

Engineering Probability & Statistics (AGE 1150)

Chapter 5: Some Discrete Probability Distributions

Discrete Uniform Distribution

- If the discrete random variable X assumes the values x_1, x_2, \dots, x_k with equal probabilities, then X has the discrete uniform distribution given by:

$$f(x) = P(X = x) = f(x; k) = \begin{cases} \frac{1}{k} & ; x = x_1, x_2, \dots, x_k \\ 0 & ; \text{elsewhere} \end{cases}$$

- Note:
- $f(x) = f(x; k) = P(X = x)$
- k is called the parameter of the distribution.

Example 5.2:

- Experiment: tossing a balanced die.
- Sample space: $S=\{1,2,3,4,5,6\}$
- Each sample point of S occurs with the same probability $1/6$.
- Let X = the number observed when tossing a balanced die.
- The probability distribution of X is:

$$f(x) = P(X = x) = f(x;6) = \begin{cases} \frac{1}{6} & ; x = 1, 2, \dots, 6 \\ 0 & ; \text{elsewhere} \end{cases}$$

Theorem 5.1:

- If the discrete random variable X has a discrete uniform distribution with parameter k , then the mean and the variance of X are:

$$E(X) = \mu = \frac{\sum_{i=1}^k x_i}{k}$$

$$\text{Var}(X) = \sigma^2 = \frac{\sum_{i=1}^k (x_i - \mu)^2}{k}$$

Example 5.3:

Find $E(X)$ and $\text{Var}(X)$ in Example 5.2.

$$E(X) = \mu = \frac{\sum_{i=1}^k x_i}{k} = \frac{1 + 2 + 3 + 4 + 5 + 6}{6} = 3.5$$

$$\begin{aligned} \text{Var}(X) = \sigma^2 &= \frac{\sum_{i=1}^k (x_i - \mu)^2}{k} = \frac{\sum_{i=1}^k (x_i - 3.5)^2}{6} \\ &= \frac{(1 - 3.5)^2 + (2 - 3.5)^2 + \cdots + (6 - 3.5)^2}{6} = \frac{35}{12} \end{aligned}$$

Binomial Distribution

- **Bernoulli Trial:**

- Bernoulli trial is an experiment with only two possible outcomes.
- The two possible outcomes are labeled:

success (s) and failure (f)

- The probability of success is $P(s)=p$ and the probability of failure is $P(f)=q=1-p$.
- Examples:
 - 1. Tossing a coin (success=H, failure=T, and $p=P(H)$)
 - 2. Inspecting an item (success=defective, failure=non-defective, and $p=P(\text{defective})$)

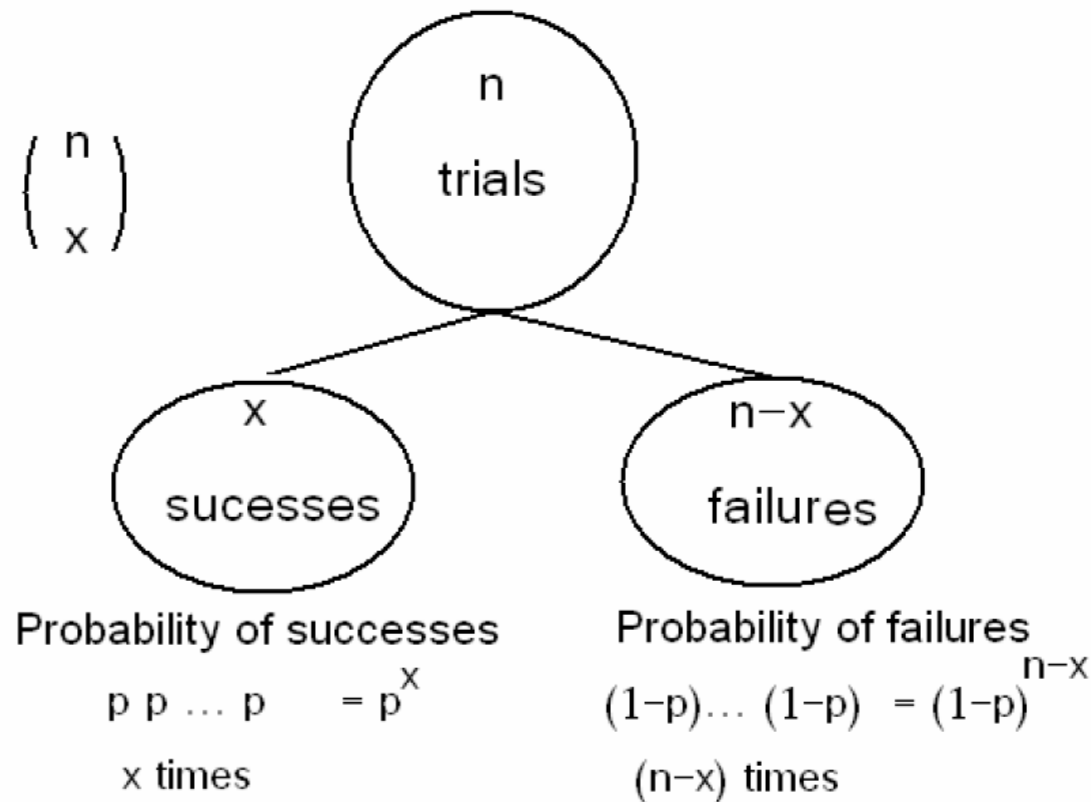
Bernoulli Process

- Bernoulli process is an experiment that must satisfy the following properties:
 1. The experiment consists of n repeated Bernoulli trials.
 2. The probability of success, $P(s)=p$, remains constant from trial to trial.
 3. The repeated trials are independent; that is the outcome of one trial has no effect on the outcome of any other trial
- **Binomial Random Variable:**
- Consider the random variable :
- X = The number of successes in the n trials in a Bernoulli process
- The random variable X has a binomial distribution with parameters n (number of trials) and p (probability of success), and we write:

$$X \sim \text{Binomial}(n,p) \text{ or } X \sim b(x;n,p)$$

- The probability distribution of X is given by:

$$f(x) = P(X = x) = b(x; n, p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} ; & x = 0, 1, 2, \dots, n \\ 0 ; & \text{otherwise} \end{cases}$$



- We can write the probability distribution of X as a table as follows

x	$f(x)=P(X=x)=b(x;n,p)$
0	$\binom{n}{0}p^0(1-p)^{n-0} = (1-p)^n$
1	$\binom{n}{1}p^1(1-p)^{n-1}$
2	$\binom{n}{2}p^2(1-p)^{n-2}$
\vdots	\vdots
$n-1$	$\binom{n}{n-1}p^{n-1}(1-p)^1$
n	$\binom{n}{n}p^n(1-p)^0 = p^n$
Total	1.00

Example:

Suppose that 25% of the products of a manufacturing process are defective. Three items are selected at random, inspected, and classified as defective (D) or non-defective (N). Find the probability distribution of the number of defective items.

Solution:

- Experiment: selecting 3 items at random, inspected, and classified as (D) or (N).

- The sample space is

$S=\{DDD,DDN,DND,DNN,NDD,NDN,NND,NNN\}$

- Let X = the number of defective items in the sample
- We need to find the probability distribution of X .

(1) First Solution:

Outcome	Probability	x
NNN	$\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{27}{64}$	0
NND	$\frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} = \frac{9}{64}$	1
NDN	$\frac{3}{4} \times \frac{1}{4} \times \frac{3}{4} = \frac{9}{64}$	1
NDD	$\frac{3}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{3}{64}$	2
DNN	$\frac{1}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{9}{64}$	1
DND	$\frac{1}{4} \times \frac{3}{4} \times \frac{1}{4} = \frac{3}{64}$	2
DDN	$\frac{1}{4} \times \frac{1}{4} \times \frac{3}{4} = \frac{3}{64}$	2
DDD	$\frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{64}$	3

The probability distribution
of X is

.x	.f(x)=P(X=x)
0	$\frac{27}{64}$
1	$\frac{9}{64} + \frac{9}{64} + \frac{9}{64} = \frac{27}{64}$
2	$\frac{3}{64} + \frac{3}{64} + \frac{3}{64} = \frac{9}{64}$
3	$\frac{1}{64}$

(2) Second Solution:

- Bernoulli trial is the process of inspecting the item. The results are success=D or failure=N, with probability of success $P(s)=25/100=1/4=0.25$.
- The experiments is a Bernoulli process with:
- number of trials: $n=3$
- Probability of success: $p=1/4=0.25$
- $X \sim \text{Binomial}(n,p)=\text{Binomial}(3,1/4)$
- The probability distribution of X is given by:

$$f(x) = P(X = x) = b(x; 3, \frac{1}{4}) = \begin{cases} \binom{3}{x} (\frac{1}{4})^x (\frac{3}{4})^{3-x}; & x = 0, 1, 2, 3 \\ 0 & ; \text{otherwise} \end{cases}$$

$$f(0) = P(X = 0) = b(0; 3, \frac{1}{4}) = \binom{3}{0} (\frac{1}{4})^0 (\frac{3}{4})^3 = \frac{27}{64}$$

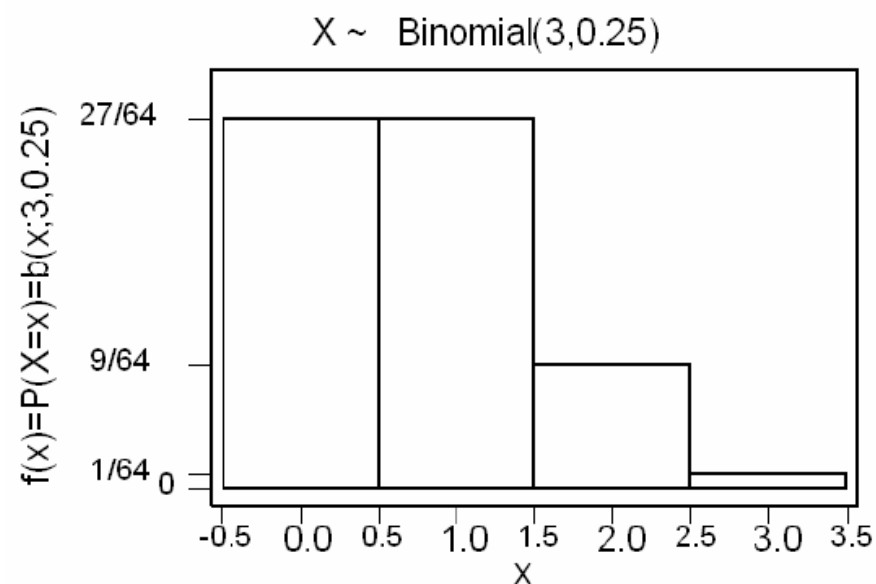
$$f(1) = P(X = 1) = b(1; 3, \frac{1}{4}) = \binom{3}{1} (\frac{1}{4})^1 (\frac{3}{4})^2 = \frac{27}{64}$$

$$f(2) = P(X = 2) = b(2; 3, \frac{1}{4}) = \binom{3}{2} (\frac{1}{4})^2 (\frac{3}{4})^1 = \frac{9}{64}$$

$$f(3) = P(X = 3) = b(3; 3, \frac{1}{4}) = \binom{3}{3} (\frac{1}{4})^3 (\frac{3}{4})^0 = \frac{1}{64}$$

The probability distribution of X is

x	f(x)=P(X=x) =b(x;3,1/4)
0	27/64
1	27/64
2	9/64
3	1/64



- **Theorem 5.2:**

The mean and the variance of the binomial distribution $b(x;n,p)$ are:

$$\begin{aligned}\mu &= n p \\ \sigma^2 &= n p (1 - p)\end{aligned}$$

- **Example:**

In the previous example, find the expected value (mean) and the variance of the number of defective items.

- X = number of defective items
- We need to find $E(X)=\mu$ and $\text{Var}(X)=\sigma^2$
- We found that $X \sim \text{Binomial}(n,p)=\text{Binomial}(3,1/4)$
- $n=3$ and $p=1/4$

The expected number of defective items is

- $E(X)=\mu = n p = (3) (1/4) = 3/4 = 0.75$

The variance of the number of defective items is

- $\text{Var}(X)=\sigma^2 = n p (1 - p) = (3) (1/4) (3/4) = 9/16 = 0.5625$

Example:

In the previous example, find the following probabilities:

- (1) The probability of getting at least two defective items.
- (2) The probability of getting at most two defective items.

• $X \sim \text{Binomial}(3, 1/4)$

$$f(x) = P(X = x) = b(x; 3, \frac{1}{4}) = \begin{cases} \binom{3}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{3-x} & \text{for } x = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

.x	.f(x)=P(X=x)=b(x;3,1/4)
0	27/64
1	27/64
2	9/64
3	1/64

- (1) The probability of getting at least two defective items:

$$P(X \geq 2) = P(X=2) + P(X=3) = f(2) + f(3) = \frac{9}{64} + \frac{1}{64} = \frac{10}{64}$$

- (2) The probability of getting at most two defective item:

$$\begin{aligned} P(X \leq 2) &= P(X=0) + P(X=1) + P(X=2) \\ &= f(0) + f(1) + f(2) = \frac{27}{64} + \frac{27}{64} + \frac{9}{64} = \frac{63}{64} \end{aligned}$$

or

$$P(X \leq 2) = 1 - P(X > 2) = 1 - P(X=3) = 1 - f(3) = 1 - \frac{1}{64} = \frac{63}{64}$$