Chapter 5: Multiple Integrals

Table of contents

Table of contents

Double integral

Review of the Definite Integral

First let's recall the basic facts concerning definite integrals of functions of a single variable. If $f(x)$ is defined for $a \leqslant x \leqslant b$, we start by dividing the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \dfrac{b-a}{n}$ and we choose sample points c_i in these subintervals. Then we form the Riemann sum

$$R_n = \sum_{i=1}^{n} f(c_i)\Delta x$$

and take the limit of such sums as $n \to \infty$ to obtain the definite integral of f from a to b:

$$\int_a^b f(x)dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x,$$

In the special case where $f(x) \geqslant 0$, the Riemann sum can be interpreted as the sum of the areas of the approximating rectangles in Figure 1, and $\int_a^b f(x)\, dx$ represents the area under the curve $y = f(x)$ from a to b.

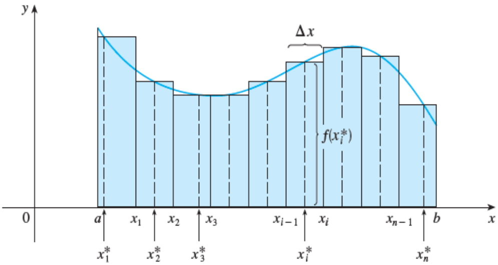


Figure 1: Reimann sum

Volumes and Double Integrals

In a similar manner we consider a function f of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | a \leqslant x \leqslant b, c \leqslant y <\leqslant\}$$

and we first suppose that $f(x, y) \geqslant 0$. The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f , that is,

$$S = \{(x, y, z) \in \mathbb{R}^3 | 0 \leqslant z \leqslant f(x, y), (x, y) \in R\}$$

(See Figure 2.) Our goal is to find the volume of S.

Double integral

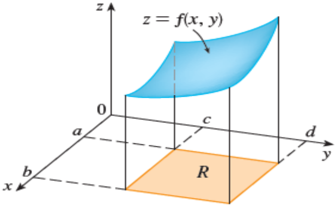


Figure 2: $z = f(x, y)$

Double integral

The first step is to divide the rectangle R into subrectangles. We accomplish this by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta_x = \dfrac{b-a}{m}$ and dividing $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta_y = \dfrac{d-c}{n}$. By drawing lines parallel to the coordinate axes through the endpoints of these subintervals, as in Figure 3, we form the subrectangles

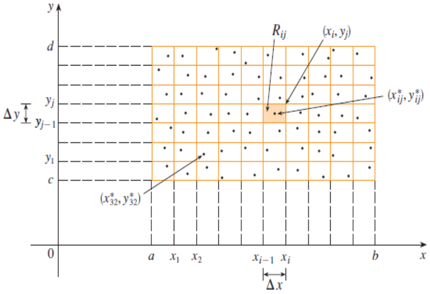


Figure 3: Dividing R into subrectangles

Double integral

If we choose a sample point (x_{ij}^*, y_{ij}^*) in each R_{ij}, then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box (or "column") with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$ as shown in Figure 4.

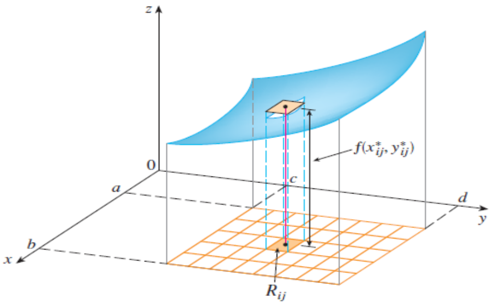


Figure 4:

Double integral

The volume of this box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*)\Delta A$$

If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of S:

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

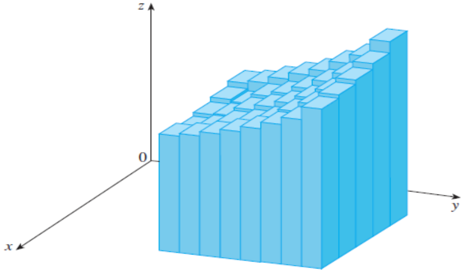


Figure 5:

Double integral

Definition 1.1

The double integral of f over the rectangle R is

$$\iint\limits_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

if this limit exists.

Definition 1.2

If $f(x,y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$\iint\limits_R f(x,y)\,dA$$

Definition 1.3 (Iterated Integral)

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a,b] \times [c,d]$, then

$$\iint\limits_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx$$

or

$$\iint\limits_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_c^d \left[\int_a^b f(x,y)\, dx \right] dy$$

Double integral

Example 1.1

Evaluate the iterated integrals.

1. $\displaystyle\int_0^3 \int_1^2 x^2 y \; dy \; dx$
2. $\displaystyle\int_1^2 \int_0^3 x^2 y \; dx \; dy$

Solution

1. $$\int_0^3 \int_1^2 x^2 y \; dy \; dx = \int_0^3 \left[\int_1^2 x^2 y \; dy \right] \; dx = \int_0^3 \left[x^2 \frac{y^2}{2} \right]_1^2 \; dx = \int_0^3 \left(\frac{3x^2}{2} \right) \; dx = \left[\frac{x^3}{2} \right]_0^3 = \frac{27}{2}$$

2. $$\int_1^2 \int_0^3 x^2 y \; dx \; dy = \int_1^2 \left[\int_0^3 x^2 y \; dx \right] \; dy = \int_1^2 \left[\frac{x^3}{3} y \right]_0^3 \; dy = \int_1^2 (9y) \; dy = \left[\frac{9y^2}{2} \right]_1^2 = \frac{27}{2}$$

Double integral

Theorem 1.1 (Fubini'S Theorem)

If f is continuous on the rectangle $R = \{(x, y) | a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, *then*

$$\iint_R f(x, y)\ dA = \int_a^b \int_c^d f(x, y)\ dy\ dx = \int_c^d \int_a^b f(x, y)\ dx\ dy$$

More generally, this is true if we assume that f *is bounded on* R, f *is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.*

Example 1.2

Evaluate the double integral $\iint_R (x-3y^2)dA$, where $R = \{(x,y) | 0 \leqslant x \leqslant 2, 1 \leqslant y \leqslant 2\}$.

Solution (Method 1)

Fubini's Theorem gives

$$\iint_R (x-3y^2)dA = \int_0^2 \int_1^2 (x-3y^2)dydx = \int_0^2 \left[xy - y^3\right]_1^2 \, dx$$
$$= \int_0^2 (x-7)dx = \left[\frac{x^2}{2} - 7x\right]_0^2 = -12$$

Solution (Method 2)

Fubini's Theorem gives

$$\iint_R (x-3y^2)dA = \int_1^2 \int_0^2 (x-3y^2)dxdy = \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_0^2 dy$$
$$= \int_1^2 (2-6y^2)dy = \left[2y - 2y^3\right]_1^2 = -12$$

Double integral

Example 1.3

Evaluate $\iint_R y\sin(xy)dA$, where $R = [1,2] \times [0,\pi]$.

Solution

$$\begin{aligned}\iint_R y\sin(xy)dA &= \int_0^{\pi}\int_1^2 y\sin(xy)\ dx\ dy \\ &= \int_0^{\pi} [-\cos(xy)]_1^2\ dy \\ &= \int_0^{\pi} (-\cos(2y)+\cos(y))\ dy \\ &= \left[-\frac{\sin(2y)}{2}+\sin(y)\right]_0^{\pi} = 0\end{aligned}$$

Example 1.4

Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

Solution

We first observe that S is the solid that lies under the surface $z = 16 - x^2 - 2y^2$ and above the square $R = [0, 2] \times [0, 2]$.

$$\begin{aligned}
\iint_R (16 - x^2 - 2y^2) dA &= \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dx\, dy \\
&= \int_0^2 \left[16x - \frac{x^3}{3} - 2xy^2\right]_0^2\, dy \\
&= \int_0^2 \left(\frac{88}{3} - 4y^2\right)\, dy \\
&= \left[\frac{88}{3}y - \frac{4y^3}{3}\right]_0^2\, dy = 48
\end{aligned}$$

Double integral

In the special case where $f(x,y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a particularly simple form. To be specific, suppose that $f(x,y) = g(x)h(y)$ and $R = [a,b] \times [c,d]$. Then,

Theorem 1.2

$$\iint_R g(x)h(y)dA = \int_a^b g(x)dx \int_c^d h(y)dy$$

Example 1.5

Evaluate the integral $\iint_R \sin x \cos y dA$, where $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$

Solution

$$\iint_R \sin x \cos y dA = \int_0^{\frac{\pi}{2}} \sin x dx \int_0^{\frac{\pi}{2}} \cos y dy$$
$$= [-\cos x]_0^{\frac{\pi}{2}} [\sin y]_0^{\frac{\pi}{2}} = 1$$

Table of contents

Double integrals over general regions

For single integrals, the region over which we integrate is always an interval. But for double integrals, we want to be able to integrate a function f not just over rectangles but also over regions D of more general shape, such as the one illustrated in Figure 6.

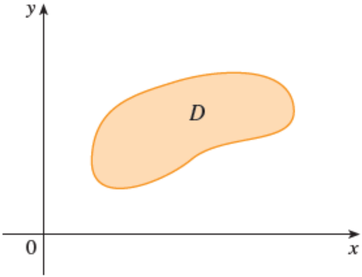


Figure 6:

Double integrals over general regions

Definition 2.1 (Type I)

A plane region D is said to be of **type I** *if it lies between the graphs of two continuous functions of x, that is,*

$$D = \{(x, y) | a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 7.

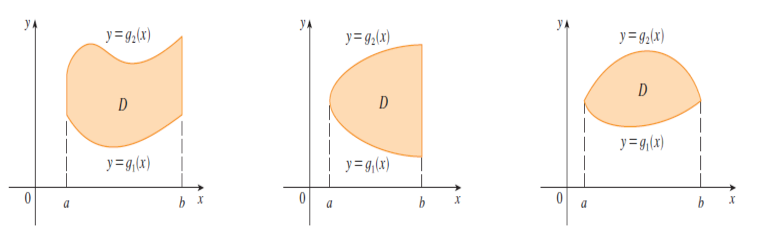


Figure 7:

Double integrals over general regions

Definition 2.2 (Type II)

A plane region D is said to be of **type II** *if it lies between the graphs of two continuous functions of y, that is,*

$$D = \{(x, y) | c \leqslant y \leqslant d, h_1(y) \leqslant x \leqslant h_2(y)\}$$

where h_1 and h_2 are continuous on $[c, g]$. Some examples of type II regions are shown in Figure 8.

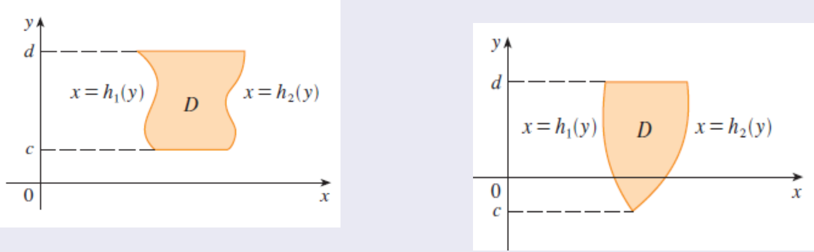


Figure 8:

Theorem 2.1

1. *If f is continuous on a type I region D such that*

$$D = \{(x,y) | a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x)\}$$

then

$$\iint_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\ dx$$

2. *If f is continuous on a type II region D such that*

$$D = \{(x,y) | c \leqslant y \leqslant d, h_1(y) \leqslant x \leqslant h_2(y)\}$$

then

$$\iint_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\ dy$$

Example 2.1

Evaluate $\iint_D (x+2y)dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

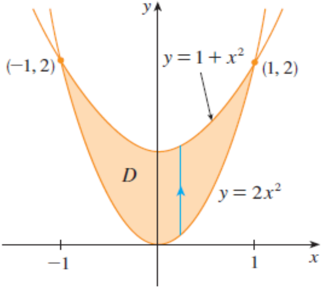


Figure 9:

Solution

The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = 1$ or $x = -1$. We note that the region D, sketched in Figure 9, is a type I region but not a type II region and we can write
$D = \{(x, y) | -1 \leqslant x \leqslant 1, 2x^2 \leqslant y \leqslant 1 + x^2\}$

$$\begin{aligned}
\iint_D (x + 2y)dA &= \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\ dy\ dx = \int_{-1}^{1} \left[xy + y^2\right]_{2x^2}^{1+x^2}\ dx \\
&= \int_{-1}^{1} \left(x(1 + x^2) + (1 + x^2)^2 - 2x^3 - 4x^4\right)\ dx \\
&= \int_{-1}^{1} \left(-3x^4 - x^3 + 2x^2 + x + 1\right)\ dx \\
&= \left[-\frac{3x^5}{5} - \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^{1} = \frac{32}{15}
\end{aligned}$$

Example 2.2

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

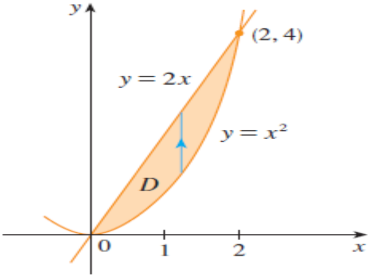


Figure 10: D as a type I region

Double integrals over general regions

Solution

From Figure 10 we see that D is a type I region and
$D = \{(x, y)|0 \leqslant x \leqslant 2, x^2 \leqslant y \leqslant 2x\}$
Therefore the volume under $z = x^2 + y^2$ and above D is

$$\iint_D (x^2+y^2)dA = \int_0^2 \int_{x^2}^{2x} (x^2+y^2)\ dy\ dx = \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_{x^2}^{2x}\ dx$$

$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3}\right)\ dx$$

$$= \left[-\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6}\right]_0^2 = \frac{216}{35}$$

Double integrals over general regions

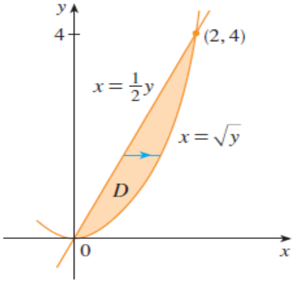


Figure 11: D as a type II region

Solution

From Figure 11 we see that D is a type II region and
$D = \{(x, y)|0 \leqslant y \leqslant 4, \frac{y}{2} \leqslant y \leqslant \sqrt{y}\}$
Therefore the volume under $z = x^2 + y^2$ and above D is

$$\iint_D (x^2 + y^2) dA = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (x^2 + y^2)\ dx\ dy = \int_0^4 \left[\frac{x^3}{3} + y^2 x \right]_{\frac{y}{2}}^{\sqrt{y}}\ dy$$

$$= \int_0^4 \left(\frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} - \frac{y^3}{24} - \frac{y^3}{2} \right)\ dy$$

$$= \int_0^4 \left[\frac{2y^{\frac{5}{2}}}{15} + \frac{2y^{\frac{7}{2}}}{7} - \frac{13y^4}{96} \right)\ dy = \frac{216}{35}$$

Example 2.3

Evaluate $\iint_D xy dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

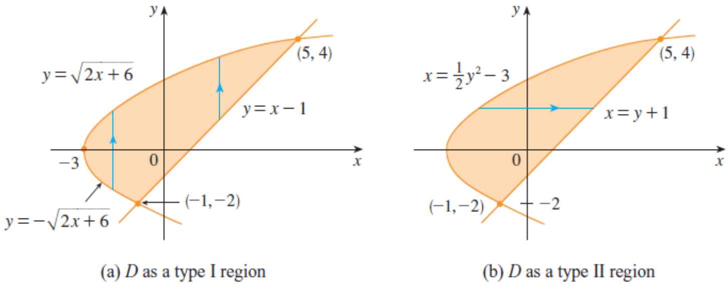


Figure 12:

Solution

The region D is shown in Figure 12. Again D is both type I and type II, but the description of D as a type I region is more complicated because the lower boundary consists of two parts. Therefore we prefer to express D as a type II region:

$$D = \{(x, y) | -2 \leqslant y \leqslant 4, \frac{y^2}{2} - 3 \leqslant x \leqslant y + 1\}$$

$$\iint_D xydA = \int_{-2}^{4} \int_{\frac{y^2}{2}-3}^{y+1} xy \, dxdy = \int_{-2}^{4} \left[\frac{x^2}{2} y\right]_{\frac{y^2}{2}-3}^{y+1} dy$$

$$= \frac{1}{2} \int_{-2}^{4} \left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy$$

$$= \frac{1}{2} \left[-\frac{y^6}{24} + y^4 + \frac{2y^3}{3} - 4y^2\right]_{-2}^{4} dy = 36$$

Exercise 2.1

Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2) dy dx$.

Double integrals over general regions

Properties 2.1

1. $\displaystyle\iint_D cf(x,y)\;dA = c\iint_D f(x,y)\;dA$, *for every real number* c.
2. $\displaystyle\iint_D [f(x,y)+g(x,y)]\;dA = \iint_D f(x,y) + \iint_D g(x,y)\;dA$
3. *If* $f(x,y)\geq 0$ *throughout* D, *then* $\displaystyle\iint_D f(x,y)\;dA \geq 0$
4. *If* $f(x,y)\geq g(x,y)$ *throughout* D, *then* $\displaystyle\iint_D f(x,y)\;dA \geq \iint_D g(x,y)\;dA$

5. If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 13), then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y) + \iint_{D_2} f(x,y)\,dA$$

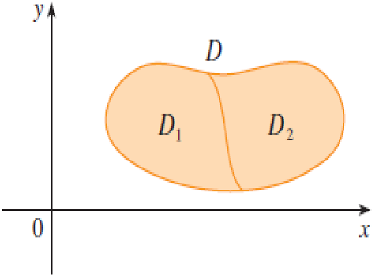
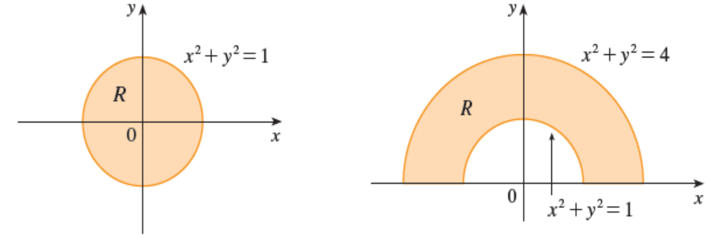


Figure 13: $D = D_1 \cup D_2$

Table of contents

Double integrals in polar coordinates

Suppose that we want to evaluate a double integral $\iint_R f(x,y)\,dA$, where R is one of the regions shown in Figure 14. In either case the description of R in terms of rectangular coordinates is rather complicated, but R is easily described using polar coordinates.



(a) $R=\{(r,\theta)\mid 0\leq r\leq 1, 0\leq\theta\leq 2\pi\}$ (b) $R=\{(r,\theta)\mid 1\leq r\leq 2, 0\leq\theta\leq \pi\}$

Figure 14: Region R

Double integrals in polar coordinates

Recall from Figure 15 that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$r^2 = x^2 + y^2, \quad x = r\cos(\theta) \quad y = r\sin(\theta)$$

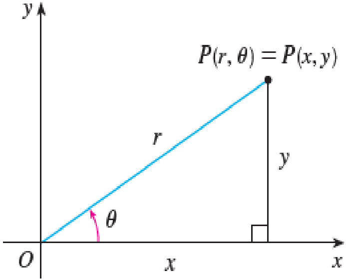


Figure 15: Polar coordinates

The regions in Figure 14 are special cases of a **polar rectangle**

$$R = \{(r, \theta) | a \leqslant r \leqslant b, \alpha \leqslant \theta \leqslant \beta\}$$

Theorem 3.1 (Change to Polar Coodinates in a Double Integral)

If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b, \alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r dr d\theta$$

Example 3.1

Evaluate $\iint_R (3x + 4y^2)dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Double integrals in polar coordinates

Solution

The region R can be described as

$$R = \{(x, y) | y \geqslant 0, 1 \leqslant x^2 + y^2 \leqslant 4\}$$

It is the half-ring shown in Figure 14(b), and in polar coordinates it is given by $1 \leqslant r \leqslant 2$, $0 \leqslant \theta < \pi$. Therefore,

$$\begin{aligned}
\iint_R (3x + 4y^2) dA &= \int_0^{\pi} \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta) r dr d\theta \\
&= \int_0^{\pi} \int_1^2 (3r^3\cos\theta + 4r^3\sin^2\theta) dr d\theta \\
&= \int_0^{\pi} \left[r^3\cos\theta + r^4\sin^2\theta\right]_1^2 d\theta]] \\
&= \int_0^{\pi} \left(7\cos\theta + 15\sin^2\theta\right) d\theta
\end{aligned}$$

Double integrals in polar coordinates

$$= \int_0^{\pi} \left(7\cos\theta + \frac{15}{2}(1 - \cos 2\theta) \right) d\theta$$
$$= \left[7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin 2\theta \right]_0^{\pi} d\theta = \frac{15\pi}{2}$$

Example 3.2

Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

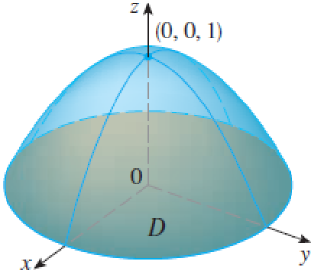


Figure 16:

Double integrals in polar coordinates

Solution

If we put $z = 0$ in the equation of the paraboloid, we get $x^2 + y^2 = 1$. This means that the plane intersects the paraboloid in the circle $x^2 + y^2 = 1$, so the solid lies under the paraboloid and above the circular disk D given by $x^2 + y^2 \leqslant 1$ [see Figures 16 and 14(a)]. In polar coordinates D is given by $0 \leqslant r \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$. Since $1 - x^2 - y^2 = 1 - r^2$, the volume is

$$V = \iint_D (1 - x^2 - y^2)\ dA = \int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (r - r^3) dr = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \frac{\pi}{2}$$

Double integrals in polar coordinates

Theorem 3.2

If f is continuous on a polar region of the form

$$D = \{(r,\theta) | \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

$$\iint_D f(x,y)\ dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta)) r dr d\theta$$

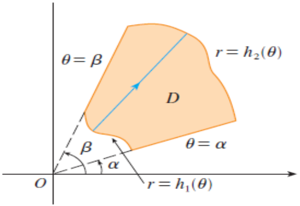


Figure 17: $D = \{(r,\theta) | \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$

Example 3.3

Use a double integral to find the area enclosed by one loop of the four leaved rose $r = \cos 2\theta$.

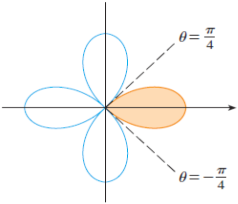


Figure 18: Four leaved rose

Solution

From the sketch of the curve in Figure 8, we see that a loop is given by the region

$$D = \{(r,\theta) | -\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}, 0 \leqslant r \leqslant \cos 2\theta\}$$

So the area is

$$A(D) = \iint_D dA = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} r dr d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_0^{\cos 2\theta} d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta d\theta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1+\cos 4\theta) d\theta = \frac{1}{4}\left[\theta + \frac{1}{4}\sin 4\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{8}$$

Example 3.4

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

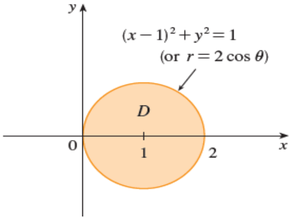


Figure 19: Four leaved rose

Double integrals in polar coordinates

Solution

The solid lies above the disk D whose boundary circle has equation $x^2 + y^2 = 2x$ or, after completing the square, $(x-1)^2 + y^2 = 1$ In polar coordinates we have $x^2 + y^2 = r^2$ and $x = r\cos\theta$, so the boundary circle becomes $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$. Thus the disk D is given by

$$D = \{(r,\theta) | -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}, 0 \leqslant r \leqslant 2\cos\theta\}$$

$$V = \iint_D (x^2 + y^2)\, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 r dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta = 8\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = 8\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left(1 + 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\right) d\theta = \frac{3\pi}{2}$$

Table of contents

Triple Integrals

Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables. Let's first deal with the simplest case where f is defined on a rectangular box:

$$B = \{(x, y, z) | a \leqslant x \leqslant b, c \leqslant y \leqslant d, r \leqslant z \leqslant s\}$$

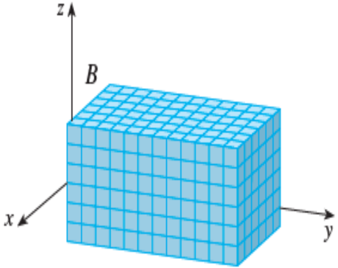


Figure 20:

Theorem 4.1 (Fubini's Theorem for Triple Integrals)

If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\begin{aligned}
\iiint_B f(x,y,z)dV &= \int_a^b \int_c^d \int_r^s f(x,y,z)dzdydx \\
&= \int_a^b \int_r^s \int_c^d f(x,y,z)dydzdx \\
&= \int_c^d \int_a^b \int_r^s f(x,y,z)dzdxdy \\
&= \int_c^d \int_r^s \int_a^b f(x,y,z)dxdzdy \\
&= \int_r^s \int_a^b \int_c^d f(x,y,z)dydxdz \\
&= \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz.
\end{aligned}$$

Example 4.1

Evaluate the triple integral $\iiint_B xyz^2 dV$, where B is the rectangular box given by

$$B = \{(x, y, z) | 0 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 2, 0 \leqslant z \leqslant 3\}$$

Solution

We could use any of the six possible orders of integration. If we choose to integrate with respect to x, then y, and then z, we obtain

$$\iiint_B xyz^2 dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 \; dxdydz = \int_0^3 \int_{-1}^2 \left[\frac{x^2}{2} yz^2\right]_0^1 \; dydz$$

$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2} \; dydz = \int_0^3 \left[\frac{y^2 z^2}{4}\right]_{-1}^2 \; dz = \int_0^3 \frac{3z^2}{4} dz$$

$$= \left[\frac{z^3}{4}\right]_0^3 = \frac{27}{4}$$

A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is the projection of E onto the xy-plane as shown in Figure 21. Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$.

If E is a type 1 region

$$\iiint_E f(x, y, z) dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) dz \right] dA$$

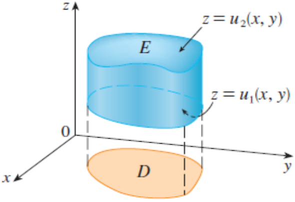


Figure 21: A type 1 solid region

Triple Integrals

In particular, if the projection D of E onto the xy-plane is a type I plane region (as in Figure 22), then

$$E = \{(x, y, z) | a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x), u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

we have

$$\iiint_E f(x, y, z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dzdydx$$

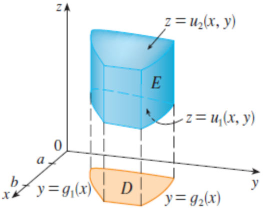


Figure 22: A type 1 solid region where the projection D is a type I plane region

Triple Integrals

If, on the other hand, D is a type II plane region (as in Figure 23), then

$$E = \{(x, y, z) | c \leqslant y \leqslant d, h_1(y) \leqslant x \leqslant h_2(y), u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

we have

$$\iiint_E f(x, y, z) dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) dz dx dy$$

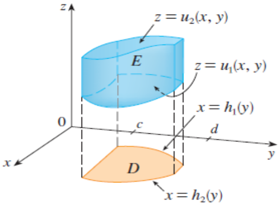


Figure 23: A type 1 solid region where the projection D is a type II plane region

Example 4.2

Evaluate $\iiint\limits_E z dV$ where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z + 1$.

Solution

When we set up a triple integral it's wise to draw two diagrams: one of the solid region E (see Figure 24)

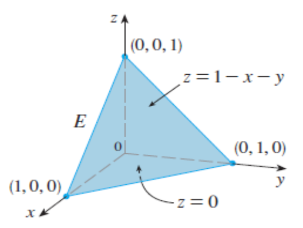


Figure 24:

and one of its projection D onto the xy-plane (see Figure 25).

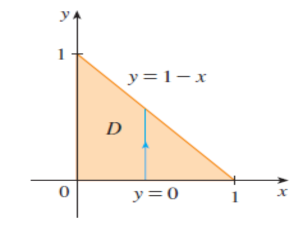


Figure 25:

The lower boundary of the tetrahedron is the plane $z = 0$ and the upper boundary is the plane $x + y + z = 1$ (or $z = 1 - x - y$), so we use $u_1(x, y) = 0$ and $u_2(x, y) = 1 - x - y$. Notice that the planes $x + y + z = 1$ and $z = 0$ intersect in the line $x + y = 1$ (or $y = 1 - x$) in the xy-plane. So the projection of E is the triangular region shown in Figure 25, and we have

$$E = \{(x,y,z) | 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1-x, 0 \leqslant z \leqslant 1-x-y\}$$

$$\iiint\limits_E z dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz dy dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_0^{1-x-y} dy dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} (1-x-y)^2 \, dy dx = -\frac{1}{2}\int_0^1 \left[\frac{(1-x-y)^3}{3}\right]_0^{1-x} dx$$

$$= \frac{1}{6}\int_0^1 (1-x)^3 \, dx = \frac{1}{6}\left[-\frac{(1-x)^4}{4}\right]_0^1 = \frac{1}{24}$$

A solid region E is said to be of **type 2** if it is of the form,

$$E = \{(x, y, z) | (y, z) \in D, u_1(y, z) \leqslant x \leqslant u_2(y, z)\}$$

where D is the projection of E onto the yz-plane as shown in Figure 26. Notice that the upper boundary of the solid E is the surface with equation $x = u_2(y, z)$, while the lower boundary is the surface $x = u_1(y, z)$.

If E is a type 2 region

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)dx\right] dA$$

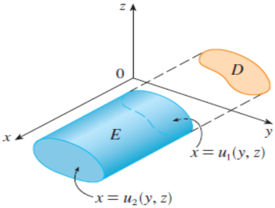


Figure 26: A type 2 solid region

Triple Integrals

A solid region E is said to be of **type 3** if it is of the form,

$$E = \{(x, y, z) | (x, z) \in D, u_1(x, z) \leqslant y \leqslant u_2(x, z)\}$$

where D is the projection of E onto the xz-plane as shown in Figure 27. Notice that the upper boundary of the solid E is the surface with equation $y = u_2(x, z)$, while the lower boundary is the surface $y = u_1(x, z)$.
If E is a type 2 region

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)dy\right] dA$$

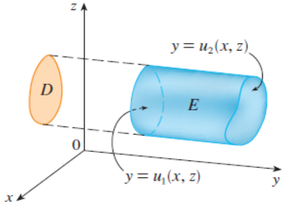


Figure 27: A type 3 solid region

Example 4.3

Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

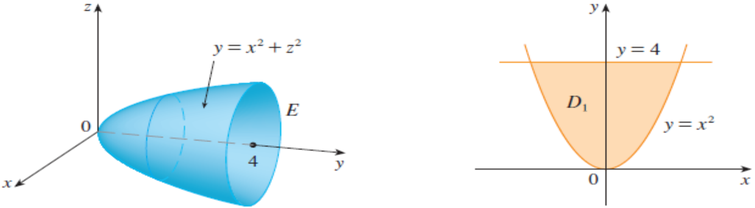


Figure 28: Region of integration and Projection onto xy-plane

Solution

The solid E is shown in Figure 28. If we regard it as a type 1 region, then we need to consider its projection D_1 onto the xy-plane, which is the parabolic region in Figure 28. (The trace of $y = x^2 + z^2$ in the plane $z = 0$ is the parabola $y = x^2$.)

From $y = x^2 + z^2$ we obtain $z = \pm\sqrt{y - x^2}$, so the lower boundary surface of E is $z = -\sqrt{y - x^2}$ and the upper surface is $z = \sqrt{y - x^2}$.

Therefore the description of E as a type 1 region is

$$E = \left\{(x, y, z) | -2 \leqslant x \leqslant 2, x^2 \leqslant y \leqslant 4, -\sqrt{y - x^2} \leqslant z \leqslant \sqrt{y - x^2}\right\}$$

and so we obtain $\displaystyle\iiint_E \sqrt{x^2 + z^2} dV = \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2 + z^2}$

Although this expression is correct, it is extremely difficult to evaluate. So let's instead consider E as a type 3 region. As such, its projection D_3 onto the xz-plane is the disk $x^2 + z^2 \leqslant 4$ shown in Figure 29.

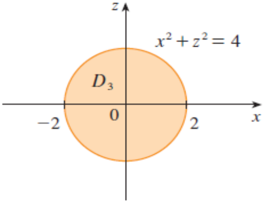


Figure 29: Region of integration and Projection onto xy-plane

Then the left boundary of E is the paraboloid $y = x^2 + z^2$ and the right boundary is the plane $y = 4$, so taking $u_1(x,z) = x^2 + z^2$ and $u_2(x,z) = 4$, we have.

$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_{D_3}\left[\int_{x^2+z^2}^{4}\sqrt{x^2+z^2}\,dy\right]dA$$
$$= \iint_{D_3}(4-x^2-z^2)\sqrt{x^2+z^2}\,dA$$

it's easier to convert to polar coordinates in the xz-plane: $x = r\cos\theta$, $z = r\sin\theta$. This gives

$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_{D_3}(4-x^2-z^2)\sqrt{x^2+z^2}\,dA$$
$$= \int_0^{2\pi}\int_0^2 (4-r^2)r\ r\,dr\,d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^2 (4r^2-r^4)\,dr$$
$$= 2\pi\left[\frac{4r^3}{3}-\frac{r^5}{5}\right]_0^2 = \frac{128\pi}{15}$$

Table of contents

Triple integrals in cylindrical coordinates

Definition 5.1 (Cylindrical coordinates)

In the cylindrical coordinate system, a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P. (See Figure 30.)

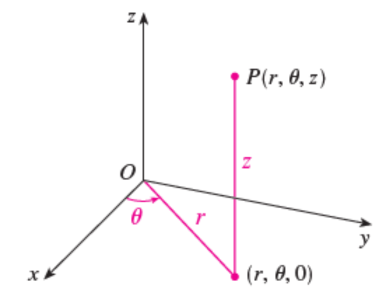


Figure 30: *The cylindrical coordinates of a point*

Triple integrals in cylindrical coordinates

To convert from cylindrical to rectangular coordinates, we use the equations

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

whereas to convert from rectangular to cylindrical coordinates, we use

$$x^2 + y^2 = r^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

Example 5.1

1. Plot the point with cylindrical coordinates $(2, \dfrac{2\pi}{3}, 1)$ and find its rectangular coordinates.
2. Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

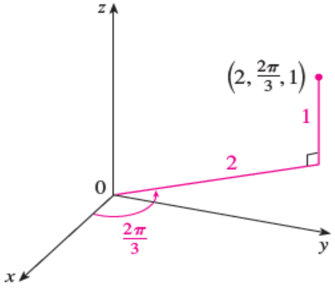


Figure 31:

Solution

1. The point with cylindrical coordinates $(2, \frac{2\pi}{3}, 1)$ is plotted in Figure 31. Its rectangular coordinates are

$$x = 2\cos\left(\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2}\right) = -1$$
$$y = 2\sin\left(\frac{2\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$
$$z = 1$$

2. We have

$$r = \sqrt{3^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$$
$$\tan\theta = \frac{-3}{3} = -1, \text{ so } \theta = -\frac{\pi}{4} + 2k\pi$$
$$z = -7$$

Triple integrals in cylindrical coordinates

Suppose that E is a type 1 region whose projection D onto the xy-plane is conveniently described in polar coordinates (see Figure 32).

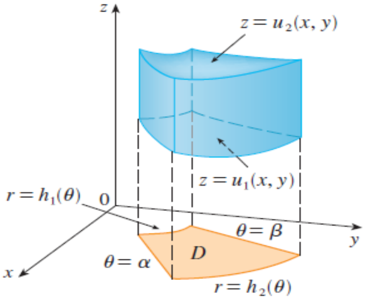


Figure 32:

In particular, suppose that f is continuous and

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is given in polar coordinates by

$$D = \{(r, \theta) | \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

$$\iiint_E f(x, y, z) dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta$$

Example 5.2

Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\ dzdydx.$

Solution

This iterated integral is a triple integral over the solid region

$$E = \{(x, y, z) | 2 \leqslant x \leqslant 2, -\sqrt{4 - x^2} \leqslant y \leqslant \sqrt{4 - x^2}, \sqrt{x^2 + y^2} \leqslant z \leqslant 2\}$$

and the projection of E onto the xy-plane is the disk $x^2 + y^2 \leqslant 4$. The lower surface of E is the cone $z = \sqrt{x^2 + y^2}$ and its upper surface is the plane $z = 2$. See Figure 33.

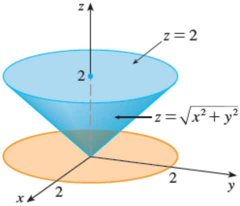


Figure 33:

Triple integrals in cylindrical coordinates

This region has a much simpler description in cylindrical coordinates:

$$E = \{(r, \theta, z) | 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant r \leqslant 2, r \leqslant z \leqslant 2\}$$

$$\begin{aligned}
\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\ dzdydx &= \iiint_E (x^2 + y^2) dV \\
&= \int_0^{2\pi} \int_0^2 \int_r^2 r^2\ r\ dzdrd\theta \\
&= \int_0^{2\pi} d\theta \int_0^2 r^3 (2 - r) dr \\
&= 2\pi \left[\frac{2r^4}{4} - \frac{r^5}{5} \right]_0^2 = \frac{16\pi}{5}
\end{aligned}$$

Exercise 5.1

1. Evaluate $\iiint_E (x^2 + y^2)dV$, where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.
2. Evaluate $\iiint_E z dV$, where E is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

Table of contents

Definition 6.1 (Spherical coordinates)

The spherical coordinates (ρ, θ, ϕ) of a point P in space are shown in Figure 34, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP. Note that

$$\rho \geqslant 0, \quad 0 \leqslant \leqslant \pi$$

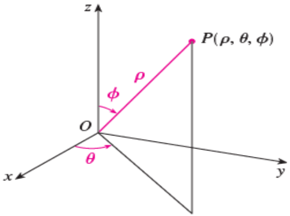


Figure 34: The spherical coordinates of a point

The relationship between rectangular and spherical coordinates can be seen from Figure 35.

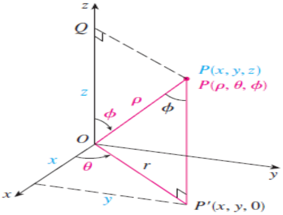


Figure 35: The relationship between rectangular and spherical coordinates

We have

$$z = \rho \cos \phi, \qquad r = \rho \sin \phi$$

But $x = r \cos \theta$ and $y = r \sin \theta$, so to convert from spherical to rectangular coordinates, we use the equations

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

Also, the distance formula shows that

$$\rho^2 = x^2 + y^2 + z^2$$

Example 6.1

The point $(2, \frac{\pi}{4}, \frac{\pi}{3})$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

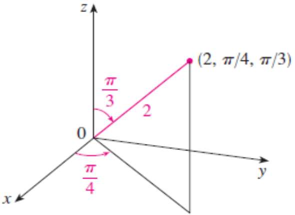


Figure 36: The relationship between rectangular and spherical coordinates

Solution

We plot the point in Figure 36. We have

$$x = \rho \sin\phi \cos\theta = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = 2\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin\phi \sin\theta = 2\sin(\frac{\pi}{3})\sin(\frac{\pi}{4}) = 2\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \sqrt{\frac{3}{2}}$$

$$z = \rho\cos\phi = 2\cos(\frac{\pi}{3})) = 2\frac{1}{2} = 1$$

The point $(2, \frac{\pi}{4}, \frac{\pi}{3})$ is $(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1)$

Example 6.2

The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find spherical coordinates for this point.

Solution

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 12 + 4} = 4$$

$$\cos\phi = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2} \qquad \phi = \frac{2\pi}{3}$$

$$\cos\theta = \frac{x}{\rho\sin\phi} = 0 \qquad \theta = \frac{\pi}{2}$$

Therefore spherical coordinates of the given point are $(4, \frac{\pi}{2}, \frac{2\pi}{3})$.

Triple integrals in spherical coordinates

Theorem 6.1 (Evaluating Triple Integrals with Spherical Coordinates)

$$\iiint_E f(x,y,z)dV =$$
$$\int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi \ d\rho \ d\theta \ d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho,\theta,\phi) | a \leqslant \rho \leqslant b, \alpha \leqslant \theta \leqslant \beta, c \leqslant \phi \leqslant d\}$$

This formula can be extended to include more general spherical regions such as

$$E = \{(\rho, \theta, \phi) | \alpha \leqslant \theta \leqslant \beta, c \leqslant \phi \leqslant d, g_1(\theta, \phi) \leqslant \rho \leqslant g_2(\theta, \phi)\}$$

In this case the formula is the same as the previous except that the limits of integration for ρ are $g_1(\theta, \phi)$ and $g_2(\theta, \phi)$.
Usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

Triple integrals in spherical coordinates

Example 6.3

Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\, dV$, where B is the unit ball:

$$B = \{(x, y, z) | x^2 + y^2 + z^2 \leqslant 1\}$$

Solution

Since the boundary of B is a sphere, we use spherical coordinates:

$$B = \{(\rho, \theta, \phi) | 0 \leqslant \rho \leqslant 1, 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant \phi \leqslant \pi\}$$

In addition, spherical coordinates are appropriate because

$$\rho^2 = x^2 + y^2 + z^2$$

Triple integrals in spherical coordinates

$$\begin{aligned}\iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dV &= \int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{(\rho^2)^{\frac{3}{2}}}\,\rho^2\sin\phi\ d\rho\ d\theta\ d\phi \\ &= \int_0^{\pi}\sin\phi\ d\phi\int_0^{2\pi} d\theta\int_0^1 \rho^2 e^{\rho^3}\ d\rho \\ &= [-\cos\phi]_0^{\pi}\,[\theta]_0^{2\pi}\left[\frac{1}{3}e^{\rho^3}\right]_0^1 = \frac{4}{3}\pi(e-1)\end{aligned}$$

Remark 6.1

It would have been extremely awkward to evaluate the integral in Example 6.3 without spherical coordinates. In rectangular coordinates the iterated integral would have been

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\ dz\ dy\ dx$$

Triple integrals in spherical coordinates

Example 6.4

Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. (See Figure 37.)

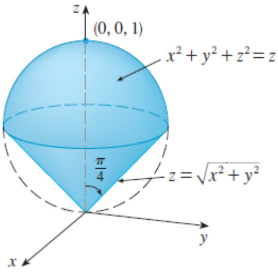


Figure 37:

Solution

Notice that the sphere passes through the origin and has center $(0, 0, \frac{1}{2})$.
We write the equation of the sphere in spherical coordinates as

$$x^2 + y^2 + z^2 = z \text{ gives } \rho^2 = \rho cos\phi \text{ or } \rho = cos\phi$$

The equation of the cone can be written as

$$\rho \cos\phi = \sqrt{\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta} = \rho \sin\phi$$

This gives $\sin\phi = \cos\phi$, so $\phi = \frac{\pi}{4}$. Therefore the description of the solid E in spherical coordinates is

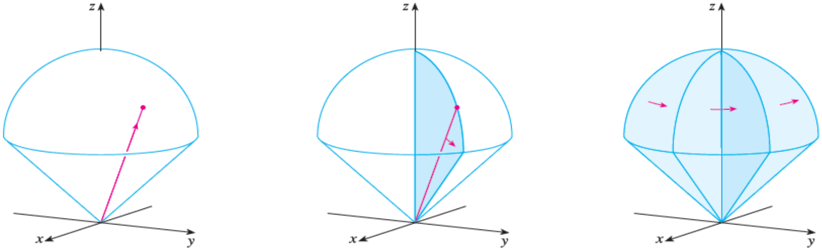
$$B = \{(\rho, \theta, \phi) | 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant \phi \leqslant \frac{\pi}{4}, 0 \leqslant \rho \leqslant \cos\phi\}$$

Triple integrals in spherical coordinates

Figure 38 shows how E is swept out if we integrate first with respect to ρ, then ϕ, and then θ. The volume of E is

$$\begin{aligned} V(E) = \iiint_E dV &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\cos\phi} \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\phi \left[\frac{\rho^3}{3}\right]_0^{\cos\phi} \; d\phi \\ &= \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} \sin\phi \cos^3\phi \; d\phi \\ &= \frac{2\pi}{3} \left[-\frac{\cos^4\phi}{4}\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8} \end{aligned}$$

Triple integrals in spherical coordinates



ρ varies from 0 to $\cos\phi$ while ϕ and θ are constant.

ϕ varies from 0 to $\pi/4$ while θ is constant.

θ varies from 0 to 2π.

Figure 38: