

Engineering Probability & Statistics (AGE 1150)

Chapter 2: Probability – Part 1

Dr. O. Phillips Agboola

Sample Space (S)

- **Experiment**: is some procedure (or process) that we do and it results in an outcome.

Definition 2.1:

- The set of all possible outcomes of a statistical experiment is called the **sample space** and is denoted by S .
- Each outcome (element or member) of the sample space S is called a **sample point**.
- Example: the sample space S , of possible outcomes when a coin is flipped, may be written

$$S = \{H, T\},$$

where H and T correspond to heads and tails, respectively

Events

Definition 2.2:

An event A is a subset of the sample space S . That is $A \subseteq S$.

- We say that an event A occurs if the outcome (the result) of the experiment is an element of A .
- $\phi \subseteq S$ is an event (ϕ is called the impossible event)
- $S \subseteq S$ is an event (S is called the sure event)

Example

- Experiment: Tossing a die, or Selecting a ball from a box containing 6 balls numbered 1,2,3,4,5 and 6.
- This experiment has 6 possible outcomes
- The sample space is $S = \{1,2,3,4,5,6\}$.
- Consider the following events:
 - $E_1 =$ getting an even number $= \{2,4,6\} \subseteq S$
 - $E_2 =$ getting a number less than 4 $= \{1,2,3\} \subseteq S$
 - $E_3 =$ getting 1 or 3 $= \{1,3\} \subseteq S$
 - $E_4 =$ getting an odd number $= \{1,3,5\} \subseteq S$
 - $E_5 =$ getting a negative number $= \{\} = \varnothing \subseteq S$
 - $E_6 =$ getting a number less than 10 $= \{1,2,3,4,5,6\} = S \subseteq S$
- $n(S) =$ no. of outcomes (elements) in S . in the above example $= 6$
- $n(E) =$ no. of outcomes (elements) in the event E . for $E_1 = 3$, $E_5 = 0$, and $E_3 = 2$,

Example

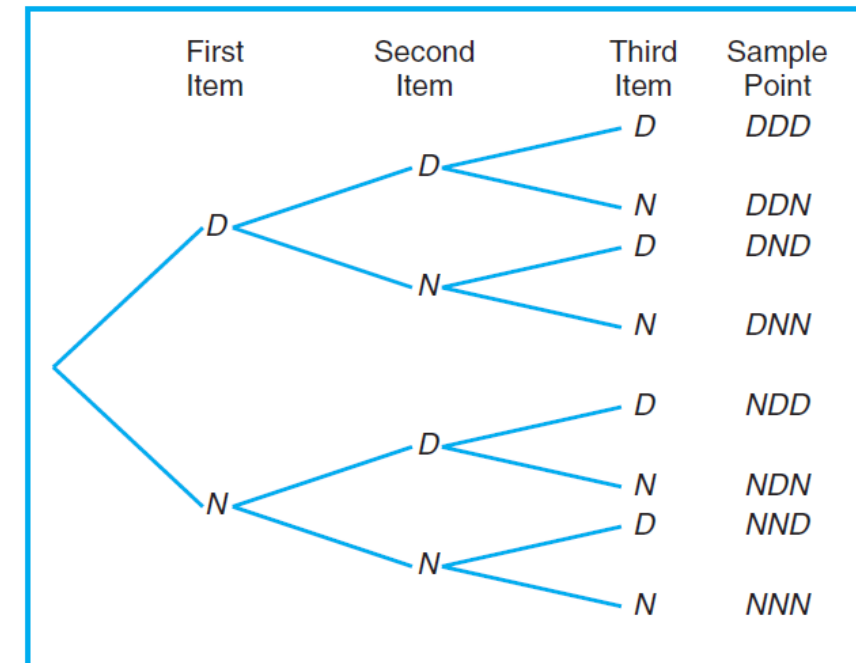
- Experiment: Selecting 3 items from manufacturing process; each item is inspected and classified as defective (D) or non-defective (N).
- Draw tree diagram of the possible outcomes as

- This experiment has 8 possible outcomes

$S = \{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\}$

- Consider the following events:

- $A = \{\text{at least 2 defectives}\} = \{DDD, DDN, DND, NDD\} \subseteq S$
- $B = \{\text{at most one defective}\} = \{DNN, NDN, NND, NNN\} \subseteq S$
- $C = \{3 \text{ defectives}\} = \{DDD\} \subseteq S$

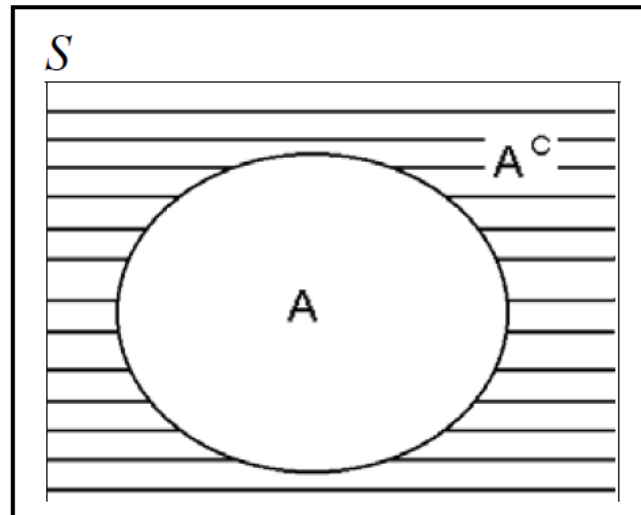


Some Operations on Events 1

- Let A and B be two events defined on the sample space S.

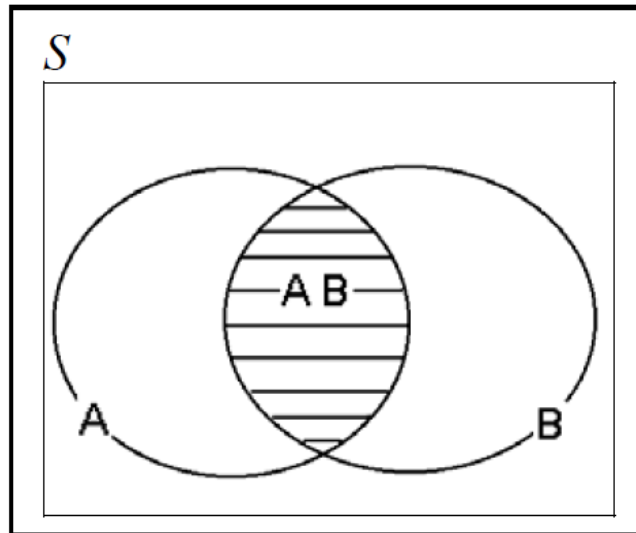
Definition 2.3: **Complement of The Event A:**

- A^c or A' or \overline{A}
- $A^c = \{x \in S: x \notin A\}$
- A^c consists of all points of S that are not in A.
- A^c occurs if A does not.



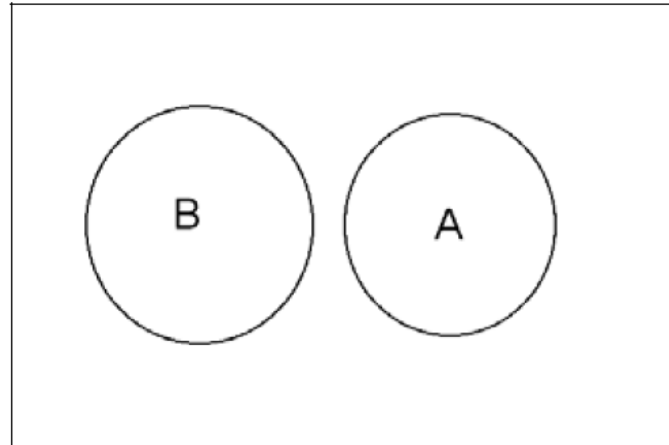
Some Operations on Events 2

- Definition 2.4: **Intersection**:
- $A \cap B = AB = \{x \in S: x \in A \text{ and } x \in B\}$
- $A \cap B$ Consists of all points in both A and B.
- $A \cap B$ Occurs if both A and B occur together.

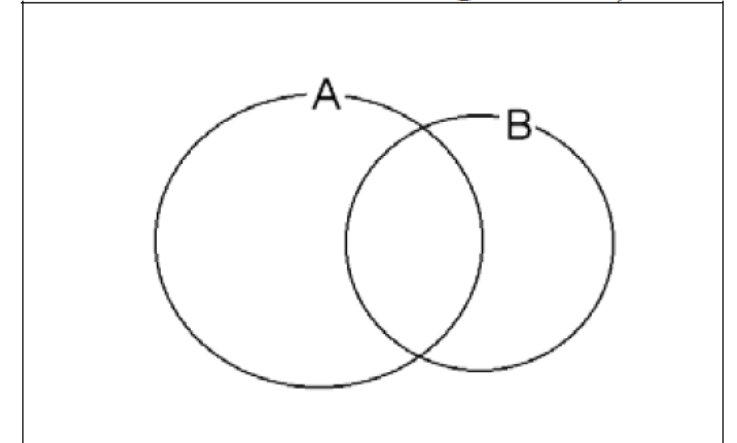


Some Operations on Events 3

- **Definition 2.5: Mutually Exclusive (Disjoint) Events:**
- Two events A and B are mutually exclusive (or disjoint) if and only if $A \cap B = \phi$; that is, A and B have no common elements (they do not occur together).



$A \cap B = \phi$
 A and B are mutually
exclusive (disjoint)

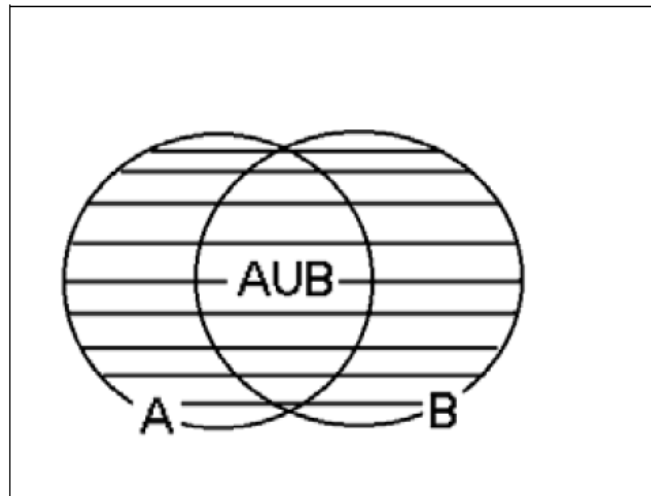


$A \cap B \neq \phi$
 A and B are not
mutually exclusive

Some Operations on Events 4

- **Definition 2.6: Union:**

- $A \cup B = \{x \in S: x \in A \text{ or } x \in B\}$
- $A \cup B$ Consists of all outcomes in A **or** in B **or** in both A and B .
- $A \cup B$ Occurs if A occurs, **or** B occurs, **or** both A and B occur.
- That is $A \cup B$ Occurs if at least one of A and B occurs.



Counting Sample Points

- There are many counting techniques which can be used to count the number points in the sample space (or in some events) without listing each element.
- In many cases, we can compute the probability of an event by using the counting techniques.
- **multiplication rule:** If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.

Counting Sample Points

Example: How many sample points are there in the sample space when a pair of dice is thrown once?

- ***Solution*** : The first die can land face-up in any one of $n_1 = 6$ ways. For each of these 6 ways, the second die can also land face-up in $n_2 = 6$ ways. Therefore, the pair of dice can land in $n_1 n_2 = (6)(6) = 36$ possible ways.

If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \cdots n_k$ ways.

Counting Sample Points

Combinations:

- In many problems, we are interested in the number of ways of selecting r objects from n objects without regard to order. These selections are called combinations.
- Notation:
- n factorial is denoted by $n!$ and is defined by:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times (2) \times (1) \quad \text{for } n = 1, 2, \dots$$

$$0! = 1$$

Example: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Theorem 2.8:

- The number of combinations of n distinct objects taken r at a time is denoted by $\binom{n}{r}$ and is given by:

$$\binom{n}{r} = \frac{n!}{r! (n-r)!}; \quad r = 0, 1, 2, \dots, n$$

- And it is read as “ n ” choose “ r ”.
- The number of different ways of selecting r objects from n distinct objects.
- The number of different ways of dividing n distinct objects into two subsets; one subset contains r objects and the other contains the rest $(n-r)$ objects.

Example

- If we have 10 equal–priority operations and only 4 operating rooms are available, in how many ways can we choose the 4 patients to be operated on first?

$$n = 10 \quad r = 4$$

The number of different ways for selecting 4 patients from 10 patients is

$$\begin{aligned} \binom{10}{4} &= \frac{10!}{4!(10-4)!} = \frac{10!}{4! \times 6!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1) \times (6 \times 5 \times 4 \times 3 \times 2 \times 1)} \\ &= 210 \quad (\text{different ways}) \end{aligned}$$

Probability of an Event

- To every point (outcome) in the sample space of an experiment S , we assign a weight (or probability), ranging from 0 to 1, such that the sum of all weights (probabilities) equals 1.
- The weight (or probability) of an outcome measures its likelihood (chance) of occurrence.
- To find the probability of an event A , we sum all probabilities of the sample points in A . This sum is called the probability of the event A and is denoted by $P(A)$.
- **Definition 2.8:**
- The probability of an event A is the sum of the weights (probabilities) of all sample points in A . Therefore,
 1. $0 \leq P(A) \leq 1$
 2. $P(S) = 1$
 3. $P(\phi) = 0$

Example

A balanced coin is tossed twice. What is the probability that at least one head occurs?

Solution: H = Head, T = Tail.

- $S = \{HH, HT, TH, TT\}$
- $A = \{\text{at least one head occurs}\} = \{HH, HT, TH\}$
- Since the coin is balanced, the outcomes are equally likely; i.e., all outcomes have the same weight or probability.

Outcome	Weight (Probability)
HH	$P(HH) = w$
HT	$P(HT) = w$
TH	$P(TH) = w$
TT	$P(TT) = w$
sum	$4w=1$

$$4w = 1 \Leftrightarrow w = 1/4 = 0.25$$

$$P(HH)=P(HT)=P(TH)=P(TT)=0.25$$

The probability that at least one head occurs is:

$$P(A) = P(\{\text{at least one head occurs}\}) = P(\{HH, HT, TH\})$$

$$= P(HH) + P(HT) + P(TH)$$

$$= 0.25 + 0.25 + 0.25$$

$$= 0.75$$

Theorem 2.9:

- If an experiment has $n(S)=N$ equally likely different outcomes, then the probability of the event A is:

$$P(A) = \frac{n(A)}{n(S)} = \frac{n(A)}{N} = \frac{\text{no. of outcomes in } A}{\text{no. of outcomes in } S}$$

Example

A mixture of candies consists of 6 mints, 4 toffees, and 3 chocolates. If a person makes a random selection of one of these candies, find the probability of getting:

(a) a mint

(b) a toffee or chocolate.

- Define the following events:

- $M = \{\text{getting a mint}\}$

- $T = \{\text{getting a toffee}\}$

- $C = \{\text{getting a chocolate}\}$

- Experiment: selecting a candy at random from 13 candies,

$n(S)$ = no. of outcomes of the experiment of selecting a candy.

= no. of different ways of selecting a candy from 13 candies.

$$= \binom{13}{1} = 13$$

Cont.

- The outcomes of the experiment are equally likely because the selection is made at random.

(a) $M = \{\text{getting a mint}\}$

$n(M)$ = no. of different ways of selecting a mint candy from 6 mint candies $= \binom{6}{1} = 6$

$$P(M) = P(\{\text{getting a mint}\}) = \frac{n(M)}{n(S)} = \frac{6}{13}$$

(b) $T \cup C = \{\text{getting a toffee or chocolate}\}$

$n(T \cup C)$ = no. of different ways of selecting a toffee **or** a chocolate candy
= no. of different ways of selecting a toffee candy + no. of different ways of selecting a chocolate candy

$$= \binom{4}{1} + \binom{3}{1} = 4 + 3 = 7 \rightarrow P(T \cup C) = P(\{\text{getting a toffee or chocolate}\}) = \frac{n(T \cup C)}{n(S)} = \frac{7}{13}$$