**Chapter- 3**

**Basic Concepts of Theory of Probability**

**Objectives:**

* To examine the use of probability theory in decision making;
* To explain the different ways probabilities arise; and
* To develop rules for calculating different types of probabilities.

**Chapter Contents:**

* Basic Terminology;
* Three Types of Probability;
* Probability Rules;
* Probabilities under Conditions of Statistical Independence; and
* Probabilities under Conditions of Statistical Dependence.

**Basic Terminology**

***Probability:***

Probability is the chance something will happen. It is expressed as fractions or as decimals between zero and one.

Assigning a probability of zero means that something can never happen; a probability of 1 indicates that something will be always happen.

***Event:***

In probability theory, an event is one or more of the possible outcomes of doing something. If we toss a coin, getting a tail would be an event, and getting a head would be another event.

***Experiment:***

The activity that produces such an event is called an experiment in probability theory. In a coin- toss experiment, what is the probability of the event head? The answer is ½ or 0.5.

***Sample Space:***

The set of all possible outcomes of an experiment is called the sample space for the experiment. . In a coin- toss experiment, the sample space is

S = {head, tail}

***Mutually exclusive events:***

Events are said to be mutually exclusive if one and only one of them can take place at a time. In a coin- toss experiment, for example, we have two possible outcomes, heads and tails. On any toss, either heads or tails may turn up, but not both. As a result, the events heads and tails on a single toss are said to be mutually exclusive.

**Types of Probability**

There are three basic ways of classifying probability which are as follow:

1. Classical approach of probability (Classical Probability);
2. Relative frequency approach;
3. Subjective approach.

**Classical probability**

It defines the probability that an event will occur as-

Probability of an event = $\frac{number of outcomes where the event occurs}{total number of possible outcomes}$

***Example:***

What is the probability of getting a head on one toss?

**Solution:** P (head) = $\frac{1}{1+1}$ = $\frac{1}{2}$ = 0.5

**Note:** *Classical probability is often called a priori probability because if we keep using orderly examples such as fair coins, unbiased dice, and standard decks of cards, we can state the answer in advance (a priori) without tossing a coin, rolling a die, or drawing a card.*

**Shortcomings of the classical probability:**

This approach is useful when we deal with card games, dice games, coin tosses, and the like, but has serious problems when we try to apply it to the less orderly decision problems we encounter in management.

**Relative frequency approach**

In the 1800s, British statisticians, interested in a theoretical foundation for calculating risk of losses in life insurance and commercial insurance, began defining probabilities from statistical data collected on births and deaths.

This method uses the relative frequencies of past occurrences as probabilities. We determine how often something has happened in the past and use that figure to predict the probability that it will happen again in the future. For example, suppose an insurance company knows from past actuarial data that of all males 40 years old, about 60 out of every 100,000 will die within a year period. Using this method, the company estimates the probability of death for that age group as

$\frac{60}{100,000}$ = 0.0006

Note: when we use the relative frequency approach to establish probabilities, our probability figure will gain accuracy as we increase the number of observations.

**Subjective Probabilities**

Subjective probabilities are based on the beliefs of the person making the probability assessment. In fact, subjective probability can be defined as the probability assigned to an event by an individual, based on whatever evidence is available. This evidence may be in the form of relative frequency of past occurrences, or it may be just an educated guess.

Subjective probability assignments are often found when events occur only once or at most a very few times.

**Probability Rules**

1. **Addition Rules; and**
2. **Multiplication Rules.**

Probability of Event A Happening = P(A)

**Marginal or Unconditional Probability:**

A single probability means that only one event can take place. It is called a marginal or unconditional probability. For example, let us suppose that 50 members of a college class drew tickets to see which student would get a free trip to the National Museum. Any one of the students could calculate his or her chances of winning as:

P (Winning) = $\frac{1}{50}$ = 0.02

In this case, a student’s chance is 1 in 50 because we are certain that the possible events are mutually exclusive, that is, only one student can win at a time.

**Addition Rule for Mutually Exclusive Events:**

Probability of Either A or B Happening **=** P (A or B) = P(A) + P(B)

***Example: Five equally capable students (A, B, C, D and E) are waiting for a summer internship in a company that has announced that it will hire only one of the five by random drawing. What is the probability that A will get internship in the company?***

**Solution:** P (A) = $\frac{1}{5}$ = 0.2

However, if we ask, “what is the probability that either A or B will get internship in the company?”

Probability of Either A or B will get Internship **=** P (A or B) = P(A) + P(B)

 = $\frac{1}{5}+ \frac{1}{5}$ = $\frac{2}{5}$ = 0.4

***Example: Following table gives data on the sizes of families in a certain town. What is the probability that a family chosen at random from this town will have four or more children?***

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| ***No. of Children*** | ***0*** | ***1*** | ***2*** | ***3*** | ***4*** | ***5*** | ***6 or more*** |
| ***Proportion of Families having this many children*** | ***0.05*** | ***0.10*** | ***0.30*** | ***0.25*** | ***0.15*** | ***0.10*** | ***0.05*** |

**Solution:** P(4,5, 6 or more) = P(4) +P(5) + P(6 or more)

 = 0.15 + 0.10 + 0.05

 = 0.30

**Note:** For any event A, either A happens or it doesn’t. So the events A and not A are exclusive and exhaustive.

P(A) + P(not A) = 1

Or, P(A) = 1- P(not A)

**Example: On the basis of the above table, what is the probability of a family having 5 or fewer children?**

**Solution:** P(0, 1, 2, 3, 4, 5) = P(0) +P(1) + P(2) + P(3) +P(4) + P(5)

 = 0.05 + 0.10 + 0.30 + 0.25 + 0.15 + 0.10

 = 0.95

**OR,**

P(0, 1, 2, 3, 4, 5) = 1- P(6 or more)

= 1- 0.05 = 0.95

**Addition Rule for Events that are Not Mutually Exclusive:**

P (A or B) = P (A) + P (B) – P (AB)

[Probability of A or B happening when A and B are not mutually exclusive = Probability of A happening + Probability of B happening – Probability of A and B happening together]

***Example: Determine the probability of drawing either an ace or a heart.***

**Solution: P(Ace or Heart) = P(Ace) + P(Heart) – P(Ace and Heart)**

 **=** $\frac{4}{52}+ \frac{13}{52}$ - $\frac{1}{52}$ = $\frac{16}{52}$ = $\frac{4}{13}$

***Example:*** The employees of a certain company have elected 5 of their number to represent them on the employee- management productivity council. Profiles of the 5 are as follows:

1. Male – age 30 years;
2. Male – age 32 years;
3. Female – age 45 years;
4. Female – age 20 years; and
5. Male – age 40 years.

This group decides to elect a spokesperson by drawing a name from a hat. *What is the probability the spokesperson will be either female or over 35?*

**Solution:** P(Female or over 35 years) = P(Female) + P(Over 35 years) – P(Female and Over 35)

 = $\frac{2}{5}+ \frac{2}{2}$ - $\frac{1}{5}$ = $\frac{3}{5}$

**Probabilities under Conditions of Statistical Independence**

When two events happen, the outcome of the first even may or may not have an effect on the outcome of the second event. That is, the events may be either dependent or independent. Here, we first examine events that are statistically independent. The occurrence of one event has no effect on the probability of the occurrence of any other event is called statistically independent probability.

There are three types of probabilities under statistical independence:

1. Marginal Probability;
2. Joint Probability; and
3. Conditional Probability.

***Marginal Probability under Statistical Independence:***

A marginal or unconditional probability is the simple probability of the occurrence of an event. In a fair coin toss, P(H) = 0.5, and P(T)= 0.5. This is true for every toss, no matter how many tosses have been made or what their outcomes have been. Every toss stands alone and in no way connected with any other toss. Thus, the outcomes of each toss of a fair coin is an event that is statistically independent of the outcomes of every other toss of the coin.

***Joint Probability under Statistical Independence:***

The probability of two or more independent events occurring together or in succession is the product of their marginal probabilities. Mathematically,

P(AB) = P(A) × P(B)

Where

P(AB) = probability of events A and B occurring together or in succession (this is known as a *joint probability*);

 P(A) = marginal probability of event A occurring; and

 P(B) = marginal probability of event B occurring.

In terms of the fair coin example, the probability of heads on two successive tosses is the probability of heads on first toss (H1) times the probability of heads on the second toss (H2). That is, *P(H1H2) = P(H1) × P(H2).* We have shown that the events are statistically independent, because the probability of any outcome is not affected by any preceding outcome. Therefore, the probability of heads on any toss is 0.5, and *P(H1H2) = 0.5 × 0.5 = 0.25.*

Likewise, the probability of getting heads on three successive tosses is *P*(*H1H2H3*) = *0.5×0.5×0.5=* 0.125.

We can make the probabilities of events even more explicit using a probability tree.

|  |  |  |
| --- | --- | --- |
| Probability tree of one toss | Probability tree of two tosses | Probability tree of third toss |
| http://flylib.com/books/3/287/1/html/2/images/11fig03.jpg |

|  |  |  |
| --- | --- | --- |
| **1 Toss** | **2 Tosses** | **3 Tosses** |
| Possible Outcomes | Probability | Possible Outcomes | Probability | Possible Outcomes | Probability |
| H1 | 0.5 | H1H2 | 0.25 | H1H2H3 | 0.125 |
| T1 | 0.5 | H1T2 | 0.25 | H1H2T3 | 0.125 |
|  | **1.0** | T1H2 | 0.25 | H1T2H3 | 0.125 |
|  |  | T1T2 | 0.25 | H1T2T3 | 0.125 |
|  |  |  | **1.00** | T1H2H3 | 0.125 |
|  |  |  |  | T1H2T3 | 0.125 |
|  |  |  |  | T1T2H3 | 0.125 |
|  |  |  |  | T1T2T3 | 0.125 |
| **The sum of the probabilities of all the possible outcomes must always equal 1.** | **1.000** |

**Example 1:** ***What is the probability of getting tails, heads, tails in that order on three successive tosses of a fair coin?***

**Solution:** *P(T1H2T3) = P(T1) × P(H2) × P(T3)= 0.5×0.5×0.5= 0.125*

**Example 2:** ***What is the probability of getting tails, tails, heads in that order on three successive tosses of a fair coin?***

**Solution:** *P(T1T2H3) = P(T1) × P(T2) × P(H3)= 0.5×0.5×0.5= 0.125*

**Example 3:** ***What is the probability of at least two heads on three tosses?***

**Solution:** Recalling that the probabilities of mutually exclusive events are additive, we can note the possible ways that at least two heads on three tosses can occur, and we can sum their individual probabilities. The outcomes satisfying the requirement are H1H2H3, H1H2T3, H1T2H3, and T1H2H3. Because each of these has an individual probability of 0.125, the sum is 0.5. Thus, the probability of at least two heads on three tosses is 0.5.

**Example 4:** ***What is the probability of at least one tail on three tosses?***

**Solution:** there is only one case in which no tails occur, namely H1H2H3. Therefore, we can simply subtract for the answer:

1. P(H1H2H3) = 1- 0.125 = 0.875

The probability of at least one tail occurring in three successive tosses is 0.875.

Example 5: what is the probability of at least one head on two tosses?

Solution: The possible ways at least one head may occur are H1H2, H1T2, T1H2. Each of these has a probability of 0.25. Therefore, the probability of at least one head on two tosses is 0.75. Alternatively, we could consider the case in which no head occurs- namely, T1T2- and subtract its probability from 1; that is,

1. P(T1T2) = 1- 0.25 = 0.75

***Conditional Probability under Statistical Independence***

Thus far, we have discussed two types of probabilities, marginal (or unconditional) probability and joint probability. Symbolically, marginal probability is P (A) and joint probability is P (AB). Besides these two, there is one other type of probability, known as conditional probability. Symbolically, conditional probability is written as- P (B│A) and is read as “the probability of *event B* given that *event A* has occurred.”

For statistically independent events, the conditional probability of even B given that event A has occurred is simply the probability of event B. That is-

P (B│A) = P(B)

Independent events are those whose probabilities are in no way affected by the occurrence of each other. Symbolically, P (B│A) = P(B).

***Summary of three types of probabilities under statistical independence:***

|  |  |  |
| --- | --- | --- |
| **Types of Probability** | **Symbol** | **Formula** |
| Marginal | P(A) | (PA) |
| Joint | P(AB) | P(A)×P(B) |
| Conditional | P (B│A) | P(B) |

**Probabilities under Conditions of Statistical Dependence**

Statistical dependence exists when the probability of some event is dependent on or affected by the occurrence of some other event. Just as with independent events, there are three types of probabilities under statistical dependence:

1. Conditional;
2. Joint; and
3. Marginal.

***Conditional Probabilities under Statistical Dependence:***

The formula for conditional probability under statistical dependence is given as:

P (B│A) = $\frac{P(BA)}{P(A)}$

Suppose we have a box containing 10 balls distributed as follows:

* 3 are coloured and dotted;
* 1 is coloured and striped;
* 2 are gray and dotted; and
* 4 are gray and striped.

 The probability of drawing any one ball from this box is 0.1, since there are 10 balls, each with equal probability of being drawn.

**Example 1:** ***Suppose someone draws a coloured ball from the box. What is the probability that it is dotted? What is the probability it is striped?***

**Solution:** P(D│C) = $\frac{P(DC)}{P(C)}$ = $\frac{0.3}{0.4}$ = 0.75

 P(S│C) = $\frac{P(SC)}{P(C)}$ = $\frac{0.1}{0.4}$ = 0.25

**Example 2:** ***On the basis of above example what is the probability of getting dotted ball given the probability of gray ball? What is the probability of getting striped ball given the probability of gray ball?***

**Solution:** P (D│G) = $\frac{P(DG)}{P(G)}$ = $\frac{0.2}{0.6}$ = $\frac{1}{3}$

 P (S│G) = $\frac{P(SG)}{P(G)}$ = $\frac{0.4}{0.6}$ = $\frac{2}{3}$

***Explanation:*** The total probability of gray is 0.6 (6 out of 10 balls). To determine the probability that the ball (which we know is gray) will be dotted, we divide the probability of gray and dotted (0.2) by the probability of gray (0.6), or 0.2/0.6 = 1/3. Similarly, to determine the probability that the ball will be striped, we divide the probability of gray and striped (0.4) by the probability of gray (0.6), or 0.4/0.6 = 2/3.

**Example 3:** ***Calculate* P (G│D) *and* P (C│D*) on the basis of above example.***

**Solution:** P (G│D) ***=*** $\frac{P(GD)}{P(D)}$ = $\frac{0.2}{0.5}$ = $\frac{2}{5}$ = 0.4

 P (C│D) ***=*** $\frac{P(CD)}{P(D)}$ = $\frac{0.3}{0.5}$ = $\frac{3}{5}$ = 0.6

**Example 4:** ***Calculate* P (C│S) *and* P (G│S) *on the basis of above example.***

**Solution:** P (C│S) ***=*** $\frac{P(CS)}{P(S)}$ = $\frac{0.1}{0.5}$ = $\frac{1}{5}$ = 0.2

 P (G│S) ***=*** $\frac{P(GS)}{P(S)}$ = $\frac{0.4}{0.5}$ = $\frac{4}{5}$ = 0.8

***Joint Probability under Statistical Dependence:***

The formula for calculation of joint probability for statistical dependence is given as-

P (BA) = P (B│A) × P (A)

i.e., *Joint probability of events B and A happening together or in succession = Probability of event B given that event A has happened × Probability that event A will happen.*

Converting the general formula P (BA) = P (B│A) × P (A) to our example and to the terms of coloured, gray, dotted and striped, we have P(CD) = P (C│D) × P (D) = 0.6×0.5 = 0.3. Here, 0.6 is the probability of coloured; given dotted and 0.5 is the probability of dotted.

The following joint probabilities are computed in the same manner and can also be verified by direct observation:

P (CS) = P (C│S) × P (S) = 0.2×0.5 = 0.1

P (GD) = P (G│D) × P (D) = 0.4×0.5 = 0.2

P (GS) = P (G│S) × P (S) = 0.8×0.5 = 0.4

***Marginal Probabilities under Statistical Dependence:***

Marginal probabilities under statistical dependence are computed by summing up the probabilities of all the joint events in which the simple event occurs. In the example above, we can compute the marginal probability of the event colored by summing the probabilities of the two joint events in which colored occurred:

P(C) = P(CD) + P(CS) = 0.3 + 0.1 = 0.4

Similarly, the marginal probability of the event gray can be computed by summing the probabilities of the two joint events in which gray occurred:

 P(G) = P(GD) + P(GS) = 0.2 + 0.4 = 0.6

In the same way, we can compute the marginal probability of the event dotted by summing the probabilities of the two joint events in which dotted occurred:

 P(D) = P(CD) + P(GD) = 0.3 + 0.2 = 0.5

And finally, the marginal probability of the event striped can be computed by summing the probabilities of the two joint events in which gray occurred:

 P(S) = P(CS) + P(GS) = 0.1 + 0.4 = 0.5

***Summary of three types of probabilities under statistical dependence:***

|  |  |  |  |
| --- | --- | --- | --- |
| **Types of Probability** | **Symbol** | **Formula under Statistical Independence** | **Formula under Statistical Dependence** |
| Marginal | P(A) | P(A) | Sum of the probabilities of the joint events in which A occurs. |
| Joint | P(AB) Or, P(BA) | P(A)×P(B)P(B)×P(A) | P(A⃒B)×P(B)P(B⃒A)×P(A) |
| Conditional | P(B⃒A)Or, P(A⃒B) | P(B)P(A) | $$\frac{P(BA)}{P(A)}$$$$\frac{P(AB)}{P(B)}$$ |

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