# Chapter 3: Multiple Random Variables

| CLO3     | Define multiple random variables in terms of their PDF and CDF and calculate joint moments such as the correlation and covariance. |

## Objectives

1. Introduce Joint distribution, joint density, conditional distribution and density, Statistical independence
2. Introduce Expectations, correlations and joint characteristic functions
1. Vector Random Variables

- Let two random variables $X$ with value $x$ and $Y$ with value $y$ are defined on a sample space $S$, then the random point $(x, y)$ is a random vector in the XY plane.

- In the general case where $N$ r.v’s. $X_1, X_2, \ldots X_N$ are defined on a sample space $S$, they become $N$-dimensional random vector or $N$-dimensional r.v.

2. Joint distribution

- Let two events $A = \{X \leq x\} \text{ and } B = \{Y \leq y\}, (X \text{ and } Y \text{ two random variables })$ with probability distribution functions $F_X(x)$ and $F_Y(y)$, respectively:

\[
F_X(x) = P(X \leq x) \\
F_Y(y) = P(Y \leq y)
\]  

(1) (2)

- The probability of the joint event $\{X \leq x, Y \leq y\}$, which is a function of the members $x$ and $y$ is called the joint probability distribution function $P\{X \leq x, Y \leq y\} = P(A \cap B)$. It is given as follows:

\[
F_{X,Y}(x, y) = P(X \leq x, Y \leq y)
\]  

(3)

- If $X$ and $Y$ are two discrete random variables, where $X$ have $N$ possible values $x_n$ and $Y$ have $M$ possible values $y_m$, then:

\[
F_{X,Y}(x, y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m) u(x - x_n) u(y - y_m)
\]  

(4)

- If $X$ and $Y$ are two continues random variables, then:
Example 1:
Let $X$ and $Y$ be two discrete random variables. Let us assume that the joint space has only three possible elements $(1, 1)$, $(2, 1)$ and $(3, 3)$. The probabilities of these events are:

\[ P(1, 1) = 0.2, \quad P(2, 1) = 0.3, \quad \text{and} \quad P(3, 3) = 0.5. \]

Find and plot $F_{X,Y}(x,y)$.

Solution:

\[
F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u,v) \, du \, dv
\]  

(5)

Properties of the joint distribution $F_{X,Y}(x,y)$:
1. $F_{X,Y}(-\infty, -\infty) = F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$
2. $F_{X,Y}(\infty, \infty) = 1$
3. $0 \leq F_{X,Y}(x,y) \leq 1$
4. $P(x_1 < X \leq x_2, \ y_1 < Y \leq y_2) = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1) + F_{X,Y}(x_1, y_1)$
5. $F_{X,Y}(x,y)$ is a non decreasing function of both $x$ and $y$
Marginal distributions:

- The marginal distribution functions of one random variable is expressed as:
  
  \[ F_{X,Y}(x, \infty) = F_X(x) \]
  
  \[ F_{X,Y}(\infty, y) = F_Y(y) \]

Example 2:

\[ S = \{(1,1), (2,1), (3,3)\} \]

\[ P(1,1) = 0.2, \quad P(2,1) = 0.3, \quad P(3,3) = 0.5 \]

Find \( F_{X,Y}(x, y) \) and the marginal distributions \( F_X(x) \) and \( F_Y(y) \) of example 1

**Solution:**

\[ F_{X,Y}(x, y) = P(1,1) u(x-1) u(y-1) + P(2,1) u(x-2) u(y-1) + P(3,3) u(x-3) u(y-3) \]

\[ = 0.2 u(x-1) u(y-1) + 0.3 u(x-2) u(y-1) + 0.5 u(x-3) u(y-3) \]

\[ F_X(x) = F_{X,Y}(x, \infty) = 0.2 u(x-1) + 0.3 u(x-2) + 0.5 u(x-3) \]

\[ F_Y(y) = F_{X,Y}(\infty, y) = 0.2 u(y-1) + 0.3 u(y-1) + 0.5 u(y-3) \]

\[ = 0.5 u(y-1) + 0.5 u(y-3) \]
3. Joint density

• For two continuous random variables $X$ and $Y$, the joint probability density function is given by:

$$f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y}$$

(6)

• If $X$ and $Y$ are two discrete random variables then the joint probability density function is given by:

$$f_{X,Y}(x, y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m) \delta(x - x_n) \delta(y - y_m)$$

(7)

Properties of the joint density function $f_{X,Y}(x, y)$:

1. $f_{X,Y}(x, y) \geq 0$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$
3. $F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x', y') dx' dy'$
4. $F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{X,Y}(x', y) dy' dx'$, $F_Y(y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{X,Y}(x, y') dx dy$
5. $P\{x_1 < X \leq x_2 , y_1 < Y \leq y_2\} = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{X,Y}(x, y) dx dy$
6. $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$ , $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx$ (Marginal density function)

Example 3:

Let us consider the following joint probability density function of two random variables $X$ and $Y$:

$$f_{X,Y}(x, y) = u(x)u(y)xe^{-x(y+1)}$$

Find the marginal probability density functions $f_X(x)$ and $f_Y(y)$
Solution:

\[ f_X(x) = \int_0^\infty u(x) \times e^{-x(y+1)} \, dy = u(x) \times e^{-x} \frac{e^{-xy}}{-y} \bigg|_0^\infty = u(x) e^{-x} \]

\[ f_Y(y) = \int_0^\infty u(y) \times e^{-y(x+1)} \, dx \]

But \( \int xe^{ax} \, dx = e^{ax} \left[ \frac{x}{a} - \frac{1}{a^2} \right] \)

\[ f_Y(y) = u(y) \left[ e^{-y(x+1)} \left( \frac{-x}{y+1} \right) - \frac{1}{(y+1)^2} \right]_0^\infty = \frac{u(y)}{(y+1)^2} \]

4. Conditional distribution and density

- The conditional distribution function of a random variable \( X \), given event \( B \) with \( P(B) \neq 0 \) is:
  \[ F_X(x|B) = P\{X \leq x|B\} = \frac{P\{X \leq x \cap B\}}{P(B)} \quad (8) \]

- The corresponding conditional density function is:
  \[ f_X(x|B) = \frac{dF_X(x|B)}{dx} \quad (9) \]

- Often, we are interested in computing the distribution function of one random variable \( X \) conditioned by the fact that the second variable has some specific values.

4.1 Point conditioning: The random variable has some specific value

- For continuous random variables:
  \[ f_X(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} \quad (10) \]
Also, we have:
\[
f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}
\]  \hspace{1cm} (11)

- For discrete random variables:

Suppose we have:
\[
f_Y(y) = \sum_{j=1}^{M} P(y_j)\delta(y - y_j)
\]  \hspace{1cm} (12)
\[
f_{X,Y}(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{M} P(x_i, y_j)\delta(x - x_i)\delta(y - y_j)
\]  \hspace{1cm} (13)

Assume \( y = y_k \) is the specific value of \( y \), then:
\[
F_X(x|Y = y_k) = \sum_{i=1}^{N} \frac{P(x_i, y_k)}{P(y_k)} u(x - x_i)
\]  \hspace{1cm} (14)
and
\[
f_X(x|Y = y_k) = \sum_{i=1}^{N} \frac{P(x_i, y_k)}{P(y_k)} \delta(x - x_i)
\]  \hspace{1cm} (15)

**Example 4:**

Let us consider the joint pdf: 
\[f_{X,Y}(x,y) = u(x)u(y)x e^{-x(y+1)}\]

Find \( f_Y(y|x) \) if the marginal pdf of \( X \) is given by: \( f_X(x) = u(x)e^{-x} \)

**Solution:**

\( f_{X,Y}(x,y) \) and \( f_X(x) \) are nonzero only for \( y > 0 \) and \( x > 0 \), \( f_Y(y|x) \) is nonzero only for \( y > 0 \) and \( x > 0 \), therefore, we keep \( u(x) \).

\[
f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = u(x)u(y)x e^{-xy}
\]
Example 5:

What represents the corresponding figure?

Find \( f_x(x|y = y_3) \)

Solution:

\[
 f_x(x|Y = y_3) = \sum_{i=1}^{2} \frac{p(x_i,y_3)}{p(y_3)} \delta(x - x_i)
\]

\[
P(y_3) = \frac{4}{15} + \frac{5}{15} = \frac{9}{15}
\]

\[
f_x(x|Y = y_3) = \frac{p(x_1,y_3)}{9/15} \delta(x - x_1) + \frac{p(x_2,y_3)}{9/15} \delta(x - x_2)
\]

\[
= \frac{4}{9} \delta(x - x_1) + \frac{5}{9} \delta(x - x_2)
\]
4.2 Interval conditioning: The random variable in the interval \( \{y_a < Y \leq y_b\} \)

\( y_a \) and \( y_b \) are real numbers.

\[
f_x(x | y_a < Y \leq y_b) = \frac{\int_{y_a}^{y_b} f_{x,Y}(x,y) dy}{\int_{y_a}^{y_b} f_Y(y) dy} = \frac{F_{x,Y}(x,y_b) - F_{x,Y}(x,y_a)}{\int_{y_a}^{y_b} f_Y(y) dy}
\]

and

\[
F_x(x | y_a < Y \leq y_b) = \frac{\int_{y_a}^{y_b} \int_{-\infty}^{x} f_{x,Y}(u,v) dudv}{\int_{y_a}^{y_b} \int_{-\infty}^{\infty} f_{x,Y}(u,v) dudv}
\]

**Example 6:**

Let \( f_{x,Y}(x,y) = u(x)u(y)x e^{-x(y+1)} \)

\[
f_x(x) = u(x)e^{-x}, \quad \text{and} \quad f_y(y) = \frac{u(y)}{(y+1)^2}
\]

Find \( f_x(x | Y \leq y) \)?

**Solution 6:**

\[
f_x(x | y \leq y) = \frac{\int_{-\infty}^{y} f_{x,Y}(x,y) dy}{\int_{-\infty}^{\infty} f_Y(y) dy}
\]
5. Statistical independence

- Two random variables X and Y are independent if:

\[ F_{X,Y}(x,y) = F_X(x)F_Y(y) \]  
(19)

This means that

\[ f_{X,Y}(x,y) = f_X(x)f_Y(y) \]  
(20)

- Note that if X and Y are independent, then

\[ f_X(x|y) = f_X(x) \quad \text{and} \quad f_Y(y|x) = f_Y(y) \]  
(21)

**Example 7:**
In example 6, are X and Y independent?

**Solution**

\[ f_{X,Y}(x,y) = u(x)u(y)x e^{-x(y+1)} \]

\[ f_X(x)f_Y(y) = u(x)u(y)e^{-x} \cdot \frac{1}{(y+1)^2} \neq f_{X,Y}(x,y) \]
Example 8: Let \( f_{X,Y}(x, y) = \frac{1}{12} u(x)u(y)e^{-\frac{x+y}{3}} \) are X and Y independent?

Solution

\[
\begin{align*}
f_X(x) &= \int_0^\infty \frac{1}{12} u(x)e^{-\frac{x+y}{3}} dy = \frac{u(x)}{12} \left[ e^{-\frac{x}{4}} - 3e^{-\frac{y}{3}} \right]_0^\infty = \frac{u(x)}{4} e^{-\frac{x}{4}} \\
f_Y(y) &= \int_0^\infty \frac{1}{12} u(y)e^{-\frac{x+y}{3}} dx = \frac{u(y)}{3} e^{-\frac{y}{3}} \\
f_X(x)f_Y(y) &= \frac{1}{12} u(x)u(y)e^{-\frac{x+y}{3}} = f_{X,Y}(x, y) \Rightarrow X & Y \text{ are independent.}
\end{align*}
\]

6. Sum of two random variables

- Here the problem is to determine the probability density function of the sum of two independent random variables X and Y:

\[
W = X + Y
\]

- The resulting probability density function of \( W \) can be shown to be the convolution of the density functions of X and Y:

\[
f_W(w) = f_X(x) * f_Y(y)
\]

\[
f_W(w) = \int_{-\infty}^{\infty} f_Y(y)f_X(w-y) dy = \int_{-\infty}^{\infty} f_X(x)f_Y(w-x) dx
\]
Proof:

- Let W equals the sum of two independent r.v.s X and Y

\[ W = X + Y \]

Then, \( F_W(w) = P\{W \leq w\} = P\{X + Y \leq w\} \)

\( X + Y \leq w \) corresponds to the shaded area, therefore:

\[ F_W(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{w-y} f_{X,Y}(x,y)\,dxdy \]

Since X and Y are independent:

\[ F_W(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{w-y} f_Y(y) f_X(x)\,dxdy \]

\[ = \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{w-y} f_X(x)\,dxdy \]

\[ = \int_{-\infty}^{\infty} f_Y(y) F_X(w-y)\,dy \]

Differentiating w.r.t. w:

\[ f_W(w) = \int_{-\infty}^{\infty} f_Y(y) f_X(w-y)\,dy = f_Y(w) * f_X(w) \]

So, the density function of the sum of two independent r.v.s is the convolution of their density functions.

**Example 9:**

Let us consider two independent random variables X and Y with the following pdfs:

\[ f_X(x) = \frac{1}{a} [u(x) - u(x - a)] \]

\[ f_Y(y) = \frac{1}{b} [u(y) - u(y - b)] \] Where \( 0 < a < b \)

Find the pdf of \( W = X + Y \)
Solution:

7. Central Limit Theorem

The probability distribution function of the sum of a large number of random variables approaches a Gaussian distribution.
8. Expectations and Correlations

- If $g(x, y)$ is a function of two random variables $X$ and $Y$ (their joint probability density function $f_{X,Y}(x, y)$). Then the expected value of $g(x, y)$, a function of the two random variables $X$ and $Y$ is given by:

$$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)f_{X,Y}(x, y)\,dxdy$$

\hspace{1cm} (24)

**Example 10**

Let $g(X, Y) = aX + bY$, find $E[g(X, Y)]$?

**Solution:**

$$\bar{g} = E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (aX + bY)f_{X,Y}(x, y)\,dxdy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} axf_{X,Y}(x, y)\,dxdy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} byf_{X,Y}(x, y)\,dxdy$$

$$= a\int_{-\infty}^{\infty} x f_X(x)\,dx + b\int_{-\infty}^{\infty} y f_Y(y)\,dy$$

$$= aE[X] + bE[Y]$$

- If we have $n$ functions of random variables $g_1(x, y), g_2(x, y), \ldots, g_n(x, y)$ then:

$$E[g_1(x, y) + g_2(x, y) + \ldots + g_n(x, y)] = E[g_1(x, y)] + E[g_2(x, y)] + \ldots + E[g_n(x, y)]$$

Which means that the expected value of the sum of the functions is equal to the sum of the expected values of the functions.
8.1 Joint Moments about the origin

\[ m_{nk} = E[X^nY^k] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^n y^k f_{X,Y}(x,y) \, dx \, dy \]  \hspace{1cm} (25)

We have the following properties:

- \( m_{n0} = E[X^n] \) are the moments of \( X \)
- \( m_{0k} = E[Y^k] \) are the moments of \( Y \)
- The sum \( n+k \) is called the order of the moments. (\( m_{20}, m_{02} \) and \( m_{11} \) are called second order moments).
- \( m_{10} = E[X] \) and \( m_{01} = E[Y] \) are called first order moments.

8.2 Correlation

- The second-order joint moment \( m_{11} \) is called the correlation between \( X \) and \( Y \).
- The correlation is a very important statistic and its denoted by \( R_{XY} \):

\[ R_{XY} = E[XY] = m_{11} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \, y \, f_{X,Y}(x,y) \, dx \, dy \]  \hspace{1cm} (26)

- If \( R_{XY} = E[X]E[Y] \) then \( X \) and \( Y \) are said to be uncorrelated.
- If \( X \) and \( Y \) are independent then \( f_{X,Y}(x,y) = f_X(x)f_Y(y) \) and

\[ R_{XY} = \int_{-\infty}^{\infty} x f_X(x) \, dx \int_{-\infty}^{\infty} y f_Y(y) \, dy = E[X]E[Y] \]  \hspace{1cm} (27)

Therefore, if \( X \) and \( Y \) are independent then they are uncorrelated.

- However, if \( X \) and \( Y \) are uncorrelated, it is not necessary that they are independent.
• If $R_{XY} = 0$ then $X$ and $Y$ are called orthogonal.

**Example 11**

Let $E[X] = 3$, $\sigma_x^2 = 2$, let also $Y = -6X + 22$

Find $R_{XY}$?

Are $X$ and $Y$ orthogonal?

Are $X$ and $Y$ uncorrelated?

**Solution:**

$$R_{XY} = E[XY] = E[X(-6X + 22)]$$

$$= E[-6X^2] + E[22X]$$

we know $\sigma_x^2 = E[X^2] - \bar{X}^2$

$$E[X^2] = \sigma_x^2 + \bar{X}^2 = 11$$

$$R_{XY} = -6(11) + 22(3) = 0 \Rightarrow X \text{ and } Y \text{ are Orthogonal}$$

$$E[Y] = -6E[X] + 22 = -6(3) + 22 = 4 \Rightarrow E[X]E[Y] = 12$$

$$R_{XY} \neq E[X]E[Y] \Rightarrow X \text{ and } Y \text{ are not uncorrelated.}$$

### 8.3 Joint central moments

• The joint central moment is defined as:

$$u_{nk} = E[(X - \bar{X})^n(Y - \bar{Y})^k]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{X})^n (y - \bar{Y})^k f_{X,Y}(x, y) dx dy \quad (28)$$

We note that:

$$u_{20} = E[(X - \bar{X})^2] = \sigma_x^2 \quad \text{is the variance of } X.$$
\[ u_{o2} = E[(Y - \bar{Y})^2] = \sigma_Y^2 \] is the variance of \( Y \).

### 8.4 Covariance

- The second order joint central moment \( u_{11} \) is called the covariance of \( X \) and \( Y \) and denoted by \( C_{XY} \).

\[
C_{XY} = u_{11} = E[(X - \bar{X})(Y - \bar{Y})] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{X})(y - \bar{Y}) f_{X,Y}(x,y) \, dx \, dy \tag{29}
\]

We have

\[
C_{XY} = E[XY] - \bar{X}\bar{Y} = R_{XY} - E[X]E[Y] \tag{30}
\]

- If \( X \) and \( Y \) are uncorrelated (or independent), then \( C_{XY} = 0 \)
- If \( X \) and \( Y \) are orthogonal, then \( C_{XY} = -\bar{X}\bar{Y} \)
- If \( X \) and \( Y \) are correlated, then the **correlation coefficient** \( \rho \) measures the degree of correlation between \( X \) and \( Y \):

\[
\rho = \frac{\mu_{11}}{\sqrt{\mu_{20}\mu_{02}}} = \frac{C_{XY}}{\sigma_X\sigma_Y} = E\left[ \left( \frac{X - \bar{X}}{\sigma_X} \right)\left( \frac{Y - \bar{Y}}{\sigma_Y} \right) \right] \tag{31}
\]

It is important to note that: \(-1 \leq \rho \leq 1\)

#### Example 12

Let \( g = aX + bY \) find \( \sigma_g^2 \) when \( X \) and \( Y \) are uncorrelated.
Solution:

\[ \sigma_g^2 = E[g^2] - E[g]^2 \]

\[ E[g^2] = E[(aX + bY)^2] = a^2 E[X^2] + 2ab E[XY] + b^2 E[Y^2] \]

\[ E[g]^2 = (aE[X] + bE[Y])^2 = a^2 X^2 + 2ab X Y + b^2 Y^2 \]

\[ \sigma_g^2 = E[X^2] - X^2 \]

\[ = a^2 \sigma_x^2 + 2ab C_{XY} + b^2 \sigma_y^2 \]

if \( X \) and \( Y \) are uncorrelated, \( C_{XY} = 0 \) \( \Rightarrow \sigma_g^2 = a^2 \sigma_x^2 + b^2 \sigma_y^2 \)

9 Joint Characteristic functions

- The joint characteristic function of two random variables is defined by:

\[ \phi_{XY}(\omega_1, \omega_2) = E\left[e^{j\omega_1 X + j\omega_2 Y}\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j\omega_1 x + j\omega_2 y} f_{XY}(x, y) \, dx \, dy \quad (32) \]

\( \omega_1 \) and \( \omega_2 \) are real numbers.

- By setting \( \omega_1 = 0 \) or \( \omega_2 = 0 \), we obtain the marginal characteristic function:

\[ \phi_X(\omega_1) = \phi_{XY}(\omega_1, 0) \]

\[ \phi_Y(\omega_2) = \phi_{XY}(0, \omega_2) \quad (33) \]

- The joint moments are obtained as:
Example 1

Let us consider the joint characteristic function: \( \phi_{XY}(\omega_1, \omega_2) = e^{-2\omega_1^2 - 8\omega_2^2} \)

Find \( E[X] \) and \( E[Y] \)

Are \( X \) and \( Y \) uncorrelated?

Solution:

\[
E[X^1 Y^0] = m_{10} = -j \frac{d \phi_{XY}(w_1, w_2)}{dw_1} \bigg|_{w_1 = w_2 = 0} = -j(-4w_1 e^{-2w_1^2 - 8w_2^2}) \bigg|_{w_1 = w_2 = 0} = 0
\]

\[
E[X^0 Y^1] = m_{01} = -j(-16w_2 e^{-2w_1^2 - 8w_2^2}) \bigg|_{w_1 = w_2 = 0} = 0
\]

\[
R_{XY} = E[XY] = m_{11} = (-j)^2 \frac{d^2 \phi_{XY}(w_1, w_2)}{dw_1 dw_2} \bigg|_{w_1 = w_2 = 0} = 0
\]

\[
= (-4w_1)(-16w_2)e^{-2w_1^2 - 8w_2^2} \bigg|_{w_1 = w_2 = 0} = 0
\]

\( R_{XY} = 0 = E[X]E[Y] \Rightarrow X \) and \( Y \) are uncorrelated.