Chapter 2

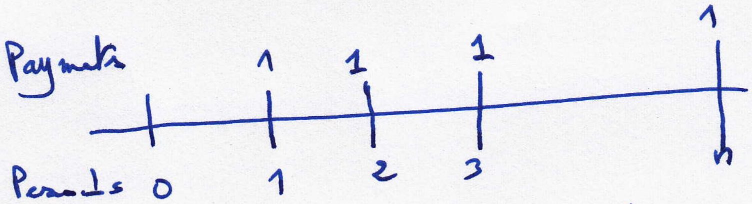
Annuities

The objective of this chapter is to study the different type of annuitys and their applications in investment or loans

Annuity:
- Annuity Certain (Level annuity) : fixed nb of payments
- Contingent annuity : The number of payments depend on some other events → probability

1) Level Annuity - Immediate

An annuity immediate is a regular series of payments at the end of every period.



. The present value $a_{\overline{n|}}$ of the annuity is the sum of the present value of each of the n payments:

$$v = \frac{1}{1+r}$$

$$a_{\overline{n|}} = \underset{\text{Pr } 1^{st} \text{ payment}}{\underset{\downarrow}{v}} + \underset{\text{PV of } 2^{nd} \text{ pay}}{\underset{\downarrow}{v^2}} + v^3 + \dots + v^n = \sum_{k=1}^{n} v^k$$

$$a_{\overline{n|}} = v \, \frac{1-v^n}{1-v} \quad \text{or} \quad \frac{v}{1-v} = \frac{1}{r}$$

$$\Rightarrow \boxed{a_{\overline{n|}} = \frac{1-v^n}{r}}$$

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$$\Rightarrow \left(1+\frac{r}{m}\right)^m < e^r$$

$\Rightarrow$ Conclusion

<u>Note</u> For continuous compounding

$$r_e = e^{r} - 1$$

<u>Ex</u> Suppose P is deposited with annual rate of 6.15 compounded continuous. Then find r_e

$$r_e = e^{0.0615} - 1$$

$$= 6.34\ \%$$

=

<u>Note</u>

$$V(t) = Pe^{rt}$$

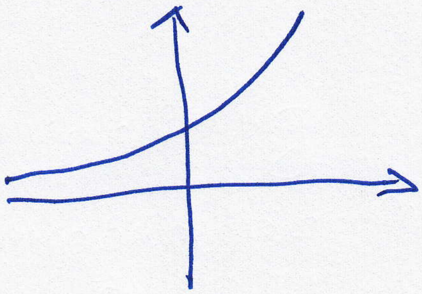
$$V'(t) = Pre^{rt} = \cancel{E}e^{r}$$

$$= r\,V(t)$$

$\Rightarrow$

(1) rate of change of the future value is proportional to the value itself

Continuous compounding : $V(t) = P e^{rt}$



Proposition

Continuous compounding produces higher future value than periodic compounding with any frequency m

Proof $P e^{rt} \geq P(1+r/m)^{mt}$

It suffices to show that

$$e^{r} \geq (1+r/m)^{m}$$

$\{a_m\} = \{1+r/m\}^{m}$ is an increasing sequence and

$$\lim_{m\to\infty} (1+r/m)^{m} = e^{r}$$

3) Continuous compounding

If a future value $V(t)$ of a Principal P attracting interest at a rate r compounded m times with m very large

$$V(t) = \lim_{m \to \infty} P\left(1+\frac{r}{m}\right)^{mt}$$

Note $\lim_{m \to \infty} P\left(1+\frac{r}{m}\right)^{mt} = 1^{\infty} = (IF)$

L'Hôpital rule

$$\lim_{m \to \infty} \left(1+\frac{r}{m}\right)^{m} ?$$

$$\lim_{m \to \infty} m \ln\left(1+r/m\right) = \lim_{m \to \infty} \frac{\ln(1+r/m)}{1/m} \quad \frac{0}{0}$$

$$= \lim_{m \to \infty} \frac{-r/m^2}{(1+r/m)} \cdot (-m^2)$$

$$= r$$

$$\Rightarrow \lim_{m \to \infty} V(t) = P e^{rt}$$

Note

$$a_{\overline{\infty}|} = \lim_{n \to \infty} a_{\overline{n}|}$$

$$\ddot{a}_{\overline{\infty}|} = \lim_{n \to \infty} \ddot{a}_{\overline{n}|}$$

Ex You want to endow a fund wich pays out a scholarship of \$ 1000 every year in perpetuity. The first scholarship will be paid out in 5 years time. Assuming an interest rate of 7%, how much do you need to pay into the fund.

Ans We need to discount by 5 years $\ddot{a}_{\overline{\infty}|}$ at $t=0$. So we need to pay into the fund:

$$v^5 \, 1000 \, \ddot{a}_{\overline{\infty}|} = \frac{v^5}{d} 1000 = \frac{0.934579}{0.0654206} \times 1000$$

$$= 10,899.50$$

Properties

(i) $\ddot{a}_{\overline{n|}} = (1+r)\, a_{\overline{n|}}$

(ii) $a_{\overline{n|}} = v\, \ddot{a}_{\overline{n|}}$

(iii) $\ddot{s}_{\overline{n|}} = (1+i)\, s_{\overline{n|}}$

(iv) $s_{\overline{n|}} = v\, \ddot{s}_{\overline{n|}}$

(v) $\dfrac{1}{\ddot{a}_{\overline{n|}}} = \dfrac{1}{\ddot{s}_{\overline{n|}}} + d$

A **perpetuity - due** is an annuity with infinite number of payments with the first payment occuring at the begining of the first periode

Note: Since the term of annuity is infinite, perpetuityes do not have accumulated values

The present value of a perpetuity - due is given by

$$\ddot{a}_{\overline{\infty|}} = \lim_{n \to \infty} \ddot{a}_{\overline{n|}} = \frac{1}{d}$$

Example: What would you be willing to pay for an infinite stream of \$37 annual payments begining now if the interest rate is 8% per annum

Ans: $37\, \ddot{a}_{\overline{\infty|}} = \dfrac{37}{0.08\,(1.08)^{-1}} = \$499.50$

The present value of an annuity - due is given by

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + \cdots + v^{n-1} = \frac{1 - v^n}{1 - v}$$

$$= \frac{1 - v^n}{d}$$

<u>Example</u> What amount must you invest today at 6% interest rate compounded annually so that you can withdraw \$ 5000 at the begining of each year for the next 5 years

<u>Sol</u>

$$= 5000 \; \ddot{a}_{\overline{5}|} = 5000 \; \frac{1 - (1.06)^{-5}}{\underbrace{0.06\,(1.06)^{-1}}_{d = rv}} = 22.325,53 \; \$$$

The accumulated value at time n of an annuity - due is given by:

$$\ddot{s}_{\overline{n}|} = (1+r)^n + (1+r)^{n-1} \cdots + (1+r)$$

$$\ddot{s}_{\overline{n}|} = (1+r)^n \; \ddot{a}_{\overline{n}|}$$

$$= \frac{1+r}{r}\left((1+r)^n - 1\right) = \frac{(1+r)^n - 1}{d}$$

<u>Ex</u> What amount will accumulate if we deposit \$ 5000 at the begining of each year for the next 5 years? Assume an interest of 6% compounded annually

<u>Sol</u>

$$= 5000 \; \ddot{s}_{\overline{5}|} = 5000 \; \frac{(1.06)^5 - 1}{0.06\,(1.06)^{-1}} = 29,876.59 \; \$$$

A **perpetuity immediate** is an annuity with infinite number of payments with the first payment occuring at the end of the ~~first~~ period

$$a_{\overline{\infty}|} = \lim_{n \to \infty} a_{\overline{n}|} = \frac{1}{r}$$

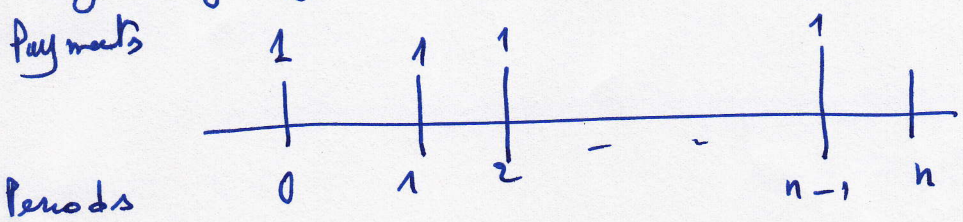
__Example__ Suppose a company issues a stock that pays a divident at the end of each year of \$ 10 indefinitely, and the Companys cost of capital is 6%. What is the value of the strock at the begining of the year.

__Ans__

$$= 10\, a_{\overline{\infty}|} = 10 \cdot \frac{1}{0.06} = \$166.67$$

2) Level annuity due

An annuty due is an annuty for wich the payment are made at the begining of the payment periods



- Newton's Method (studied previously)
- Bisection Method

r	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07
$a_{\overline{15}\rceil}$	15.000	13.8651	12.8493	11.9..	11.11..	10.3797	9.7122	9.1075

$\Rightarrow$ $a_{\overline{15}\rceil} = 10$ for some $0.05 < r < 0.06$

Note that $a_{\overline{15}\rceil}$ decrease as $r \nearrow$

$r = 5.\frac{1}{2} = 5.05$ ~~Since~~ $a_{\overline{15}\rceil} = 10.0376 \Rightarrow$

$a_{\overline{15}\rceil} = 10$ for some $5\frac{1}{2} < r < 6$

$r = 5.\frac{3}{4} \rightarrow a_{\overline{15}\rceil} = 9.8729 \Rightarrow$

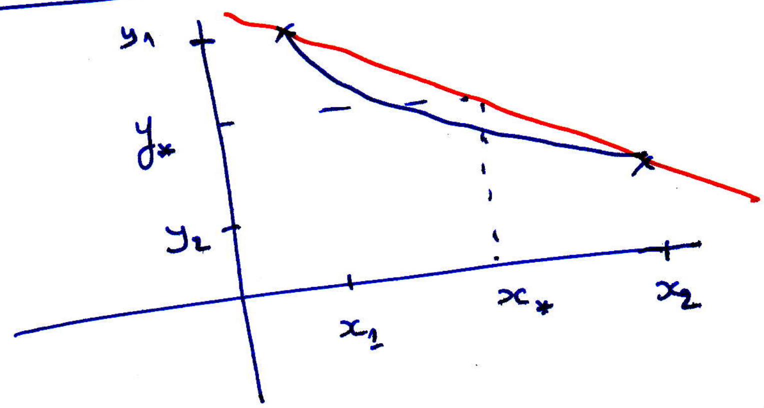
$a_{\overline{15}\rceil} = 10$ for $5\frac{1}{2} < r < 5.\frac{3}{4}$

$r = 5.\frac{5}{8} \rightarrow a_{\overline{15}\rceil} = 9.9547$

$a_{\overline{15}\rceil} = 10$ for $\Rightarrow$ $5\frac{1}{2} < r < 5\frac{5}{8}$

$(x_1, y_1) = (0.05, 10.3797$
$(x_2, y_2) = (0.06, 9.7122$

- Linear Interpolation



x_* ? $\quad y_* = 10$

Equation of the straight line passing by (x_1, y_1) (x_2, y_2)

$y - y_1 = (x - x_1)\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$

Formulas

$x_* - x_1 = \frac{x_2 - x_1}{y_2 - y_1}(y_* - y_1)$

$x_* = x_1 + (y_* - y_1)\frac{x_2 - x_1}{y_2 - y_1}$

$(10 - 10.3797)\ \frac{0.06 - 0.05}{\ldots - 10.3797} \simeq 0.055689$

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~~Unknown interest rate~~

Ex A loan of \$ 5000 is repaid by 15 annual payments of \$ 500. With the first payment due in a year, what is the interest rate?

Sol ~~$a_{\overline{15|}}$ ...~~

The value of the annuity at the time of loan (one year before the first payment)

is $500\, a_{\overline{15|}}$. So we have to solve

$$500\, a_{\overline{15|}} = 5000 \quad \text{or} \quad a_{\overline{15|}} = 0.1$$

$$\Rightarrow \quad \frac{1 - v^{15}}{r} = \frac{1}{r}\left(1 - \left(\frac{1}{1+r}\right)^{15}\right)$$

So the equation is

$$\frac{1}{r}\left(1 - \left(\frac{1}{1+r}\right)^{15}\right) = 10$$

$\Rightarrow$ solution $r = 5.56\%$?

The accumulated value of an annuity-immediate (How much do you have at the end) right after the n the payment is given by:

$$s_{\overline{n}|} = \sum_{k=0}^{n-1} (1+r)^k = \frac{(1+r)^n - 1}{r}$$

$$\boxed{s_{\overline{n}|} = \frac{(1+r)^n - 1}{r}}$$

Note that $$s_{\overline{n}|} = (1+r)^n \, a_{\overline{n}|}$$

$$a_{\overline{n}|} = v^n \, s_{\overline{n}|}$$

($a_{\overline{n}|}$ is equal to $s_{\overline{n}|}$ discounted n years)

<u>Proof</u> $$v^n s_{\overline{n}|} = v^n \frac{(1+r)^n - 1}{r} = \frac{\left(\frac{1+r}{1+r}\right)^n - v^n}{r}$$

$$= \frac{1 - v^n}{r} = a_{\overline{n}|}$$

<u>Ex1</u> Calculate the ~~future~~ present value of an annuity-immediate of amount \$100 paid annualy for 5 years ~~of~~ at the rate of interest of 9%

<u>Sol</u> $$Pv = 100 \, a_{\overline{5}|} = 100 \, \frac{(1.09)^5 - 1}{0.09}$$

$$= 598.47$$

<u>Ex2</u> A loan of \$2500 at a rate of 6.5% is paid off in 10 years. How much is each installments

$$a_{\overline{10}|} \cdot x = 2500$$

$$x = 2500 / a_{\overline{10}|}$$

$$a_{\overline{10}|} = \frac{1 - v^{10}}{r} \quad a \quad v = 0.938967 \Rightarrow a_{\overline{10}|} = 7.18830$$