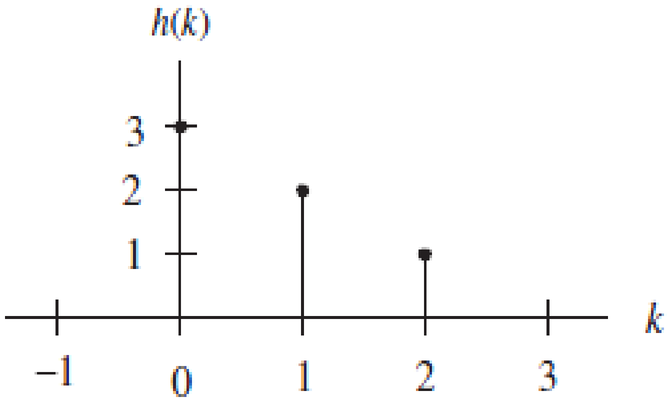
$k = -1,\ h(-k) = h(-(-1)) = h(1) = 3$

$k = -2,\ h(-k) = h(-(-2)) = h(2) = 1$

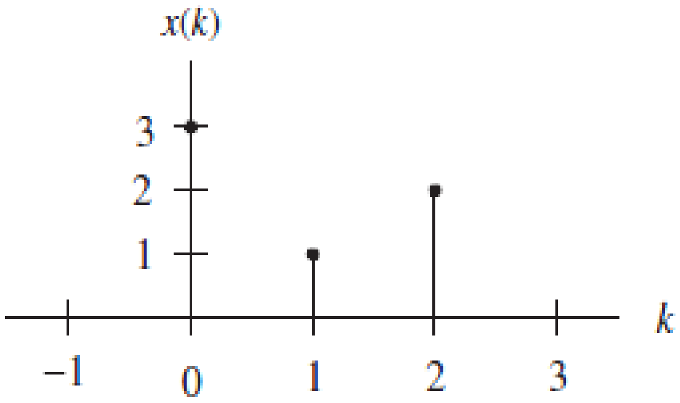
$k = -3,\ h(-k) = h(-(-3)) = h(3) = 1$



Convolution Using Table Method
Example 9



Length =3



Length =3

Solution:

Convolution sum using the table method.

k:	-2	-1	0	1	2	3	4	5	
$x(k)$:			3	1	2				
$h(-k)$:	1	2	3						$y(0) = 3 \times 3 = 9$
$h(1-k)$		1	2	3					$y(1) = 3 \times 2 + 1 \times 3 = 9$
$h(2-k)$			1	2	3				$y(2) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$
$h(3-k)$				1	2	3			$y(3) = 1 \times 1 + 2 \times 2 = 5$
$h(4-k)$					1	2	3		$y(4) = 2 \times 1 = 2$
$h(5-k)$						1	2	3	$y(5) = 0$ (no overlap)

Convolution length = 3 +3 −1 = 5

Convolution Using Table Method
Example 10

$$x(n) = \begin{cases} 1 & n = 0,1,2 \\ 0 & otherwise \end{cases} \text{ and } h(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1,2 \\ 0 & otherwise \end{cases}$$

Length = 3

Length = 2

Solution:

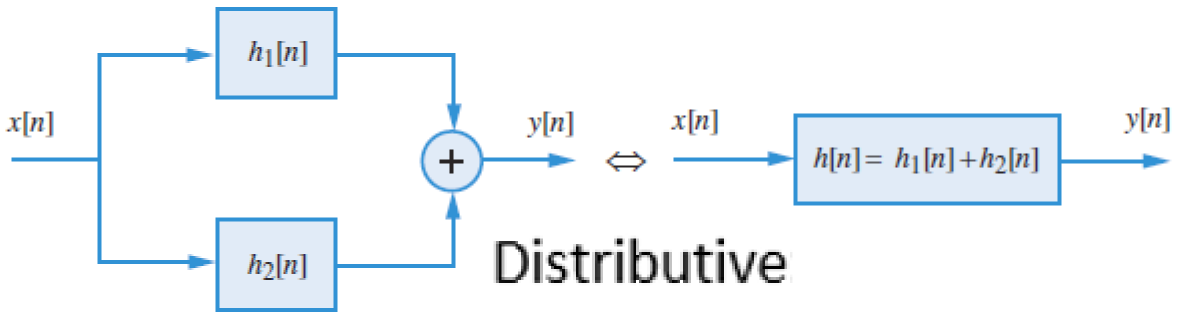
k:	-2	-1	0	1	2	3	4	5	$\ldots$	
$x(k)$:			**1**	**1**	**1**				$\ldots$	
$h(-k)$:	1	1	0							$y(0) = 0$ (no overlap)
$h(1-k)$		1	1	0						$y(1) = 1 \times 1 = 1$
$h(2-k)$			1	1	0					$y(2) = 1 \times 1 + 1 \times 1 = 2$
$h(3-k)$				1	1	0				$y(3) = 1 \times 1 + 1 \times 1 = 2$
$h(4-k)$					1	1	0			$y(4) = 1 \times 1 = 1$
$h(n-k)$						1	1	0		$y(n) = 0,\ n \geq 5$ (no overlap) Stop

Convolution length = 3 + 2 –1 = 4

Convolution Properties

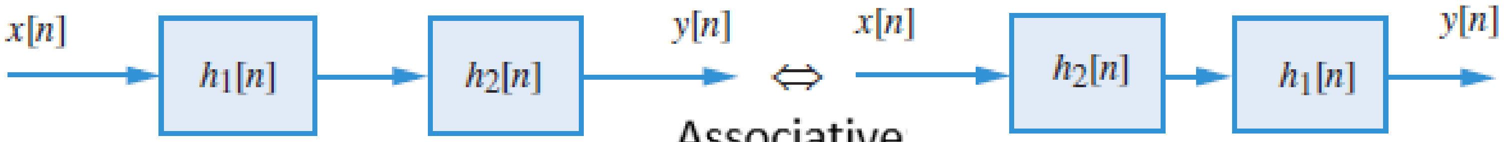
- $\delta[n]$ is the *identity element* of the convolution operation.
- Commutative: $a[n] \otimes b[n] = b[n] \otimes a[n]$
- Associative: $(a[n] \otimes b[n]) \otimes c[n] = a[n] \otimes (b[n] \otimes c[n])$
- Distributive: $a[n] \otimes (b[n] + c[n]) = a[n] \otimes b[n] + a[n] \otimes c[n]$

Commutative



Distributive

Associative



Associative