Parametric equations

- Parametric equation: x and y expressed in terms of a parameter t, for example, $x = a\cos t, y = b\sin t$
- A curve can be described by parametric equations $x=x(t)$, $y=y(t)$. Each value of t determines a point (x,y). So the parametric equations define a function.
- Typical parametric equations:

circle: $x = r\cos t, y = r\sin t, \quad 0 \le t \le 2\pi.$

ellipse: $x = a\cos t, y = b\sin t, \quad 0 \le t \le 2\pi$

Example

- **Ex**. Sketch the curve with parametric equations

$$x = \sin t, y = \sin^2 t.$$

- **Sol**. Observe that $y = x^2$ and $-1 \le x \le 1$ so the curve is part of the parabola. Since sin*t* is periodic, the point (*x*,*y*) moves back and forth infinitely often along the parabola, as *t* changes.

- **Ex**. A **cycloid** is defined by

$$x = r(\theta - \sin\theta), y = r(1 - \cos\theta), \quad \theta \in R$$

Derivative of functions defined by parametric equations

Suppose $y=y(x)$ is defined by the parametric equation $x=\varphi(t), y=\psi(t)$. Then

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\psi'(t)}{\varphi'(t)}.$$

Ex. Find an equation of the tangent line to the curve $\begin{cases} x = \ln(1+t^2) \\ y = 1 - \arctan t \end{cases}$ at the point $(\ln 2, 1-\frac{\pi}{4})$.

Sol. $$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{-\frac{1}{1+t^2}}{\frac{2t}{1+t^2}} = -\frac{1}{2t} \Rightarrow y'(1) = -\frac{1}{2}.$$

Question

Suppose $\begin{cases} x = \ln(\sin t) \\ y - e^y \sin t = 1 \end{cases}$, find $\dfrac{dy}{dx}$.

Sol. $\dfrac{dy}{dt} - (\sin t)e^y \dfrac{dy}{dt} - e^y \cos t = 0 \Rightarrow \dfrac{dy}{dt} = \dfrac{e^y \cos t}{1 - e^y \sin t}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{e^y \cos t}{1 - e^y \sin t}}{\frac{\cos t}{\sin t}} = \frac{e^y \sin t}{1 - e^y \sin t} = \frac{y-1}{2-y}.$$

Question

Find $\dfrac{d^2y}{dx^2}$ if $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$.

Sol. $$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{a \sin t}{a(1 - \cos t)} = \cot\frac{t}{2},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \frac{-\dfrac{1}{2}\csc^2\dfrac{t}{2}}{a(1 - \cos t)} = -\frac{1}{4a\sin^4\dfrac{t}{2}}.$$

Area formula

- When a function is in parametric form:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad (\alpha \le t \le \beta),$$

Then the area of the region bounded by the curve and x=a, x=b is

$$A = \int_a^b y dx = \int_\alpha^\beta y(t) dx(t) = \int_\alpha^\beta y(t) x'(t) dt.$$

Remark: in the formula, $(x(\alpha), y(\alpha))$ is the left endpoint

Example

- **Ex.** Find the area of the region bounded by $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$

- **Sol.** $x=a\cos t, y=b\sin t$

$$\Rightarrow A=4\int_0^a ydx=4\int_{\frac{\pi}{2}}^0 b\sin t(-a\sin t)dt=\pi ab.$$

Example

- Find the area under one arc of the **cycloid**

$$x = a(t - \sin t), y = a(1 - \cos t).$$

- **Sol.** $A = \int_0^{2\pi a} y dx = \int_0^{2\pi} a(1 - \cos t) d[a(t - \sin t)]$

$$= a^2 \int_0^{2\pi} (1 - \cos t)^2 dt = 3\pi a^2.$$

Example: volume

- **Ex.** Find the volume of the solid obtained by rotating about y-axis the region bounded by the cycloid $x = a(t - \sin t)$, $y = a(1-\cos t)$, $(0 \le t \le 2\pi)$ and y=0.
- **Sol.** $V = 2\pi\int_0^{2\pi a} xydx = 2\pi\int_0^{2\pi} a(t-\sin t)\cdot a(1-\cos t)d[a(t-\sin t)]$

$$= 2\pi a^3\int_0^{2\pi}(t-\sin t)(1-\cos t)^2 dt = \cdots = 6\pi^3 a^3.$$

Arc length formula

- If a smooth curve is defined by the parametric equation

$$x = x(t), y = y(t) \quad (\alpha \le t \le \beta)$$

we have $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$

$$\Rightarrow s = \int_{\alpha}^{\beta} \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt.$$

Example

- **Ex.** Find the length of one arc of the cycloid
  $$x = a(t - \sin t), y = a(1 - \cos t) \quad (0 \le t \le 2\pi).$$

- **Sol.** $s = \int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = a\int_0^{2\pi} \sqrt{2(1 - \cos t)}\, dt$

  $$= 2a\int_0^{2\pi} \sin\frac{t}{2}\, dt = 8a.$$

Example: surface area

- **Ex.** Find the area of the surface obtained by rotating the cycloid $x = a(t - \sin t), y = a(1 - \cos t)$ $(0 \le t \le 2\pi)$ about y-axis.

- **Sol.** $S = \int_0^{2\pi a} 2\pi x ds = \int_0^{2\pi} 2\pi x(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$

$$= 4\pi a^2 \int_0^{2\pi} (t - \sin t)\sin\frac{t}{2} dt = 4\pi a^2 \left[\int_0^{2\pi} t \sin\frac{t}{2} dt - \int_0^{2\pi} \sin t \sin\frac{t}{2} dt \right]$$

$$= 4\pi a^2 \left[-2t\cos\frac{t}{2} + 4\sin\frac{t}{2} - \frac{4}{3}\sin^3\frac{t}{2} \right]_0^{2\pi} = 16\pi^2 a^2$$

Polar coordinates

- A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates.
- The usual rectangular coordinate system, also called Cartesian coordinates, uses (x,y) to locate a point.
- In many situation, the **Polar coordinate system** is more convenient for some purposes. A point P is represented by an ordered pair (r,θ) where r is the distance from O to P, θ is the angle between the polar axis and the line OP.

Polar coordinates

- The polar coordinate for a point may not be unique: r can be negative, $(-r,\theta)$ and $(r,\theta+\pi)$ represent the same point.
- θ is not unique, (r,θ) can be represented by $(r,\theta+2n\pi)$
- An angle is positive if measured in the counterclockwise direction from the polar axis and negative clockwise.

- **Ex**. Plot the points whose polar coordinates are given

(a) $(1,5\pi/4)$ (b) $(2,3\pi)$ (c) $(2,-2\pi/3)$ (d) $(-3,3\pi/4)$

Polar coordinates

- Relationship between polar and Cartesian coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \quad \tan\theta = y/x$$

- Polar curves: $r = f(\theta)$
- Simple examples: $r=a$ is a circle centered origin with radius a; $\theta = \alpha$ is a line passing through origin with slope $\tan\alpha$

Polar curves

- **Ex.** Sketch the curve with polar equation $r = 2R\cos\theta$
- **Sol.**

$$x = 2R\cos^2\theta, y = 2R\cos\theta\sin\theta$$

$$\Rightarrow (x-R)^2 + y^2 = R^2$$

$$r = 2R\cos\theta = 2Rx/r \Rightarrow r^2 = 2Rx \Rightarrow (x-R)^2 + y^2 = R^2$$

- Question: $r = 2R\sin\theta$

Calculus in polar coordinate

- tangent line to a polar curve $r = f(\theta)$

$$x = f(\theta)\cos\theta, y = f(\theta)\sin\theta$$

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{r'(\theta)\sin\theta + r\cos\theta}{r'(\theta)\cos\theta - r\sin\theta}$$

- **Ex**. For the **cardioid** $r = 1+\sin\theta,$ (a) find the slope of the tangent line when $\theta = \pi/3$ (b) find the points on the cardioid where the tangent line is horizontal or vertical.

Area formula

- Area problem: the boundary of a region is given in polar coordinates: $r = r(\theta) \quad (\alpha \le \theta \le \beta).$ find the area A.

- Use **differential element** method. In the total interval $[\alpha, \beta]$,

take any element $[\theta, \theta + d\theta]$, then the sub-area corresponding to this element is $\Delta A \approx \frac{1}{2} r^2(\theta) d\theta = dA$, because $d\theta$ is small, $r(\theta)$ is approximately a constant and thus the sub-region a sector.

Therefore, $A = \int_\alpha^\beta dA = \frac{1}{2}\int_\alpha^\beta r^2(\theta) d\theta.$

Area formula

- Alternatively, by the relationship between polar coordinate and Cartesian coordinate, the polar coordinate equation can be converted to parametric equation:

$$x = r(\theta)\cos\theta, \quad y = r(\theta)\sin\theta.$$

Example

- Find the area of the region bounded by the cardioid $r = a(1+\cos\theta)$.
- **Sol I** Sketch the graph first. By symmetry,

$$A = 2\cdot\frac{1}{2}\int_0^{\pi} r^2(\theta)d\theta = a^2\int_0^{\pi}(1+\cos\theta)^2 d\theta$$

$$= a^2\int_0^{\pi}(1+2\cos\theta+\frac{1+\cos 2\theta}{2})d\theta = \frac{3a^2}{2}\pi.$$

- **Sol II** The parametric equation is $x = a(1+\cos\theta)\cos\theta$, $y = a(1+\cos\theta)\sin\theta$. So

$$A = 2\int_0^{2a} ydx = -2\int_{\pi}^{0} a^2(1+\cos\theta)(1+2\cos\theta)\sin^2\theta d\theta = \frac{3a^2}{2}\pi$$

Example

- **Ex.** Find the area of the region bounded by the **two-leaved rose** $r^2 = a^2 \cos 2\theta$.
- **Sol** Sketch the graph first. By symmetry,

$$A = 4 \cdot \frac{1}{2} \int_0^{\pi/2} a^2 \cos 2\theta d\theta = a^2.$$

Arc length

- To find the length of a polar curve $r = f(\theta) \quad (\alpha \le \theta \le \beta)$, we regard θ as a parameter and write the parametric equation of the curve as

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta.$$

- So the arc length is computed as follows

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta, \quad \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$\Rightarrow s = \int_\alpha^\beta \sqrt{[x'(\theta)]^2 + [y'(\theta)]^2}\, d\theta = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta.$$

Example

- **Ex.** Find the length of the cardioid $r = 1 + \sin\theta$.
- **Sol.**

$$s = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{2+2\sin\theta}\, d\theta$$

(I) $$= \int_0^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\, d\theta = -2\sqrt{2-2\sin\theta}\Big|_0^{2\pi} = 0$$

$$= \int_0^{2\pi} \frac{2|\cos\theta|}{\sqrt{2-2\sin\theta}}\, d\theta = 8$$

(II) $$2+2\sin\theta = 2\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2(\sin\frac{\theta}{2} + \cos\frac{\theta}{2})^2$$