Engineering Probability & Statistics (AGE 1150)
Chapter 2: Probability – Part 3

Total Probability

- **Definition:**

The events A_1, A_2,…, and A_n constitute a partition of the sample space S if:

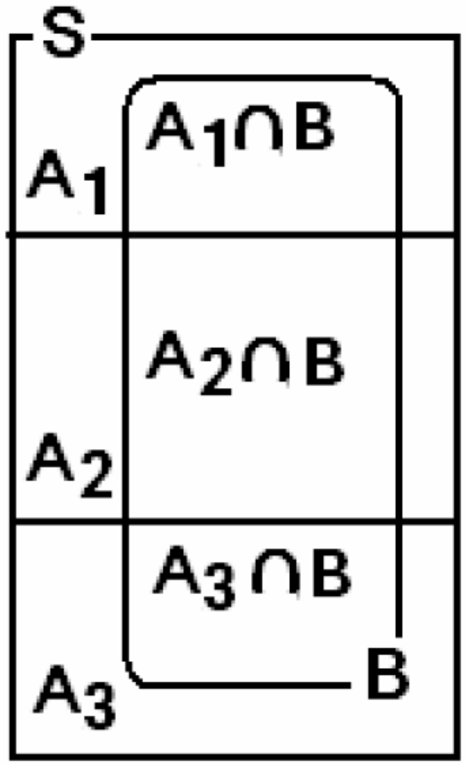
- $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ \ldots \ \cup A_n = S$
- $A_i \cap A_j = \phi, \quad \forall\ i \neq j$

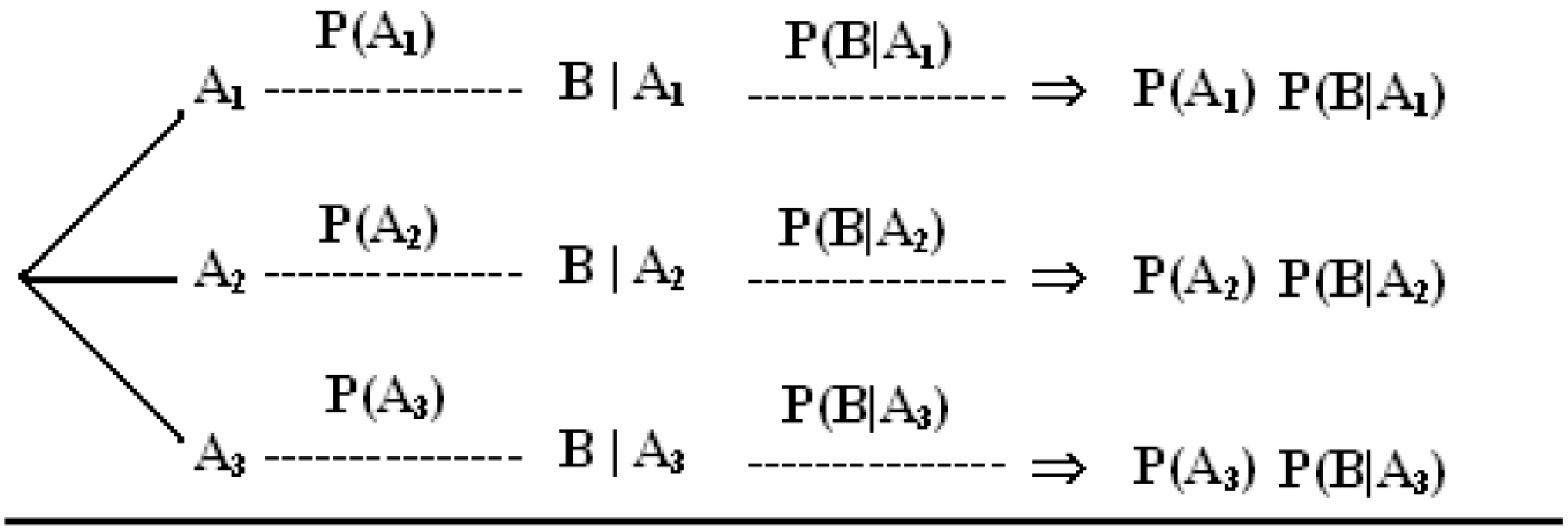
- **Theorem** (Total Probability):
- If the events A_1, A_2,…, and A_n constitute a partition of the sample space S such that $P(A_k) \neq 0$ for $k=1, 2, \ldots, n$,

then for any event B:

$$P(B) = \sum_{k=1}^{n} P(A_k \cap B)$$

$$= \sum_{k=1}^{n} P(A_k) P(B \mid A_k)$$





$$P(B) = \sum_{k=1}^{n} P(A_k)\, P(B \mid A_k)$$

Tree Diagram

Example

- Three machines A_1, A_2, and A_3 make 20%, 30%, and 50%, respectively, of the products. It is known that 1%, 4%, and 7% of the products made by each machine, respectively, are defective. If a finished product is randomly selected, what is the probability that it is defective?
- Define the following events:
- B = {the selected product is defective}
- A_1 = {the selected product is made by machine A_1}
- A_2 = {the selected product is made by machine A_2}
- A_3 = {the selected product is made by machine A_3}

$$P(A_1) = \frac{20}{100} = 0.2; \quad P(B|A_1) = \frac{1}{100} = 0.01$$

$$P(A_2) = \frac{30}{100} = 0.3; \quad P(B|A_2) = \frac{4}{100} = 0.04$$

$$P(A_3) = \frac{50}{100} = 0.5; \quad P(B|A_3) = \frac{7}{100} = 0.07$$

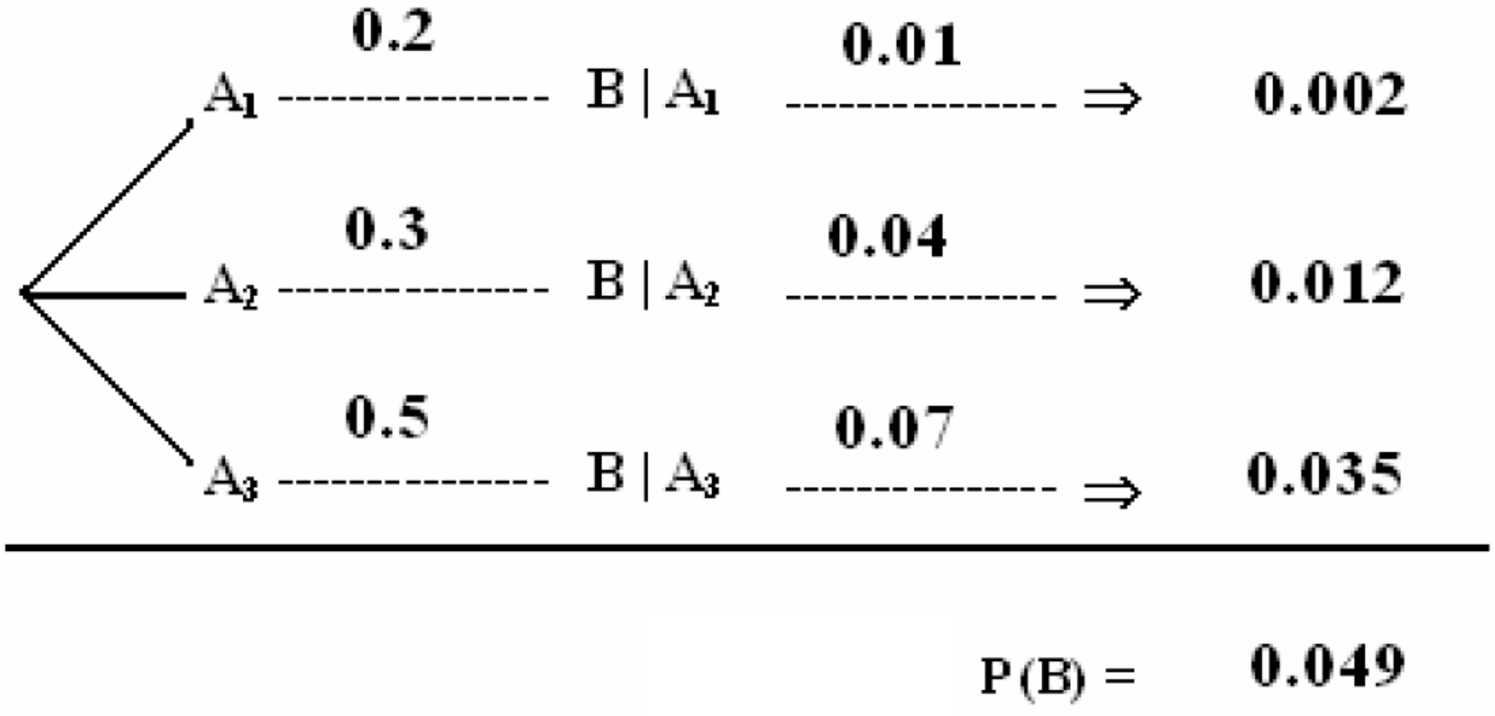
$$P(B) = \sum_{k=1}^{3} P(A_k)P(B \mid A_k)$$

$$= P(A_1)\,P(B|A_1) + P(A_2)\,P(B|A_2) + P(A_3)\,P(B|A_3)$$

$$= 0.2\times0.01 + 0.3\times0.04 + 0.5\times0.07$$

$$= 0.002 + 0.012 + 0.035$$

$$= 0.049$$



- Question:

If it is known that the selected product is defective, what is the probability that it is made by machine A^1?

- Answer:

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{P(A_1)P(B|A_1)}{P(B)} = \frac{0.2 \times 0.01}{0.049} = \frac{0.002}{0.049} = 0.0408$$

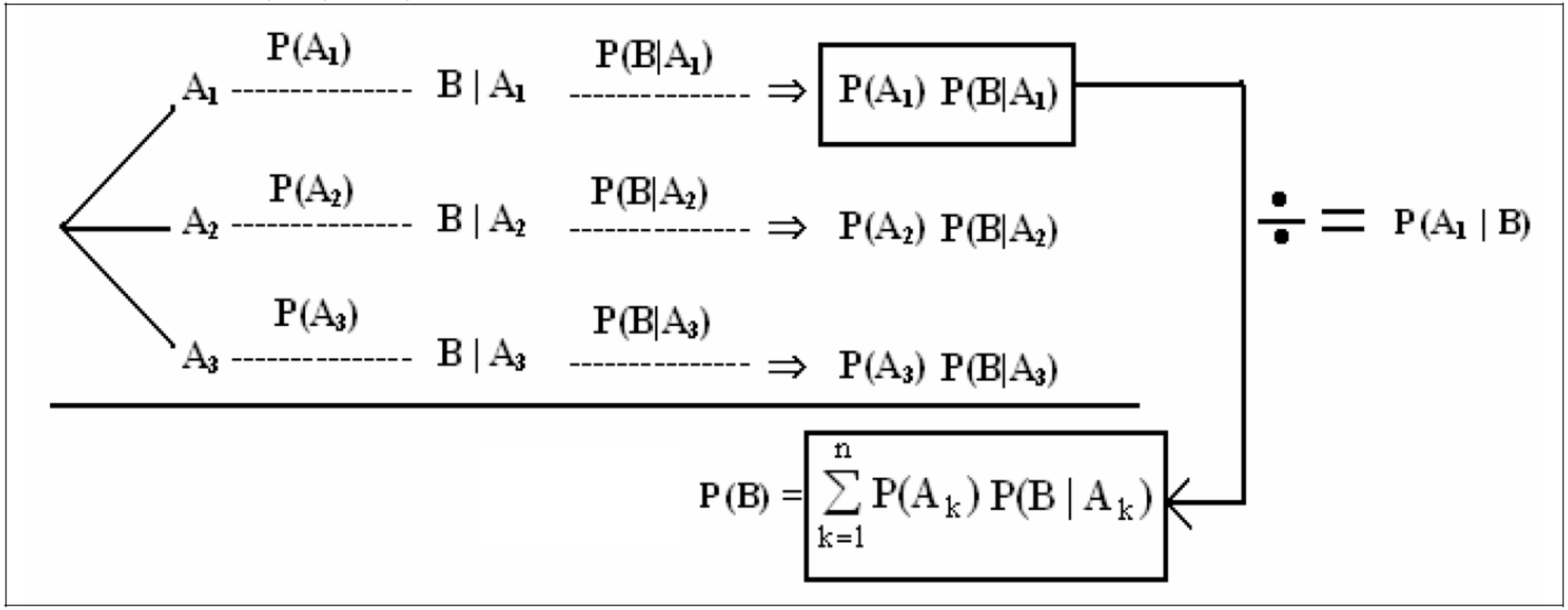
- This rule is called Bayes' rule.

Theorem (Bayes' rule)

- If the events A_1, A_2,…, and A_n constitute a partition of the sample space S such that $\mathrm{P}(A_k)\neq 0$ for k=1, 2, …, n, then for any event B such that $\mathrm{P}(B)\neq 0$:

$$\mathrm{P}(A_i \mid B) = \frac{\mathrm{P}(A_i \cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(A_i)\mathrm{P}(B \mid A_i)}{\sum_{k=1}^{n} \mathrm{P}(A_k)\mathrm{P}(B \mid A_k)} = \frac{\mathrm{P}(A_i)\mathrm{P}(B \mid A_i)}{\mathrm{P}(B)}$$

for $i = 1, 2, \ldots, n$.



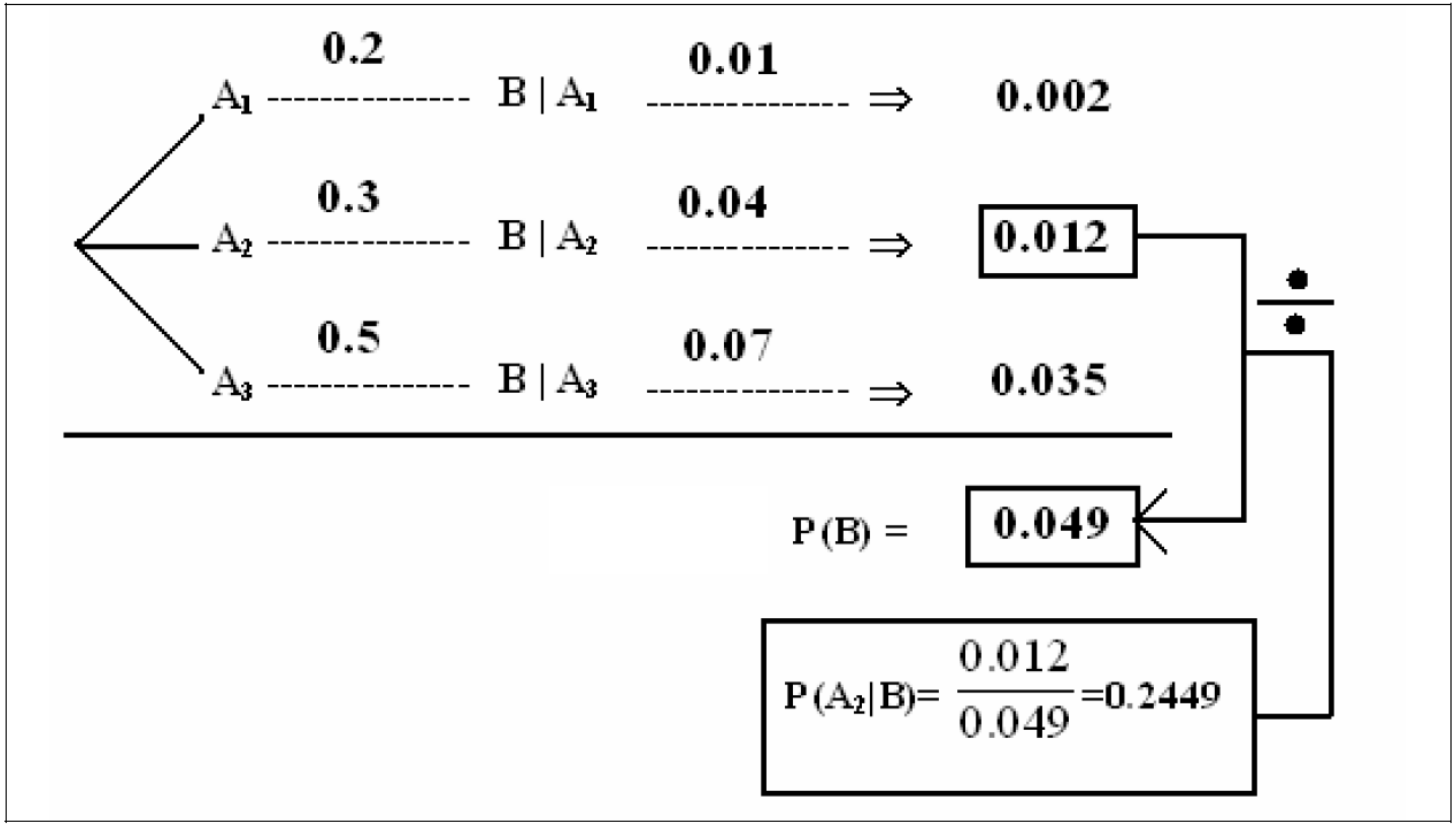
From Previous Example,

If it is known that the selected product is defective, what is the probability that it is made by:

- (a) machine A_2?
- (b) machine A_3?

$$\text{(a) } P(A_2|B)=\frac{P(A_2)P(B|A_2)}{\sum_{k=1}^{n} P(A_k)P(B|A_k)}=\frac{P(A_2)P(B|A_2)}{P(B)}$$

$$=\frac{0.3\times 0.04}{0.049}=\frac{0.012}{0.049}=0.2449$$



(b) $P(A_3|B) = \dfrac{P(A_3)P(B|A_3)}{\sum_{k=1}^{n} P(A_k)P(B|A_k)} = \dfrac{P(A_3)P(B|A_3)}{P(B)}$

$= \dfrac{0.5 \times 0.07}{0.049} = \dfrac{0.035}{0.049} = 0.7142$

Note:

$P(A_1|B) = 0.0408$, $P(A_2|B) = 0.2449$, $P(A_3|B) = 0.7142$

- $\sum_{k=1}^{3} P(A_k|B) = 1$
- If the selected product was found defective, we should check machine A_3 first, if it is ok, we should check machine A_2, if it is ok, we should check machine A_1.