Chapter 6: Sequences

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Definition 1.1 (sequences)

A sequence is a function whose domain is the set of positive integers. It is denoted by $\{a_n\} = a_1, a_2, a_3, \ldots, a_n, \ldots$ *(entire seq) and* $\{a_n\} = a_1, a_2, a_3, \ldots, a_n$ *(finite seq).*

Example 1.1

$1, 3, 5, 7, \ldots$ is a sequence.

- The three dots mean to continue forward in the pattern established.
- Each number in the sequence is called a **term**.
- In this sequence, $a_1 = 1$, $a_2 = 3$, $a_3 = 5$ and so on.

Remark 1.1

The expression a_n is referred to as the general or $\mathbf{n^{th}}$ **term** of the sequence.

Example 1.2

Write the first five terms of a sequence described by the general term $a_n = 3n + 2$.

- $a_1 = 3(1) + 2 = 5$
- $a_2 = 3(2) + 2 = 8$
- $a_3 = 3(3) + 2 = 11$
- $a_4 = 3(4) + 2 = 14$
- $a_5 = 3(5) + 2 = 17$

Therefore, the first five terms are 5, 8, 11, 14, and 17.

Example 1.3

Write the first five terms of a sequence described by the general term $a_n = 2(3^{n-1})$.

- $a_1 = 2(3^{1-1}) = 2$
- $a_2 = 2(3^{2-1}) = 6$
- $a_3 = 2(3^{3-1}) = 18$
- $a_4 = 2(3^{4-1}) = 54$
- $a_5 = 2(3^{5-1}) = 162$

Therefore, the first five terms are 2, 6, 18, 54, and 162.

Example 1.4

Find an expression for the nth term of each sequence.

1. $2, 4, 6, 8, \ldots$
2. $10, 50, 250, 1250, \ldots$
3. $3, 7, 11, 15, 19, \ldots$

Solution

1.
   - $a_1 = 2 = 2(1)$
   - $a_2 = 4 = 2(2)$
   - $a_3 = 6 = 2(3)$
   - $a_4 = 8 = 2(4)$

   Based on this pattern, $a_n = 2n$.

2.
   - $a_1 = 10 = 2(5) = 2(5^1)$
   - $a_2 = 50 = 2(25) = 2(5^2)$
   - $a_3 = 250 = 2(125) = 2(5^3)$
   - $a_4 = 1250 = 2(625) = 2(5^4)$

   Based on this pattern, $a_n = 2(5^n)$.

3.
   - $a_1 = 3$
   - $a_2 = 7 = 3 + 4(1)$
   - $a_3 = 11 = 3 + 4(2)$
   - $a_4 = 15 = 3 + 4(3)$
   - $a_5 = 19 = 3 + 4(4)$

   Based on this pattern, $a_n = 3 + 4(n - 1)$.

Exercise 1.1

Find the first four terms and nth term of each sequence:

1. $\left\{\dfrac{n}{n+1}\right\}$
2. $\{2+(0.1)^n\}$
3. $\left\{(-1)^{n+1}\dfrac{n^2}{3n-1}\right\}$
4. $\{4\}$
5. $a_1 = 3$, $a_{k+1} = 2a_k$, for all $k \geq 1$.

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Arithmetic Sequences

Definition 2.1

An arithmetic sequence is a sequence of the form

$$a, a+d, a+2d, a+3d, a+4d, \dots$$

The number a is the first term, and d is the common difference of the sequence. The $\mathbf{n^{th}}$ term of an arithmetic sequence is given by

$$a_n = a + (n-1)d$$

Example 2.1

1. If $a = 2$ and $d = 3$, then we have the arithmetic sequence

$$2, 2+3, 2+6, 2+9, \dots$$

$$\text{or } 2, 5, 8, 11, \dots$$

Any two consecutive terms of this sequence differ by $d = 3$. The nth term is

$$a_n = 2 + 3(n-1)$$

2. Consider the arithmetic sequence

$$9, 4, -1, -6, -11, \dots$$

Here the common difference is $d = -5$. The terms of an arithmetic sequence decrease if the common difference is negative. The nth term is $a_n = 9 - 5(n-1)$

Example 2.2

Find the common difference, the first six terms, the nth term, and the 300th term of the arithmetic sequence

$$13, 7, 1, -5, \ldots$$

Solution

Since the first term is 13, we have $a = 13$. The common difference is $d = 7 - 13 = -6$. Thus the nth term of this sequence is

$$a_n = 13 - 6(n - 1)$$

From this we find the first six terms:

$$13, 7, 1, -5, -11, -17, \ldots$$

The 300th term is $a_{300} = 13 - 6(300 - 1) = -1781$.

Example 2.3

The 11th term of an arithmetic sequence is 52, and the 19th term is 92. Find the 1000th term.

Solution

To find the nth term of this sequence, we need to find a and d in the formula

$$a_n = a - (n-1)d$$

From this formula we get

$$a_{11} = a + (11 - 12)d = a + 10d$$

$$a_{19} = a + (19 - 1)d = a + 18d$$

Since $a_{11} = 52$ and $a_{19} = 92$, we get the following two equations:

$$\begin{cases} 52 = a + 10d \\ 92 = a + 18d \end{cases}$$

Solving this system for a and d, we get $a = 2$ and $d = 5$. (Verify this.) Thus the nth term of this sequence is

$$an = 2 + 5(n - 1)$$

The 1000th term is $a_{1000} = 2 + 5(1000 - 1) = 4997$.

Definition 2.2 (Partial Sums of an Arithmetic Sequence)

For the arithmetic sequence given by $a_n = a + (n-1)d$*, the nth partial sum* $S_n = a + (a+d) + (a+2d) + (a+3d) + \cdots + [a+(n-1)d]$ *is given by either of the following formulas.*

1. $S_n = \dfrac{n}{2}[2a + (n-1)d]$
2. $S_n = n\left(\dfrac{a + a_n}{2}\right)$

Example 2.4

Find the sum of the first 50 odd numbers.

Solution

The odd numbers form an arithmetic sequence with $a = 1$ and $d = 2$. The n^{th} term is $a_n = 1 + 2(n-1) = 2n - 1$, so the 50^{th} odd number is $a_{50} = 99$. Substituting in Formula 2 for the partial sum of an arithmetic sequence, we get

$$S_{50} = 50\left(\frac{a + a_{50}}{2}\right) = 50\left(\frac{1 + 99}{2}\right) = 2500$$

Arithmetic Sequences

Example 2.5

Find the following partial sum of an arithmetic sequence:

$$3 + 7 + 11 + 15 + \cdots + 159$$

Solution

For this sequence $a = 3$ and $d = 4$, so $a_n = 3 + 4(n-1)$. To find which term of the sequence is the last term 159, we use the formula for the n^{th} term and solve for n.

$$\begin{aligned} 159 &= 3 + 4(n-1) \\ 39 &= n - 1 \\ n &= 40 \end{aligned}$$

To find the partial sum of the first 40 terms, we use Formula 1 for the n^{th} partial sum of an arithmetic sequence:

$$S_{40} = \frac{40}{2}[2(3) + (40-1)4] = 3240$$

Example 2.6

An amphitheater has 50 rows of seats with 30 seats in the first row, 32 in the second, 34 in the third, and so on. Find the total number of seats.

Solution

The numbers of seats in the rows form an arithmetic sequence with $a = 30$ and $d = 2$. Since there are 50 rows, the total number of seats is the sum

$$S_{50} = \frac{50}{2}[2(30) + (50 - 1)2] = 3950$$

Thus the amphitheater has 3950 seats.

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Definition 3.1 (Geometric Sequences)

A geometric sequence is a sequence of the form

$$a, ar, ar^2, ar^3, ar^4, \ldots$$

The number a *is the first term, and* r *is the common ratio of the sequence. The* n^{th} *term of a geometric sequence is given by*

$$a_n = ar^{n-1}$$

The number r is called the common ratio because the ratio of any two consecutive terms of the sequence is r.

Example 3.1

1. If $a = 3$ and $r = 2$, then we have the geometric sequence

$$3, 3.2, 3.2^2, 3.2^3, 3.2^4, \ldots$$

or

$$3, 6, 12, 24, 48, \ldots$$

Notice that the ratio of any two consecutive terms is $r = 2$. The n^{th} term is $a_n = 3(2)^{n-1}$.

2. The sequence

$$2, -10, 50, -250, 1250, \ldots$$

is a geometric sequence with $a = 2$ and $r = -5$. When r is negative, the terms of the sequence alternate in sign. The n^{th} term is $a_n = 2(-5)^{n-1}$.

Example 3.2

Find the common ratio, the first term, the n^{th} term, and the eighth term of the geometric sequence

Solution

To find a formula for the n^{th} term of this sequence, we need to find the first term a and the common ratio r. Clearly, $a = 5$. To find r, we find the ratio of any two consecutive terms. For instance, $r = \dfrac{45}{15} = 3$. Thus

$$a_n = 5(3)^{n-1}$$

The eighth term is $a_8 = 5(3)^{8-1} = 5(3)^7 = 10935$

Example 3.3

The third term of a geometric sequence is $\frac{63}{4}$, and the sixth term is $\frac{1701}{32}$. Find the fifth term.

Solution

Solution–Since this sequence is geometric, its n^{th} term is given by the formula $a_n = ar^{n-1}$. Thus

$$\begin{aligned} a_3 &= ar^2 \\ a_6 &= ar^5 \end{aligned}$$

From the values we are given for these two terms, we get the following system of equations:

$$\begin{cases} \dfrac{63}{4} = ar^2 \\ \dfrac{1701}{32} = ar^5 \end{cases}$$

We solve this system by dividing.

$$\frac{ar^5}{ar^2} = \frac{\frac{1701}{32}}{\frac{63}{4}}$$
$$r^3 = \frac{27}{8}$$
$$r = \frac{3}{2}$$

Substituting for r in the first equation gives

$$\frac{63}{4} = a\left(\frac{3}{2}\right)^2$$
$$a = 7$$

Geometric Sequences

It follows that the n^{th} term of this sequence is

$$a_n = 7\left(\frac{3}{2}\right)^{n-1}$$

Thus the fifth term is

$$a_5 = 7\left(\frac{3}{2}\right)^4 = \frac{567}{16}$$

Definition 3.2 (Partial Sums of a Geometric Sequence)

For the geometric sequence defined by $a_n = ar^(n-1)$, *the* n^{th} *partial sum*

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \qquad r \neq 1$$

is given by

$$S_n = a\frac{1-r^n}{1-r}$$

Example 3.4

Find the following partial sum of a geometric sequence:

$$1 + 4 + 16 + \cdots + 4096$$

Solution

Solution–For this sequence $a = 1$ and $r = 4$, so $a_n = 4^{n-1}$. Since $4^6 = 4096$, we use the formula for S_n with $n = 7$, and we have

$$S_7 = \frac{1 - 4^7}{1 - 4} = 5461$$

Thus this partial sum is 5461.

Example 3.5

Find the sum $\sum_{k=1}^{6} 7\left(-\frac{2}{3}\right)^{k-1}$.

Solution

The given sum is the sixth partial sum of a geometric sequence with first term $a = 7\left(-\frac{2}{3}\right)^{0} = 7$ and $r = -\frac{2}{3}$. Thus by the formula for S_n with $n = 6$ we have

$$S_6 = 7\frac{1-\left(-\frac{2}{3}\right)^6}{1-\left(-\frac{2}{3}\right)} = 7\frac{1-\frac{64}{729}}{\frac{5}{3}} = \frac{931}{243} \approx 3.83$$

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Expanding $(a+b)^n$

To find a pattern in the expansion of $(a+b)^n$, we first look at some special cases.

$$\begin{aligned}
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{aligned}$$

$$\vdots$$

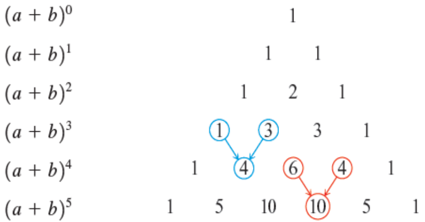
The following simple patterns emerge for the expansion of $(a+b)^n$.

1. There are $n+1$ terms, the first being a^n and the last being b^n.
2. The exponents of a decrease by 1 from term to term, while the exponents of b increase by 1.
3. The sum of the exponents of a and b in each term is n.
   For instance, notice how the exponents of a and b behave in the expansion of $(a+b)^5$

With these observations we can write the form of the expansion of $(a+b)^n$ for any natural number n. For example, writing a question mark for the missing coefficients, we have

$$(a+b)^8 = a^8 + ?a^7b + ?a^6b^2 + ?a^5b^3 + ?a^4b^4 + ?a^3b^5 + ?a^2b^6 + ?ab^7 + b^8$$

To complete the expansion, we need to determine these coefficients. To find a pattern, let's write the coefficients in the expansion of $(a+b)^n$ for the first few values of n in a triangular array as shown in the following array, which is called Pascal's triangle.



Expanding $(a+b)^n$

Key Property of Pascal's Triangle

Every entry (other than a 1) is the sum of the two entries diagonally above it.

From this property it is easy to find any row of Pascal's triangle from the row above it. For instance, we find the sixth and seventh rows, starting with the fifth row:

$(a+b)^5$			1		5		10		10		5		1		
$(a+b)^6$		1		6		15		20		15		6		1	
$(a+b)^7$	1		7		21		35		35		21		7		1

To see why this property holds, let's consider the following expansions:

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Expanding $(a+b)^n$

Example 4.1

Find the expansion of $(a+b)^7$ using Pascal's triangle.

Solution

The first term in the expansion is a^7, and the last term is b^7. Using the fact that the exponent of a decreases by 1 from term to term and that of b increases by 1 from term to term, we have

$$(a+b)^7 = a^7 + ?a^6b + ?a^5b^2 + ?a^4b^3 + ?a^3b^4 + ?a^2b^5 + ?ab^6 + b^7$$

The appropriate coefficients appear in the seventh row of Pascal's triangle. Thus

$$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

Example 4.2

Use Pascal's triangle to expand $(2-3x)^5$.

Solution

We find the expansion of $(a+b)^5$ and then substitute 2 for a and $-3x$ for b. Using Pascal's triangle for the coefficients, we get

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Substituting $a=2$ and $b=-3x$ gives

$$\begin{aligned}(2-3x)^5 &= 2^5 + 5(2)^4(-3x) + 10(2)^3(-3x)^2 + 10(2)^2(-3x)^3 \\ &+ 5(2)(-3x)^4 + (-3x)^5 \\ &= 32 - 240x + 720x^2 - 1080x^3 + 810x^4 - 243x^5\end{aligned}$$

The Binomial Coefficient

Although Pascal's triangle is useful in finding the binomial expansion for reasonably small values of n, it isn't practical for finding $(a+b)^n$ for large values of n.

Definition 4.1 (The Binomial Coefficient)

Let n and r be nonnegative integers with $r \leqslant n$. The binomial coefficient is denoted by $\binom{n}{r}$ and is defined by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where $n!$ is the product of the first n natural numbers, called $\mathbf{n}$ **factorial**.

$$n! = 1.2.3.\ldots.(n-1)n$$

We also define $0!$ as follows:

$$0! = 1$$

Example 4.3

Calculating Binomial Coefficients

$$\textbf{(a)}\quad \binom{9}{4} = \frac{9!}{4!(9-4)!} = \frac{9!}{4!\,5!} = \frac{\cancel{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot 6\cdot 7\cdot 8\cdot 9}{(1\cdot 2\cdot 3\cdot 4)(\cancel{1\cdot 2\cdot 3\cdot 4\cdot 5})}$$

$$= \frac{6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4} = 126$$

$$\textbf{(b)}\quad \binom{100}{3} = \frac{100!}{3!(100-3)!} = \frac{\cancel{1\cdot 2\cdot 3\cdot \cdots \cdot 97}\cdot 98\cdot 99\cdot 100}{(1\cdot 2\cdot 3)(\cancel{1\cdot 2\cdot 3\cdot \cdots \cdot 97})}$$

$$= \frac{98\cdot 99\cdot 100}{1\cdot 2\cdot 3} = 161{,}700$$

$$\textbf{(c)}\quad \binom{100}{97} = \frac{100!}{97!(100-97)!} = \frac{\cancel{1\cdot 2\cdot 3\cdot \cdots \cdot 97}\cdot 98\cdot 99\cdot 100}{(\cancel{1\cdot 2\cdot 3\cdot \cdots \cdot 97})(1\cdot 2\cdot 3)}$$

$$= \frac{98\cdot 99\cdot 100}{1\cdot 2\cdot 3} = 161{,}700$$

Remark 4.1

Notice that the binomial coefficients in parts (b) and (c) of Example 4.3 are equal. This is a special case of the following relation

$$\binom{n}{r} = \binom{n}{n-r}$$

To see the connection between the binomial coefficients and the binomial expansion of $(a+b)^n$, let's calculate the following binomial coefficients:

$$\binom{5}{0} = 1 \quad \binom{5}{1} = 5 \quad \binom{5}{2} = 10 \quad \binom{5}{3} = 10 \quad \binom{5}{4} = 5 \quad \binom{5}{5} = 1$$

The Binomial Coefficient

These are precisely the entries in the fifth row of Pascal's triangle. In fact, we can write Pascal's triangle as follows.

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$. \quad . \quad . \quad . \quad . \quad . \quad .$$

$$\binom{n}{0} \quad \binom{n}{1} \quad \binom{n}{2} \quad . \quad . \quad . \quad \binom{n}{n-1} \quad \binom{n}{n}$$

Key Property of the Binomial Coefficients

For any nonnegative integers r and k with $r \leqslant k$,

$$\binom{k}{r-1} + \binom{k}{r} = \binom{k+1}{r}$$

Theorem 4.1 (The Binomial Theorem)

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

Example 4.4

Use the Binomial Theorem to expand $(x+y)^4$.

Solution

By the Binomial Theorem,

$$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

we have

$$\binom{4}{0} = \binom{4}{4} = 1, \quad \binom{4}{1} = \binom{4}{3} = 4, \quad \binom{4}{2} = 6$$

It follows that

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

The Binomial Coefficient

Definition 4.2 (General Term of the Binomial Expansion)

The term that contains a^r *in the expansion of* $(a+b)^n$ *is*

$$\binom{n}{r} a^r b^{n-r}$$

Example 4.5

Find the term that contains x^5 in the expansion of $(2x+y)^{20}$.

Solution

The term that contains x^5 in the expansion of $(2x+y)^{20}$ is

$$\binom{20}{5}(2x)^5y^{15} = \frac{20!}{5!15!}32x^5y^{15} = 496128x^5y^{15}$$

Example 4.6

Find the term that contains x^8 in the expansion of $\left(x^2 + \dfrac{1}{x}\right)^{10}$.

The Binomial Coefficient

Solution

Both x^2 and $\dfrac{1}{x}$ are powers of x, so the power of x in each term of the expansion is determined by both terms of the binomial. To find the required coefficient, we first find the general term in the expansion. By the formula we have

$$\binom{10}{r}(x^2)^r\left(\frac{1}{x}\right)^{10-r} = \binom{10}{r}x^{2r}x^{r-10} = \binom{10}{r}x^{3r-10}$$

Thus the term that contains x^8 is the term in which

$$3r - 10 = 8 \Longrightarrow r = 6$$

So the required coefficient is

$$\binom{10}{6} = 210$$

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Convergent or Divergent Sequences

Definition 5.1 (Convergent sequence (c'gt))

A convergent sequence is a sequence $\{a_n\}$ *that has a limit* L*, or converges to* L *denoted by either* $\lim_{n\to\infty} a_n = L$ *or* $a_n \to L$ *as* $n \to \infty$

Definition 5.2 (Divergent sequence (d'gt))

A sequence $\{a_n\}$ *is called divergent if:*

- $\lim_{n\to\infty} a_n$ *does not exist.*
- $\lim_{n\to\infty} a_n = +\infty$ *or* $\lim_{n\to\infty} a_n = -\infty$

Definition 5.3 (Constant sequence)

A sequence $\{a_n\}$ *is constant if* $a_n = c$*, for every* $n \in \mathbb{N}$ *with* $c \in \mathbb{R}$*.*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c = c$$

Convergent or Divergent Sequences

Theorem 5.1

Let $\{a_n\}$ *be a sequence and* f *be a function such that:*

- $f(n) = a_n$.
- $f(x)$ *exists for every real number* $x \geq 1$.

then

1. *If* $\lim_{x\to\infty} f(x) = L$ *then* $\lim_{n\to\infty} f(n) = L$.
2. *If* $\lim_{x\to\infty} f(x) = +\infty$ (*or* $-\infty$) *then* $\lim_{n\to\infty} f(n) = +\infty$ (*or* $-\infty$).

Example 5.1

1. If $a_n = 1 + \frac{1}{n}$, determine whether $\{a_n\}$ converges or diverges.
2. Determine whether the sequence $\left\{\frac{1}{4}n^2 - 1\right\}$ converges or diverges.

Solution

1. Let $f(x) = 1 + \frac{1}{x}$. So $f(n) = 1 + \frac{1}{n} = a_n$.
   We have $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 1 + \frac{1}{x} = 1$.
   So $\lim_{n\to\infty} a_n = 1$, thus a_n converges.
2. Let $f(x) = \frac{1}{4}x^2 - 1$. So $f(n) = \frac{1}{4}n^2 - 1 = a_n$.
   We have $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{4}x^2 - 1 = +\infty$.
   So $\lim_{n\to\infty} a_n = +\infty$, thus a_n diverges.

Theorem 5.2 (L'Hopital's Rule)

Suppose further that $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ *has the indeterminate form* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ *and that* $\lim_{n\to\infty} \frac{f'(n)}{g'(n)} = L$ *(or* $\pm\infty$*).*
Then, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$.

Example 5.2

Find the limit, if there exist, of the sequence $\{a_n\}$

1. $a_n = \dfrac{5n}{e^{2n}}$.
2. $a_n = \dfrac{2n^2}{5n^2 - 3}$.
3. $a_n = \dfrac{4n^4 + 1}{2n^2 - 1}$.
4. $a_n = \dfrac{e^n}{4}$.

Solution

1. Let $f(x) = \dfrac{5x}{e^{2x}}$. $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{5x}{e^{2x}} = \dfrac{\infty}{\infty}$.
   We use L'Hopital's Rule,
   $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{5}{2e^{2x}} = 0$. Thus, $\lim\limits_{n\to\infty} a_n = 0$.
2. Let $f(x) = \dfrac{2x^2}{5x^2-3}$. $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{2x^2}{5x^2-3} = \dfrac{\infty}{\infty}$.
   We use L'Hopital's Rule,
   $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{4x}{10x} = \lim\limits_{x\to\infty} \dfrac{2}{5} = \dfrac{2}{5}$. Thus, $\lim\limits_{n\to\infty} a_n = \dfrac{2}{5}$.
3. Let $f(x) = \dfrac{4x^4+1}{2x^2-1}$. $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{4x^4+1}{2x^2-1} = \dfrac{\infty}{\infty}$.
   We use L'Hopital's Rule,
   $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{4x^4+1}{2x^2-1} = \lim\limits_{x\to\infty} \dfrac{16x^3}{4x} = \lim\limits_{x\to\infty} 4x^2 = +\infty$. Thus,
   $\lim\limits_{n\to\infty} a_n = +\infty$.

Properties 5.1

Let a_n *and* b_n *be two sequences. If* $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} b_n = K$*, then*

1. $\lim_{n\to\infty} a_n + b_n = L + K$
2. $\lim_{n\to\infty} a_n - b_n = L - K$
3. $\lim_{n\to\infty} a_n . b_n = L.K$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{K}, \; K \neq 0.$
5. $\lim_{n\to\infty} c.a_n = c.L, \; c \in \mathbb{R}.$

Theorem 5.3 (Absolute value)

For a sequence $\{a_n\}$*, we have*

$$\lim_{n\to\infty} |a_n| = 0 \Longleftrightarrow \lim_{n\to\infty} a_n = 0$$

Theorem 5.4

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if} \quad |r| < 1 \\ \infty & \text{if} \quad |r| > 1 \end{cases}$$

Example 5.3

Find the limit, if there exist, of the sequence $\{a_n\}$

1. $a_n = \left(-\frac{2}{3}\right)^n$.
2. $a_n = (1,01)^n$.
3. $a_n = 6\left(-\frac{5}{6}\right)^n$.
4. $a_n = 8 - \left(\frac{7}{8}\right)^n$.
5. $a_n = 1 + \left(\frac{3}{2}\right)^n$.

Convergent or Divergent Sequences

Solution

1. From Theorem 2.4, $\lim_{n\to\infty} a_n = 0$.
2. From Theorem 2.4, $\lim_{n\to\infty} a_n = +\infty$.
3. From Theorem 2.4, $\lim_{n\to\infty} a_n = 6 \times 0 = 0$.
4. From Theorem 2.4, $\lim_{n\to\infty} a_n = 8 - 0 = 8$.
5. From Theorem 2.4, $\lim_{n\to\infty} a_n = 1 + \infty = +\infty$.

Convergent or Divergent Sequences

Theorem 5.5 (Sandwich)

If a_n, b_n *and* c_n *are sequences such that:*

- $a_n \leq b_n \leq c_n$,
- $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} c_n$.

Then, $\lim\limits_{n\to\infty} b_n = L$

Theorem 5.6

A bounded, monotonic sequence has limit.

Convergent or Divergent Sequences

Remark 5.1

1. $-1 \leq \cos(\theta) \leq 1.$
2. $-1 \leq \sin(\theta) \leq 1.$
3. $0 \leq \cos^2(\theta) \leq 1.$
4. $-\dfrac{\pi}{2} \leq \tan^{-1}(\theta) \leq \dfrac{\pi}{2}$
5. $\cos(\pi n) = (-1)^n$

Example 5.4

Find the limit, if there exist, of the sequence $\{a_n\}$

1. $a_n = \dfrac{\cos^2 n}{3^n}$.
2. $a_n = \dfrac{\cos n}{n}$.
3. $a_n = \dfrac{(-1)^{n+1}}{n}$.
4. $a_n = \dfrac{\ln n}{n}$.
5. $a_n = \dfrac{\tan^{-1}(n)}{n}$.

Solution

1. We have $0 \le \cos^2(n) \le 1$, then $0 \le \dfrac{\cos^2(n)}{3^n} \le \dfrac{1}{3^n}$.
   $$\lim_{n\to\infty} \frac{1}{3^n} = \lim_{n\to\infty} \left(\frac{1}{3}\right)^n = 0.$$
   From the sandwich theorem, $\displaystyle\lim_{n\to\infty} \frac{\cos^2(n)}{3^n} = 0$.
2. We have $-1 \le \cos(n) \le 1$, then $-\dfrac{1}{n} \le \dfrac{\cos(n)}{n} \le \dfrac{1}{n}$.
   $$\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0.$$
   From the sandwich theorem, $\displaystyle\lim_{n\to\infty} \frac{\cos(n)}{n} = 0$.

Solution

3. $|a_n| = \left|\dfrac{(-1)^{n+1}}{n}\right| = \dfrac{1}{n}$.

   Since $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, so $\lim\limits_{n\to\infty} \dfrac{(-1)^{n+1}}{n} = 0$

4. Since $\lim\limits_{x\to\infty} \dfrac{\ln x}{x} = 0$, so $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$.

5. We have $-\frac{\pi}{2} \leq \tan^{-1}(n) \leq \frac{\pi}{2}$ then $-\dfrac{\pi}{2n} \leq \dfrac{\tan^{-1}(n)}{n} \leq \dfrac{\pi}{2n}$.

   $\lim\limits_{n\to\infty} -\dfrac{\pi}{2n} = \lim\limits_{n\to\infty} \dfrac{\pi}{2n} = 0$.

   From the sandwich theorem, $\lim\limits_{n\to\infty} \dfrac{\tan^{-1}(n)}{n} = 0$.