

# Engineering Probability & Statistics (AGE 1150)

## Chapter 4: Mathematical Expectation Part 2

# Means and Variances of Linear Combinations of Random Variables:

- If  $X_1, X_2, \dots, X_n$  are  $n$  random variables and  $a_1, a_2, \dots, a_n$  are constants, then the random variable :

$$Y = \sum_{i=1}^n a_i X_i = a_1 X_1 + a_2 X_2 + \dots + a_n X_n$$

is called a linear combination of the random variables  $X_1, X_2, \dots, X_n$  .

## • **Theorem:**

If  $X$  is a random variable with mean  $\mu = E(X)$ , and if  $a$  and  $b$  are constants, then:

$$E(aX \pm b) = a E(X) \pm b$$

$$\Leftrightarrow$$

$$\mu_{aX \pm b} = a \mu_X \pm b$$

- **Corollary 1:**  $E(b) = b$  (a=0 in the Theorem)
- **Corollary 2:**  $E(aX) = a E(X)$  (b=0 in the Theorem)

**Example:** Let  $X$  be a random variable with the following probability density function:

$$f(x) = \begin{cases} \frac{1}{3}x^2 & ; -1 < x < 2 \\ 0 & ; \text{elsewhere} \end{cases}$$

- Find  $E(4X+3)$ .
- Sol.

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_{-1}^2 x \left[ \frac{1}{3}x^2 \right] dx = \frac{1}{3} \int_{-1}^2 x^3 dx = \frac{1}{3} \left[ \frac{1}{4}x^4 \right]_{x=-1}^{x=2} = 5/4$$

$$E(4X+3) = 4 E(X) + 3 = 4(5/4) + 3 = 8$$

Another solution:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx \quad ; \quad g(X) = 4X+3$$

$$E(4X+3) = \int_{-\infty}^{\infty} (4x+3) f(x) dx = \int_{-1}^2 (4x+3) \left[ \frac{1}{3}x^2 \right] dx = \dots = 8$$

## Theorem:

- If  $X_1, X_2, \dots, X_n$  are  $n$  random variables and  $a_1, a_2, \dots, a_n$  are constants, then:

$$E(a_1X_1 + a_2X_2 + \dots + a_nX_n) = a_1E(X_1) + a_2E(X_2) + \dots + a_nE(X_n)$$

$$\Leftrightarrow$$

$$E\left(\sum_{i=1}^n a_i X_i\right) = \sum_{i=1}^n a_i E(X_i)$$

- **Corollary:**

If  $X$ , and  $Y$  are random variables, then:

$$E(X \pm Y) = E(X) \pm E(Y)$$

## Theorem:

- If  $X$  is a random variable with variance  $Var(X) = \sigma_X^2$  and if  $a$  and  $b$  are constants, then:

$$Var(aX \pm b) = a^2 Var(X)$$

$$\Leftrightarrow$$

$$\sigma_{aX+b}^2 = a^2 \sigma_X^2$$

## • Theorem :

If  $X_1, X_2, \dots, X_n$  are  $n$  **independent** random variables and  $a_1, a_2, \dots, a_n$  are constants, then:

$$Var(a_1X_1 + a_2X_2 + \dots + a_nX_n)$$

$$= a_1^2 Var(X_1) + a_2^2 Var(X_2) + \dots + a_n^2 Var(X_n)$$

$$\Leftrightarrow$$

$$Var\left(\sum_{i=1}^n a_i X_i\right) = \sum_{i=1}^n a_i^2 Var(X_i)$$

$$\Leftrightarrow$$

$$\sigma_{a_1X_1 + a_2X_2 + \dots + a_nX_n}^2 = a_1^2 \sigma_{X_1}^2 + a_2^2 \sigma_{X_2}^2 + \dots + a_n^2 \sigma_{X_n}^2$$

## **Corollary: :**

- If  $X$ , and  $Y$  are **independent** random variables, then:
- $\text{Var}(aX+bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y)$
- $\text{Var}(aX-bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y)$
- $\text{Var}(X \pm Y) = \text{Var}(X) + \text{Var}(Y)$

## **Example:**

Let  $X$ , and  $Y$  be two independent random variables such that  $E(X)=2$ ,  $\text{Var}(X)=4$ ,  $E(Y)=7$ , and  $\text{Var}(Y)=1$ . Find:

1.  $E(3X+7)$  and  $\text{Var}(3X+7)$
2.  $E(5X+2Y-2)$  and  $\text{Var}(5X+2Y-2)$ .

**Sol.:**

$E(X)=2$ ,  $\text{Var}(X)=4$ ,  $E(Y)=7$ , and  $\text{Var}(Y)=1$ . required to find:

1.  $E(3X+7)$  and  $\text{Var}(3X+7)$
2.  $E(5X+2Y-2)$  and  $\text{Var}(5X+2Y-2)$ .

1.  $E(3X+7) = 3E(X)+7 = 3(2)+7 = 13$

$$\text{Var}(3X+7) = (3)^2 \text{Var}(X) = (3)^2 (4) = 36$$

2.  $E(5X+2Y-2) = 5E(X) + 2E(Y) - 2 = (5)(2) + (2)(7) - 2 = 22$

$$\begin{aligned} \text{Var}(5X+2Y-2) &= \text{Var}(5X+2Y) = 5^2 \text{Var}(X) + 2^2 \text{Var}(Y) \\ &= (25)(4) + (4)(1) = 104 \end{aligned}$$