The Vector Spaces

Mongi BLEL

King Saud University

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Table of contents

Introduction to Vector Spaces

Definition

A non empty set $\mathbb{V}$ is called a vector space on $\mathbb{R}$ if:

1. For any u and v in $\mathbb{V}$, $u + v$ is also in $\mathbb{V}$;
2. For any u, v, w in $\mathbb{V}$, $u + (v + w) = (u + v) + w$.
3. There is an element in $\mathbb{V}$ called the zero or null vector, which we denote by 0, such that for all u in $\mathbb{V}$ we have $0 + u = u$.
4. For every u in $\mathbb{V}$, there is a vector called the negative of u and denoted u, such that $u + u = 0$.
5. For any u and v in $\mathbb{V}$, $u + v = v + u$.

1. If k is any scalar in $\mathbb{R}$ and u is any vector in $\mathbb{V}$, then ku is a vector in $\mathbb{V}$.
2. For any scalar k in $\mathbb{R}$ and any vectors u and v in $\mathbb{V}$, $k(u+v) = ku + kv$.
3. For any scalars k and ℓ in $\mathbb{R}$ and any vector u in $\mathbb{V}$, $(k+\ell)u = ku + \ell u$.
4. For any scalars k and ℓ in $\mathbb{R}$ and any vector u in $\mathbb{V}$, $k(\ell u) = (k\ell)u$.
5. For any vector u in $\mathbb{V}$, $1u = u$.

Examples

1. $\mathbb{R}^n$ is a vector space .
2. The set $\{(x, y, 2x + 3y);\ x, y \in \mathbb{R}\}$ is a vector space .
3. The set of polynomials $\mathcal{P} = \mathbb{R}[X]$ is a vector space .
   Also the set of polynomials of degree less then n, $\mathcal{P}_n = \mathbb{R}_n[X]$ is a vector space .

Some basic identities in a vector space

Theorem

Let V be a vector space. The following statements are always true.

1. $0u = 0$;
2. $k0 = 0$;
3. $(1)u = u$;
4. If $ku = 0$, then $k = 0$ or $u = 0$.

The Vector Sub-Spaces

Definition

Let $(V, +, .)$ be a vector space and F a nonempty subset of V. F is called a sub-space of V if F is itself a vector space with the same operations of the vector space V.

Theorem

Let $(V, +, .)$ be a vector space and F be a nonempty subset of V. F is a sub-space of V if and only if

1. For any u and v in F, $u + v$ is also in F;
2. For any k in $\mathbb{R}$ and any vector u in F, the vector ku is in F.

Examples

1. The set $F = \{\begin{pmatrix} a & b \\ 0 & 2a-b \end{pmatrix};\ a, b \in \mathbb{R}\}$ is a sub-space of $V = M_2(\mathbb{R})$.
2. Let $A \in M_{m,n}(\mathbb{R})$ be a matrix and $F = \{X \in \mathbb{R}^n;\ AX = 0\}$. F is sub-space of $V = \mathbb{R}^n$. (F is the set of solutions of the homogeneous system $AX = 0$).
3. The set $F = \{(x, x+1);\ x \in \mathbb{R}\}$ is not a sub-space of $\mathbb{R}^2$ since $(0,0) \notin F$.

Example

The set $W = \{A \in M_n : \ A = -A^T\}$ is a sub-space of $M_n(\mathbb{R})$.
Indeed: if $A, B \in W$ and $\lambda \in \mathbb{R}$

$$(A+B)^T = A^T + B^T = -A - B$$

and

$$(\lambda A)^T = \lambda A^T = -\lambda A.$$

Then W is a sub-space of M_n.

Example

The set $E = \{(x, y) \in \mathbb{R}^2;\ xy = 0\}$ is not a sub-space since $(1, 0) \in E$ and $(0, 1) \in E$ but $(1, 0) + (0, 1) = (1, 1) \notin E$.

Definition

Let V be a vector space and let $v_1, \ldots, v_n$ be a finite vectors in V. We say that a vector $w \in V$ is a linear combination of the vectors $v_1, \ldots, v_n$ if there is $x_1, \ldots, x_n \in \mathbb{R}$ such that

$$w = x_1 v_1 + \ldots + x_n v_n.$$

Example

The vector $(4, 1, 1)$ is a linear combination of the vectors $(1, 0, 2)$,$(2, -1, 3)$, $(0, -1, 1)$ because

$$(4, 1, 1) = -2(1, 0, 2) + 3(2, -1, 3) - 4(0, -1, 1).$$

Example

The vector $(1, 1, 2)$ is not a linear combination of the vectors $(1, 0, 2)$, $(0, -1, 1)$ because the linear system $(1, 1, 2) = x(1, 0, 2) + y(0, -1, 1)$ don't have a solution.

Theorem

Let A be the matrix of type (m, n) and let $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be the matrix of type $(n, 1)$. If $C_1, \ldots, C_n$ are the columns of the matrix A, then

$$AX = x_1 C_1 + \ldots + x_n C_n.$$

Corollary

Let A be a matrix of type (m, n).
The linear system $AX = B$ is consistent if and only if the matrix B is a linear combination of the columns of the matrix A.

Definition

Let $S = \{v_1, \ldots, v_n\}$ be a set of vectors in a vector space V.
We say that the vector space V is generated (or spanned) by the set S if any vector in V is a linear combination of the vectors $v_1, \ldots, v_n$. (We say also that S is a spanning set of V).

Theorem

Let $S = \{v_1, \ldots, v_n\} \subset \mathbb{R}^m$ and A the matrix of type (m, n) with columns $v_1, \ldots, v_n$.
The set S spans the vector space $\mathbb{R}^m$ if and only if the system $AX = B$ is consistent for all $B \in \mathbb{R}^m$.

Example

Determine whether the vectors $v_1 = (1, -1, 4)$, $v_2 = (-2, 1, 3)$, and $v_3 = (4, -3, 5)$ span $\mathbb{R}^3$.
We solve the following linear system $AX = B$, where
$A = \begin{pmatrix} 1 & -2 & 4 \\ -1 & 1 & -3 \\ 4 & 3 & 5 \end{pmatrix}$, $B = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ for arbitrary $a, b, c \in \mathbb{R}$.

A reduced of the augmented matrix is given by:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -a-2b \\ 0 & 1 & -1 & -a-b \\ 0 & 0 & 0 & 7a+11b+c \end{array}\right].$$

This system has a solution only when $7a + 11b + c = 0$. Thus, the vectors do not span $\mathbb{R}^3$.

Example

Determine whether the vectors $v_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$, span the vector space $F = \{\begin{pmatrix} a & b \\ 0 & 2a-b \end{pmatrix}\ ;\ a, b \in \mathbb{R}\}$.

$$\begin{pmatrix} a & b \\ 0 & 2a-b \end{pmatrix} = xv_1 + yv_2 \iff \begin{cases} x+2y & = & a \\ x+y & = & b \\ x+3y & = & 2a-b \end{cases}.$$

This system has the unique solution $x = 2b - a$ and $y = a - b$.

Theorem

Let $S = \{v_1, \ldots, v_n\}$ be a set of vectors in a vector space V, then

1. the set W of linear combinations of the vectors of S is a linear sub-space in V.
2. W is the smallest sub-space of V which contains S.
   This sub-space is called the sub-space generated (or spanned) by the set S and denoted by $\langle S \rangle$ or $\mathrm{Vect}(S)$.

Example

Let $F = \{\begin{pmatrix} a & b \\ 0 & 2a-b \end{pmatrix};\ a, b \in \mathbb{R}\}$.

$\begin{pmatrix} a & b \\ 0 & 2a-b \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$. Then F is the sub-space of $V = M_2(\mathbb{R})$ spanned by $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\right\}$.

Example

In $\mathbb{R}^4$ is the vectors $(a, 1, b, 1)$ and $(a, 1, 1, b)$ are linear combination of the vectors $e_1 = (1, 2, 3, 4)$ and $e_2 = (1, -2, 3, -4)$.
The vector $(a, 1, b, 1) \in \mathrm{Vect}(e_1, e_2)$ if and only if the linear system $AX = B$ is consistent with $A = \begin{pmatrix} 1 & 1 \\ 2 & -2 \\ 3 & 3 \\ 4 & -4 \end{pmatrix}$ and $B = \begin{pmatrix} a \\ 1 \\ b \\ 1 \end{pmatrix}$.
The system is not consistent because the second and the forth equations can not be true in the same time. $((2a - 2b = 1,\ 4a - 4b = 1))$

The vector $(a, 1, 1, b) \in \mathrm{Vect}(e_1, e_2)$ if and only if the linear system $AX = B$ is consistent with $A = \begin{pmatrix} 1 & 1 \\ 2 & -2 \\ 3 & 3 \\ 4 & -4 \end{pmatrix}$ and $B = \begin{pmatrix} a \\ 1 \\ 1 \\ b \end{pmatrix}$.

The system has a unique solution and in this case $a = \dfrac{1}{3}$ and $b = 2$.

Example

Let E be the vector sub-space of $\mathbb{R}^3$ generated by the vectors $(2, 3, -1)$ and $(1, -1, -2)$ and let F be the sub-space of $\mathbb{R}^3$ generated by the vectors $(3, 7, 0)$ and $(5, 0, -7)$.
The sub-spaces E and F are equal.

$$\begin{cases} 2x + y = a \\ 3x - y = b \\ -x - 2y = c \end{cases}$$

This system is equivalent with the following system

$$\begin{cases} x + 2y = -c \\ -3y = a + 2c \\ -7y = b + 3c \end{cases}.$$

This system is consistent if and only if $7a - 3b + 5c = 0$.

We remark that the vectors $(2, 3, -1)$ and $(1, -1, -2)$ are solutions of the system, then $F \subset E$.
With the same method, the vectors $(2, 3, -1)$ and $(1, -1, -2)$ are in the sub-space F. This proves that $E = F$.

Example

Is there $a, b \in \mathbb{R}$ such that the vector $v = (-2, a, b, 5)$ is in the sub-space of $\mathbb{R}^4$ generated by the vectors $u = (1, -1, 1, 2)$ and $v = (-1, 2, 3, 1)$.

Solution

The vector $v = (-2, a, b, 5)$ is in the sub-space of $\mathbb{R}^4$ generated by the vectors $u = (1, -1, 1, 2)$ and $v = (-1, 2, 3, 1)$ if the following linear system is consistent $AX = B$, where

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 1 & 3 \\ 2 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 \\ a \\ b \\ 5 \end{pmatrix}.$$

This system is consistent if and only if $3 = a - 2 = \frac{b+2}{4}$.
Then $a = 5$ and $b = 10$.

Definition

A set of vectors $v_1, \ldots, v_n$ in a vector space V are called linearly independent if the equation

$$x_1 v_1 + \ldots, + x_n v_n = 0$$

has 0 as unique solution.

Example

The vectors $u = (1, 1, -2)$, $v = (1, -1, 2)$ and $w = (3, 0, 2)$ are linearly independent in $\mathbb{R}^3$.

$$xu + yv + zw = (0, 0, 0) \iff \begin{cases} x + y + 3z &= 0 \\ x - y &= 0 \\ -2x + 2y + 2z &= 0 \end{cases}.$$

This system has 0 as unique solution.

The matrix of this system is $\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 0 \\ -2 & 2 & 2 \end{pmatrix}$ and its determinant is -4.

Example

The set of vectors $\{P_1 = 1+x+x^2,\ P_2 = 2-x+3x^2, P_3 = x-x^2\}$ is linearly independent in $\mathcal{P}_2$.

$aP_1+bP_2+cP_3 = 0 \iff (a+2b)+(a-b+c)x+(a+3b-c)x^2 = 0 \iff \left\{\begin{array}{ccc} a+2b & = & 0 \\ a-b+c & = & 0 \\ a+3b-c & = & 0 \end{array}\right..$

Definition

We say that the vectors $v_1, \dots, v_n$ in a vector space V are linearly dependent if they are not linearly independent.

Example

The vectors $u = (0, 1, -2, 1)$, $v = (1, 0, 2, -1)$ and $w = (3, 2, 2, -1)$ are linearly dependent in $\mathbb{R}^4$.

$$xu + yv + zw = (0,0,0,0) \iff \begin{cases} y + 3z &= 0 \\ x + 2z &= 0 \\ -2x + 2y + 2z &= 0 \\ x - y - z &= 0 \end{cases}.$$

This system has infinite solutions.

The extended matrix of this system is $\left[\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ -2 & 2 & 2 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$ and the reduced row form of this matrix is : $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

Theorem

Let $S = \{v_1, \ldots, v_n\}$ be a set of vectors in a vector space V, with $n \geq 2$.
The set S is linearly independent if and only if no vector in the set is a linear combination of the other vectors in the set.
The set S is linearly dependent if and only if one of the vectors in the set is a linear combination of the other vectors in the set.

Theorem

Let $S = \{v_1, \ldots, v_n\} \subset \mathbb{R}^m$ and A the matrix of type (m, n) such that its columns are the vectors of S.
The set S is linearly independent if and only if the homogeneous system $AX = 0$ has 0 as unique solution.

Examples

1. If A is a matrix of type (m, n) with $m < n$. Then the homogeneous system $AX = 0$ has an infinite solutions.
2. If $S = \{v_1, \ldots, v_n\} \subset \mathbb{R}^m$ with $m < n$, then the set S is linearly dependent.

Bases and Dimension

Definition

Let $S = \{v_1, \ldots, v_n\}$ be a set of vectors in a vector space V.
We say that S is a basis of the vector space V if :

1. The set S generates the vector space V
2. The set S is linearly independent.

Theorem

If $S = \{v_1, \ldots, v_n\}$ is a basis of the vector space V.
Any vector $v \in V$ can be written uniquely as a linear combination of vectors in the basis S.

Remark

Let $S = \{e_1, \dots, e_n\}$ be the set of the vectors in the vector space $\mathbb{R}^n$, where

$$e_1 = (1, 0, \dots, 0), \dots, e_k = (0, \dots, 0, 1, 0, \dots 0), \dots, e_n = (0, \dots, 0, 1).$$

The set S is a basis of $\mathbb{R}^n$ and is called the natural basis of $\mathbb{R}^n$.

Exercise

Prove that $S = \{1, X, \dots, X^n\}$ is a basis of the vector space $\mathcal{P}_n$.

Example

Let $v_1 = (\lambda, 1, 1)$, $v_2 = (1, \lambda, 1)$ and $v_3 = (1, 1, \lambda)$.
Find the values of $\lambda \in \mathbb{R}$ such that $\{v_1, v_2, v_3\}$ is a basis of the vector space $\mathbb{R}^3$.

Solution

The set $\{v_1, v_2, v_3\}$ is linearly independent if 0 the unique solution of the equation

$$xv_1 + yv_2 + zv_3 = 0.$$

This is equivalent that the following matrix has an inverse :

$$A = \begin{pmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{pmatrix}.$$

Then $\lambda \notin \{-2, 1\}$.

The set $\{v_1, v_2, v_3\}$ generates the vector space $\mathbb{R}^n$ because the linear system $AX = B$ is consistent for all $B \in \mathbb{R}^n$ since the matrix A has an inverse .

Theorem

Let $S = \{v_1, \dots, v_n\}$ be a basis of the vector space V and let $T = \{u_1, \dots, u_m\}$ be a set of vectors.
If $m > n$, then T is linearly dependent .

Corollary

All bases of a finite dimensional vector space have the same number of elements.

Definition

If $S = \{v_1, \ldots, v_n\}$ is a basis of the vector space V then the number of vectors n of S is called the dimension of the vector space V and denoted by: $\dim V = n$.

Theorem

Let V is a vector space of dimension n. If $S = \{v_1, \ldots, v_n\}$ in V. Then
S is linearly independent if and only if S generates the vector space V and this is equivalent also with S is a basis of V.

Theorem

If $S = \{v_1, \ldots, v_n\}$ generates the vector space V, then it contains a basis of the vector space V.

Remark

If $S = \{v_1, \ldots, v_m\} \subset \mathbb{R}^n$ is a set of vectors and F the vector sub-space generated by S. We have the following two algorithms to construct a basis of F.

First Algorithm

1. Construct the matrix A such that its rows are the vectors of S
2. The non zeros rows of any row echelon form of the matrix A are a basis of the vector space $F = \langle S \rangle$.

Second Algorithm

1. Construct the matrix A such that its columns are the vectors of S
2. Take any row echelon form C of the matrix A.
3. Let $C_{k_1}, \dots C_{k_p}$ be the columns which contain a leading number and $k_1 < \dots < k_p$. Then $\{v_{k_1}, \dots, v_{k_p}\}$ is a basis of the vector space $F = \langle S \rangle$.

Theorem

1. If $S = \{v_1, \ldots, v_n\}$ is a generating system of the vector space V, then S contains a basis of V.
2. If $S = \{v_1, \ldots, v_n\}$ is a linearly independent system of vectors in the vector space V, then there is a basis T of V which contain the set S.

Example

Let W be the sub-space of $\mathbb{R}^5$ generated by the set of following vectors: $S = \{v_1 = (1, 0, 2, -1, 2), v_2 = (2, 0, 4, -2, 4), v_3 = (1, 2, -1, 2, 0), v_4 = (1, 4, -4, 5, -2).\}$

1. Find a basis of the sub-space W in $\{v_1, v_2, v_3, v_4\}$.
2. Find a basis of $\mathbb{R}^5$ and contains $\{v_1, v_3\}$.

Solution

1. Consider the matrix $A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 2 & 4 & -1 & -4 \\ -1 & -2 & 2 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$ with columns the coordinates of the vectors v_1, v_2, v_3, v_4 in the standard basis of $\mathbb{R}^5$.

   The reduced row form the matrix A is $\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

   Then $\{v_1, v_3\}$ is a basis of the sub-space W.

2. If $e_1 = (1, 0, 0, 0, 0)$, $e_2 = (0, 1, 0, 0, 0)$, $e_3 = (0, 0, 1, 0, 0)$. Then $\{v_1, v_3, e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^5$ and contain $\{v_1, v_3\}$.

Example

Let $W = \{(x, y, z, t) \in \mathbb{R}^4;\ 2x + y + z = 0,\ x - y + z = 0\}$

1. Prove that W is a sub-space of $\mathbb{R}^4$
2. Find a basis of the sub-space W.

Solution

1. $u = (x, y, z, t) \in W \iff AX = 0$, where

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix} \text{ and } X = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}.$$

Since the set of solutions of an homogeneous linear system is a vector sub-space, then W is vector sub-space of $\mathbb{R}^4$.

2. $AX = 0 \iff \begin{cases} 2x + y + z = 0 \\ x - y + z = 0 \end{cases} \iff \begin{cases} x = -2y \\ z = 3y \end{cases}$

$$\iff X = y \begin{pmatrix} -2 \\ 1 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then $\{(-2, 1, 3, 0), (0, 0, 0, 1)\}$ is a basis of the vector sub-space W.

Coordinate System and Change of Bases

Definition

If $S = \{v_1, \dots, v_n\}$ is a basis of the vector space V and if $v \in V$ such that

$$v = x_1 v_1 + \dots x_n v_n$$

then $(x_1, \dots x_n)$ are called the system of coordinates of the vector v in the basis S. We denote $[v]_S = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Theorem

If $B = \{v_1, \ldots, v_n\}$ and $C = \{u_1, \ldots, u_n\}$ are two bases of the vector space V. The matrix ${}_CP_B \in M_n(\mathbb{R})$ with columns $[v_1]_C, \ldots, [v_n]_C$ is called the change of bases matrix from the basis B to the basis C. This matrix ${}_CP_B$ is invertible, ${}_CP_B^{-1} = {}_BP_C$ and

$$[v]_C = {}_CP_B[v]_B, \quad \text{for all } v \in V.$$

(${}_BP_C$ is the change of bases matrix from the basis C to the basis B.)

Example

Let $B = \{v_1 = (0, 1, 1), v_2 = (1, 0, -2), v_3 = (1, 1, 0)\}$ be a basis of the vector space $\mathbb{R}^3$ and let $C = \{u_1 = (1, 0, 0), u_2 = (0, 1, 0), u_3 = (0, 0, 1)\}$ be the standard basis of $\mathbb{R}^3$.

We have ${}_C P_B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix}$ and ${}_B P_C = \begin{pmatrix} -2 & 2 & -1 \\ -1 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}$. If $[v]_C = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, then $[v]_B = {}_B P_C [v]_C = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Example

Consider B and C two bases of vector space V such that the matrix ${}_CP_B = \begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & 1 \\ 1 & 2 & -2 \end{pmatrix}$. Let u be a vector in V such that $[u]_C = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then $[u]_B = {}_BP_C[X]_C = {}_CP_B^{-1}[X]_C = \frac{1}{3}\begin{pmatrix} 6y+3z \\ -x+z \\ -x+3y+z \end{pmatrix}$.

Definition

Let A be a matrix of type (m, n).
The vector sub-space of $\mathbb{R}^n$ spanned by the rows of the matrix A is called the row vector space of the matrix A and denoted by: $\operatorname{row}(A)$.
The vector sub-space of $\mathbb{R}^m$ spanned by the columns of the matrix A is called the column vector space of the matrix A and denoted by: $\operatorname{col}(A)$.

Theorem

Let A be a matrix and B a matrix which is a result of some row operations on the matrix A, then $\mathrm{row}(A) = \mathrm{row}(B)$.

Theorem

Let A be a matrix and B any row echelon form of the matrix A. Then the set of non zero rows of the matrix B is a basis of the row vector space of A.

Definition

The dimension of the row vector space of a matrix A is called the rank of A.

$$\operatorname{rank}(A) = \dim(\operatorname{row}(A)).$$

Remark

The rank of a matrix A is the numbers of leading numbers in any row echelon form of the matrix A.

Theorem

For any matrix A:

$$\mathrm{rank}(A) = \dim(\mathrm{row}(A)) = \dim(\mathrm{col}(A)).$$

Corollary

For any matrix A:

$$\mathrm{rank}(A) = \mathrm{rank}(A^T).$$

Corollary

If A is a matrix of type (m, n) and P is any invertible matrix of type m and Q an invertible matrix of type n, then

$$\operatorname{rank}(A) = \operatorname{rank}(PAQ).$$

Proof

There exist some elementary matrices $E_1, \dots, E_p$ of order m such that $P = E_1 \dots E_p$.

We know that if E is a elementary matrix which corresponds to an elementary row operation R, then EA is the result of this elementary row operation R on the matrix A. Hence

$$\operatorname{rank}(A) = \operatorname{rank}(PA).$$

Also $\operatorname{rank}(PAQ) = \operatorname{rank}(PAQ)^T = \operatorname{rank}(Q^T A^T P^T) = \operatorname{rank}(A^T P^T) = \operatorname{rank}(PA) = \operatorname{rank}(A)$.

Theorem

If A is a matrix of type (m, n). We have the equivalence of the following statements:

1. The homogeneous system $AX = 0$ has 0 as unique solution.
2. The columns of the matrix A are linearly independent .
3. $\mathrm{rank}(A) = n$.
4. The matrix $A^T A$ has an inverse.

Theorem

Let A be a matrix of type (m, n). We have the equivalence of the following statements

1. The system $AX = B$ is consistent for all $B \in \mathbb{R}^m$.
2. The columns of the matrix A generates the vector space $\mathbb{R}^m$.
3. $\mathrm{rank}(A) = m$.
4. The matrix AA^T has an inverse.

Definition

Let A be a matrix of type (m, n). The vector sub-space $\{X \in \mathbb{R}^n;\ AX = 0\}$ is called the nullspace of the matrix A and denoted by: $N(A)$. The dimension of $N(A)$ is denoted by $\mathrm{nullity}(A)$.
Also the vector sub-space $\{AX;\ X \in \mathbb{R}^n\}$ is called the image (or the range) of the matrix A and denoted by: $\mathrm{Im}(A)$.

Theorem

For any matrix A, $\mathrm{Im}(A) = \mathrm{col}(A)$.

Rank-Nullity Theorem

For any matrix A, $\mathrm{nullity}(A) + \mathrm{rank}(A) = n$.

Finding bases for the null space, row space and column space of a matrix

Given a matrix A of type (m, n)

1. Reduce the matrix A to the reduced row echelon form R.
2. Solve the system $RX = 0$. Find a basis for the solutions space. The same basis for the solution space of $RX = 0$ is a basis for the null space of A.
3. Consider the non-zero rows of R. They form a basis for the row space of R. The same basis for the row space of R is a basis for the row space of A.
4. Take the columns of R with leading 1^s. They form a basis for the column space of R. The corresponding column vectors in A form a basis for the column space of A.

Example

Let the matrix $A = \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$

1. Find a basis of the vector space $N(A)$.
2. Find a basis of the vector space $\mathrm{Col}(A)$.
3. Find the rank of the matrix A.

Solution

The reduced row form the matrix A is $\begin{pmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

1. $(-3, 2, 1, 0)$, $(-5, 3, 0, 1)$ is basis of the vector space $N(A)$ ..
2. $(0, 1, 2, 1)$, $(-1, 2, 3, 1)$ is a basis of the vector space $\mathrm{Col}(A)$.
3. The rank of the matrix A is 2.

Example

Let $e_1 = (0,1,-2,1)$, $e_2 = (1,0,2,-1)$, $e_3 = (3,2,2,-1)$, $e_4 = (0,0,1,0)$ and $e_5 = (0,0,0,1)$ vectors in $\mathbb{R}^4$.
Is the following statements are true?

1. $\mathrm{Vect}\{e_1, e_2, e_3\} = \mathrm{Vect}\{(1,1,0,0), (-1,1,-4,2)\}$.
2. $(1,1,0,0) \in \mathrm{Vect}\{e_1, e_2\} \cap \mathrm{Vect}\{e_2, e_3, e_4\}$.
3. $\mathrm{Vect}\{e_1, e_2\} + \mathrm{Vect}\{e_2, e_3, e_4\} = \mathbb{R}^4$.

Solution

1. Let the matrix A which rows are the vectors e_1, e_2, e_3.
   The vector space $\mathrm{Vect}\{e_1, e_2, e_3\}$ is the row vector space of the matrix A.
   The reduced row form of the matrix A is
   $$A_1 = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
   Then $\dim \mathrm{Vect}\{e_1, e_2, e_3\} = 2$.
   We have $\mathrm{Vect}\{e_1, e_2, e_3\} = \mathrm{Vect}\{(1, 1, 0, 0), (-1, 1, -4, 2)\}$ if and only if the rank of the following matrix B is 2

$$B = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -2 & 1 \\ -1 & 1 & -4 & 2 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

The reduced row form of the matrix B is $A_2 = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Then
$\text{Vect}\{e_1, e_2, e_3\} = \text{Vect}\{(1,1,0,0), (-1,1,-4,2)\}$.

2. $(1,1,0,0) = e_1 + e_2$, $2(1,1,0,0) = e_3 - e_2$.
   Then $(1,1,0,0) \in \mathrm{Vect}\{e_1, e_2\} \cap \mathrm{Vect}\{e_2, e_3, e_4\}$.
3. $(1,1,0,0) \in \mathrm{Vect}\{e_1, e_2\} \cap \mathrm{Vect}\{e_2, e_3, e_4\}$ and
   $e_2 \in \mathrm{Vect}\{e_1, e_2\} \cap \mathrm{Vect}\{e_2, e_3, e_4\}$.
   Then $\dim \mathrm{Vect}\{e_1, e_2\} \cap \mathrm{Vect}\{e_2, e_3, e_4\} = 2$ and

$$\dim \mathrm{Vect}\{e_1, e_2\} + \mathrm{Vect}\{e_2, e_3, e_4\} \leq 3$$

Then $\mathrm{Vect}\{e_1, e_2\} + \mathrm{Vect}\{e_2, e_3, e_4\} \neq \mathbb{R}^4$.