Chapter 2: Application of the definite integral

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Area Between Curves.

In this section we are going to look at finding the area between two curves.

QUESTION:

How we can determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$

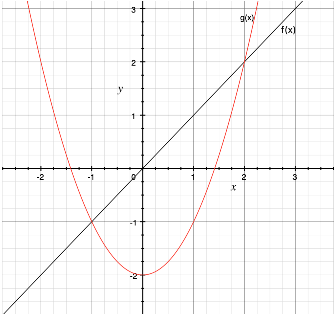
Theorem 1.1 (Area Between Curves)

Let $f(x)$ *and* $g(x)$ *be continuous functions defind on* $[a, b]$ *where* $f(x) \geq g(x)$ *for all* x *in* $[a, b]$.
The area of the region *bounded by the curves* $y = f(x), \quad y = g(x)$ *and the lines* $x = a$ *and* $x = b$ *is*

$$\int_a^b \Big[f(x) - g(x) \Big] \, dx$$

Area Between Curves.

$$A = \int_a^b \text{(upper function) - (lower function)}\ dx, \qquad a \leq x \leq b$$



$$A = \int_a^b f(x) - g(x)\ dx$$

The steps to calculate the area between curves

1. Find the intersection points between the curves.
2. determine the upper function and the lower function.
3. Calculate the integral:

$$A = \int_a^b \text{(upper function) - (lower function) } dx$$

Which give us the required area.

Area Between Two Curves (Example)

Example 1.1

Find the area enclosed between the graphs $y = x$ and $y = x^2 - 2$.

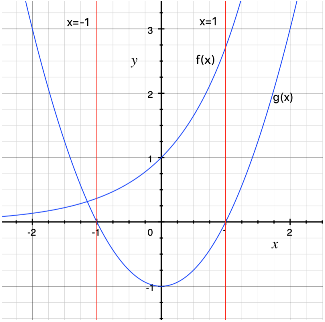
Solution

1. Points of intersection between $y = x^2 - 2$ and $y = x$ is:
   $x^2 - 2 = x \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0$
   $\Rightarrow x = -1$ and $x = 2$
2. Note that upper function is $y = x$ and lower function is $y = x^2 - 2$
   Note that $y = x^2 - 2$ is a parabola opens upward with vertex $(0, -2)$, and $y = x$ is a straight line passing through the origin.
3. $A = \displaystyle\int_{-1}^{2} x - (x^2 - 2)\, dx = \int_{-1}^{2} x - x^2 + 2\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_{-1}^{2} = \frac{27}{6}$

Area Between Two Curves (Example)

Example 1.2

Find the area enclosed between the graphs $y = e^x, y = x^2 - 1, x = -1,$ and $x = 1$



Area Between Two Curves (Example)

Solution

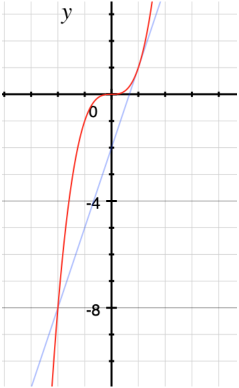
Note that upper function is $y = e^x$ and lower function is $y = x^2 - 1$

$$A = \int_{-1}^{1} e^x - (x^2 - 1)\ dx = \int_{-1}^{1} e^x - x^2 + 1\ dx = \left[e^x - \frac{1}{3}x^3 + x \right]_{-1}^{1}$$

$$= e - \tfrac{1}{e} + \tfrac{4}{3}$$

Example 1.3

Compute the area oh the region bounded by the curves
$y = x^3$ and $y = 3x - 2$



Area Between Two Curves (Example)

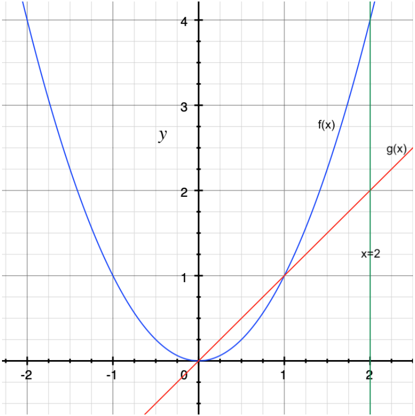
1. Points of intersection between $y = x^3$ and $y = 3x - 2$
   $x^3 - 3x + 2 = 0 \Rightarrow (x-1)(x^2 + x - 2) = 0$
   $\Rightarrow x = -2$ and $x = 1$
2. Note that upper function is $y = x^3$ and lower function is $y = 3x - 2$
3. $A = \int_{-2}^{1} x^3 - (3x - 2)\ dx = \int_{-2}^{1} x^3 - 3x + 2\ dx$

$$= \left[\frac{x^4}{4} - \frac{3}{2}x^2 + 2x \right]_{-2}^{1}$$

$$= \frac{3}{4} + 6 = \frac{27}{4}$$

Example 1.4

Find the area enclosed between the graphs $f(x) = x^2$ and $g(x) = x$ between $x = 0$, and $x = 2$.



Area Between Two Curves (Example)

1. we see that the two graphs intersect at $(0,0)$ and $(1,1)$.
2. In the interval $[0,1]$, we have $g(x) = x \geq f(x) = x^2$, and in the interval $[1,2]$, we have $f(x) = x^2 \geq g(x) = x$
3. Therefore the desired area is:

$$A = \int_0^1 (x - x^2)\ dx + \int_1^2 (x^2 - x)\ dx = \left[\frac{x^2}{2} - \frac{x^3}{0}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{6} + \frac{5}{6} = 1$$

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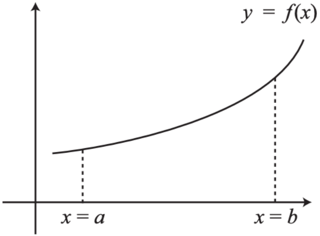
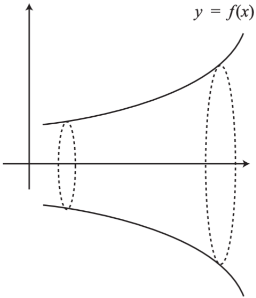
Suppose we have a curve $y = f(x)$



Imagine that the part of the curve between the ordinates $x = a$ and $x = b$ is rotated about the x-axis through 360 degree.

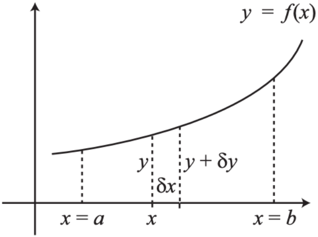
Volume Of A Solid Revolution (The Disk Method)

Now if we take a cross-section of the solid, parallel to the y-axis, this cross-section will be a circle.



But rather than take a cross-section, let us take a thin disc of thickness δx, with the face of the disc nearest the y-axis at a distance x from the origin.

Volume Of A Solid Revolution (The Disk Method)



The radius of this circular face will then be y. The radius of the other circular face will be $y + \delta y$, where δy is the change in y caused by the small positive increase in $x, \delta x$.

Volume Of A Solid Revolution (The Disk Method)

The volume δV of the disc is then given by the volume of a cylinder, $\pi r^2 h$, so that

$$\delta V = \pi r^2 \delta x$$

So the volume V of the solid of revolution is given by

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \delta V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x = \pi \int_a^b [f(x)]^2 \, dx$$

Examples

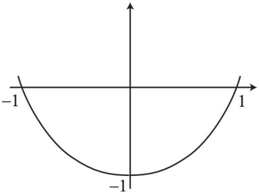
Example 2.1

The curve $y = x^2 - 1$ is rotated about the x-axis through 360 degree. Find the volume of the solid generated when the area contained between the curve and the x-axis is rotated about the x-axis by 360 degree.

$$V = \pi \int_a^b [f(x)]^2 \, dx = \pi \int_{-1}^{1} [x^2 - 1]^2 \, dx$$

$$= \pi \int_{-1}^{1} (x^4 - 2x^2 + 1) \, dx$$

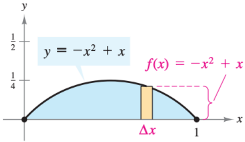
$$= \left[\frac{x^5}{5} - \frac{2x^3}{3} + x \right]_{-1}^{1} = \frac{16\pi}{15}$$

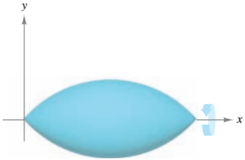


The graph of $y = x^2 - 1$

Example 2.2

Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = -x^2 + x$ and the x-axis about the x-axis.





Using the Disk Method, you can find the volume of the solid of revolution.

$$V = \pi \int_0^1 [f(x)]^2 \, dx = \pi \int_0^1 [(-x^2 + x)^2 \, dx = \pi \int_0^1 (x^4 - 2x^3 + x^2) \, dx$$

$$= \pi \left[\frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3} \right]_0^1 = \frac{\pi}{30}$$

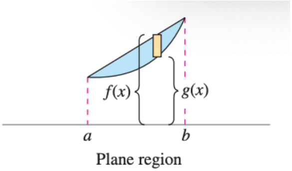
Volume Of A Solid Revolution (The Washer Method)

The Washer Method

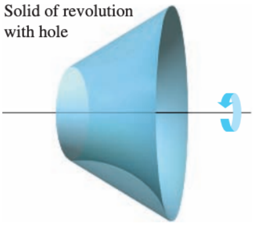
Let f and g be continuous and nonnegative on the closed interval $[a, b]$, if $f(x) \geq g(x)$ for all x in the interval, then the volume of the solid formed by revolving the region bounded by the graphs of $f(x)$ and $g(x)$ $(a \leq x \leq b)$, about the x-axis is:

$$V = \pi \int_a^b \left\{ [f(x)]^2 - [g(x)]^2 \right\} dx$$

$f(x)$ is the **outer radius** and $g(x)$ is the **inner radius**.



(a)

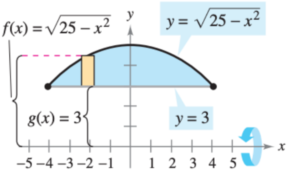


(b)

Example 2.3

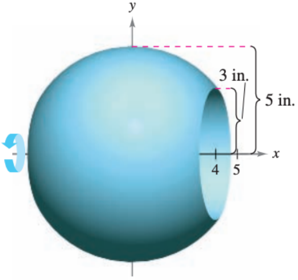
Find the volume of the solid formed by revolving the region bounded by the graphs of $f(x) = \sqrt{25 - x^2}$ and $g(x) = 3$

We sketch the bounding region and the solid of revolution:



Plane region

(a)



Solid of revolution

(b)

Examples

First find the points of intersection of f and g, by setting $f(x)$ equal to $g(x)$ and solving for x.
$\sqrt{25-x^2} = 3 \Rightarrow 25 - x^2 = 9 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$
Using $f(x)$ as the outer radius and $g(x)$ as the inner radius, you can find the volume of the solid as shown.

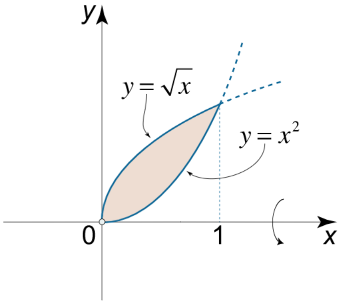
$$V = \pi \int_a^b \left\{ [f(x)]^2 - [g(x)]^2 \right\} dx = \pi \int_{-4}^{4} (\sqrt{25-x^2})^2 - (3)^2 \, dx$$

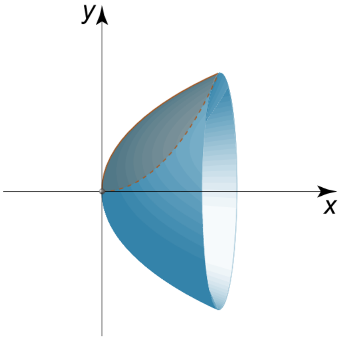
$$= \pi \int_{-4}^{4} (16 - x^2) \, dx = \pi \left[16x - \frac{x^3}{3} \right]_{-4}^{4} = \frac{256\pi}{3}$$

Example 2.4

Calculate the volume of the solid obtained by rotating the region bounded by the parabola $y = x^2$ and the square root function $y = \sqrt{x}$ around the $x-$axis

We sketch the bounding region and the solid of revolution:





Both curves intersect at the points $x = 0$ and $x = 1$. Using the washer method, we have

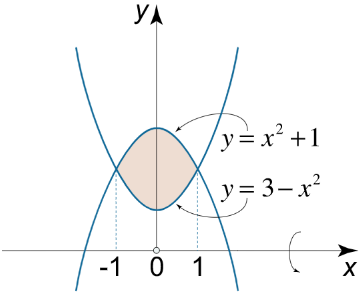
$$V = \pi \int_a^b \left\{ [f(x)]^2 - [g(x)]^2 \right\} dx = \pi \int_0^1 (\sqrt{x})^2 - (x^2)^2 \; dx$$

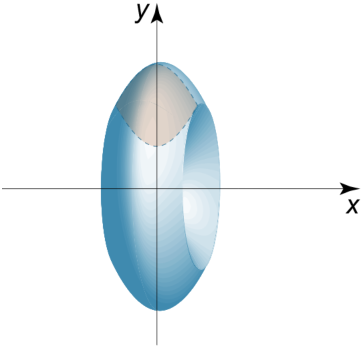
$$= \pi \int_0^1 (x - x^4) \; dx = \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \pi \left[\frac{1}{2} - \frac{1}{5} \right] = \frac{3\pi}{10}$$

Example 2.5

Find the volume of the solid obtained by rotating the region bounded by two parabolas $y = x^2 + 1$ and $y = 3 - x^2$ about the $x-$axis.

We sketch the bounding region and the solid of revolution:





First we determine the boundaries a and b:
$x^2+1=3-x^2 \Rightarrow 2x^2=2 \Rightarrow x^2=1 \Rightarrow x=\pm 1$
Hence the limits of integration are $a=1$ and $b=-1$.
Using the washer method, we find the volume of the solid:

$$V=\pi\int_a^b \left\{ [f(x)]^2-[g(x)]^2 \right\} dx$$

$$=\pi\int_{-1}^{1} \left[(3-x^2)^2-(x^2+1)^2\right]\ dx=\pi\int_{-1}^{1}\left(8-8x^2\right)\ dx$$

$$=8\pi\int_{-1}^{1}\left(1-x^2\right)\ dx=8\pi\left[x-\frac{x^3}{3}\right]_{-1}^{1}=\frac{32\pi}{3}$$

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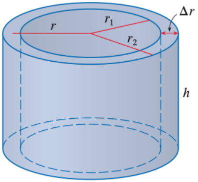
The method of cylindrical shells

the cylindrical shell with inner radius r_1, outer radius r_2 , and height h. Its volume V is calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:

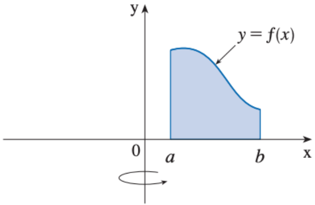
$$V = V_2 - V_1 = \pi r_2^2 h - \pi r_1^2 h$$

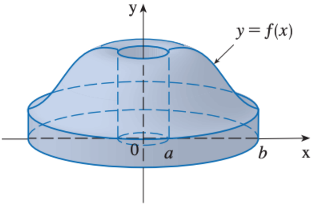
$$= \pi(r_2^2 - r_1^2)h = \pi(r_2 - r_1)(r_2 + r_1)h$$

$$= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1) \Rightarrow V = 2\pi r h \Delta r$$



let be the solid obtained by rotating about the -axis the region bounded by $y = f(x)$,
where $f(x) \geq 0$, $y = 0$, $x = a$ and $x = b$, where $b > a \geq 0$.

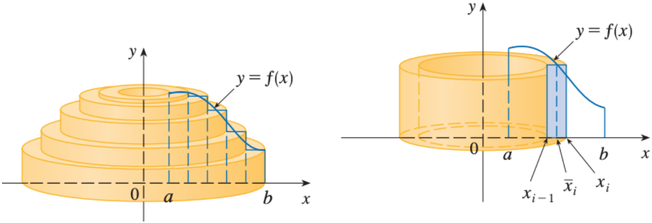




Volume Of A Solid Revolution (Cylindrical shells method)

We divide the interval into n subintervals $[x_{i-1}, x_{i+1}]$ of equal width and let $\overline{x_i}$ be the midpoint of the i th subinterval. If the rectangle with base $[x_{i-1}, x_i]$ and height $f(\overline{x}_i)$ is rotated about the $y-$ axis then the result is a cylindrical shell with average radius $\overline{x}_i$ height $f(\overline{x}_i)$ and thickness Δx so its volume is:

$$V_i = (2\pi)\overline{x}_i[f(\overline{x}_i)]\Delta x$$



Volume Of A Solid Revolution (Cylindrical shells method)

An approximation to the volume of is given by the sum of the volumes of these shells:

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi \overline{x}_i [f(\overline{x}_i)] \Delta x$$

This approximation appears to become better as $n \to \infty$ But, from the definition of an integral, we know that

$$\lim_{n \to \infty} \sum_{i=1}^{n} 2\pi \overline{x}_i [f(\overline{x}_i)] \Delta x = \int_a^b 2\pi x f(x)\, dx$$

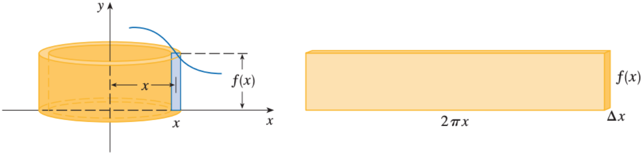
Volume Of A Solid Revolution (Cylindrical shells method)

The volume of the solid, obtained by rotating about the $y-$axis the region under the curve $y = f(x)$ from a to b, is

$$V = \int_a^b 2\pi x f(x) dx \qquad\qquad where \;\; 0 \le a < b$$

The best way to remember the last Formula is to think of a typical shell, cut and flattened as in Figure with radius x, circumference $2\pi x$, height $f(x)$ and thickness Δx or dx :

$$\int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \underbrace{[f(x)]}_{\text{height}} \, dx$$



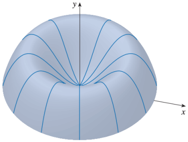
Example 3.1

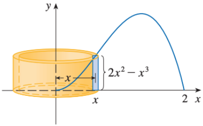
Find the volume of the solid obtained by rotating about the $y-$axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$

by the shell method, the volume is

$$V = \int_0^2 (2\pi x)(2x^2 - x^3)\ dx = 2\pi \int_0^2 (2x^3 - x^4)\ dx = 2\pi \left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2$$

$$= 2\pi(8 - \tfrac{32}{5}) = \tfrac{16}{5}\pi$$





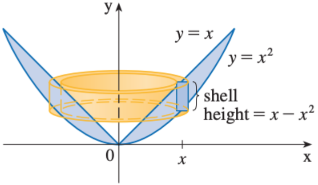
Example 3.2

Find the volume of the solid obtained by rotating about the $y-$axis the region between $y = x$ and $y = x^2$.

$$V = \int_0^1 (2\pi x)(x - x^2)\ dx$$

$$= 2\pi \int_0^1 (x^2 - x^3)\ dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}$$



Example 3.3

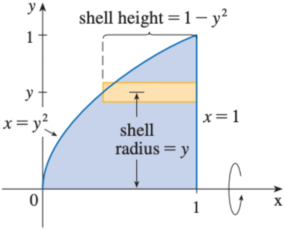
Use cylindrical shells to find the volume of the solid obtained by rotating about the $x-$axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

For rotation about the $x-$axis we see that a typical shell has radius y, circumference $2\pi y$, and height $1 - y^2$. So the volume is

$$V = \int_0^1 (2\pi y)(1 - y^2)\ dy$$

$$= 2\pi \int_0^1 (y - y^3)\ dy$$

$$= 2\pi \left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = \frac{\pi}{2}$$



Example 3.4

Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

the region and a cylindrical shell formed by rotation about the line $x = 2$. It has radius $2 - x$, circumference $2\pi(2 - x)$, and height $x - x^2$.

$$V = \int_0^1 2\pi(2 - x)(x - x^2)\ dx = 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\ dx$$
$$= 2\pi\left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = \frac{\pi}{2}$$