

Engineering Probability & Statistics (AGE 1150)

Chapter 6: Some Continuous Probability Distributions

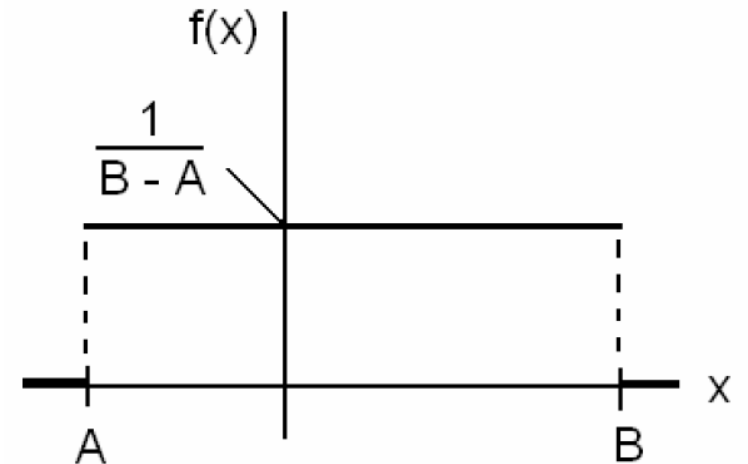
Continuous Uniform distribution (Rectangular Distribution)

- The probability density function of the continuous uniform random variable X on the interval $[A, B]$ is given by:

$$f(x) = f(x; A, B) = \begin{cases} \frac{1}{B - A} & ; A \leq x \leq B \\ 0 & ; elsewhere \end{cases}$$

- We write $X \sim \text{Uniform}(A, B)$.
- Theorem 6.1:
- The mean and the variance of the continuous uniform distribution on the interval $[A, B]$ are:

$$\mu = \frac{A + B}{2}$$
$$\sigma^2 = \frac{(B - A)^2}{12}$$



Example 6.1:

Suppose that, for a certain company, the conference time, X , has a uniform distribution on the interval $[0,4]$ (hours).

(a) What is the probability density function of X ?

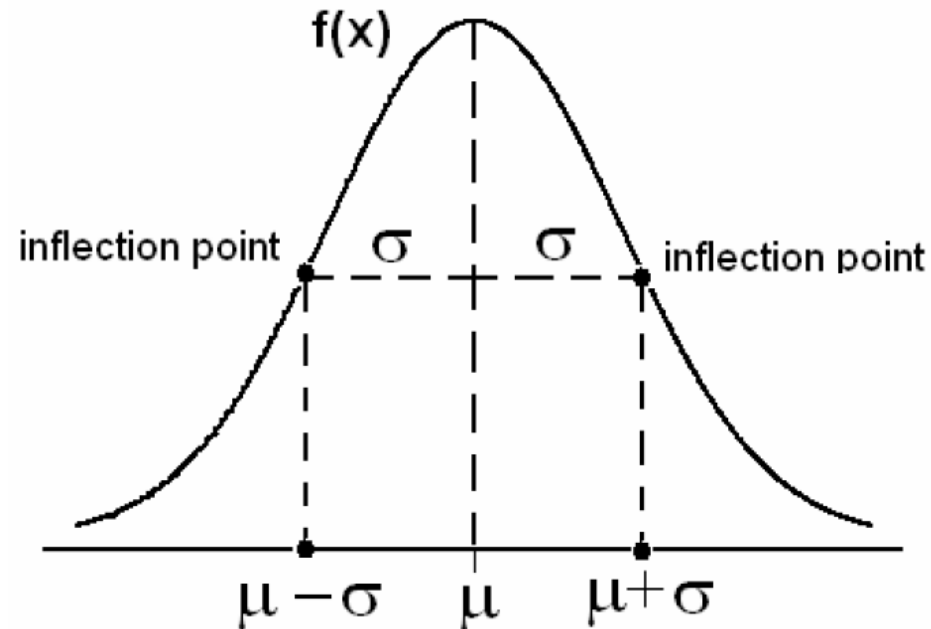
(b) What is the probability that any conference lasts at least 3 hours?

$$(a) \ f(x) = f(x;0,4) = \begin{cases} \frac{1}{4} & ; \ 0 \leq x \leq 4 \\ 0 & ; \ elsewhere \end{cases}$$

$$(b) \ P(X \geq 3) = \int_3^4 f(x) dx = \int_3^4 \frac{1}{4} dx = \frac{1}{4}$$

Normal Distribution

- The normal distribution is one of the most important continuous distributions.
- Many measurable characteristics are normally or approximately normally distributed, such as, height and weight.
- The graph of the probability density function (pdf) of a normal distribution, called the normal curve, is a bell-shaped curve.

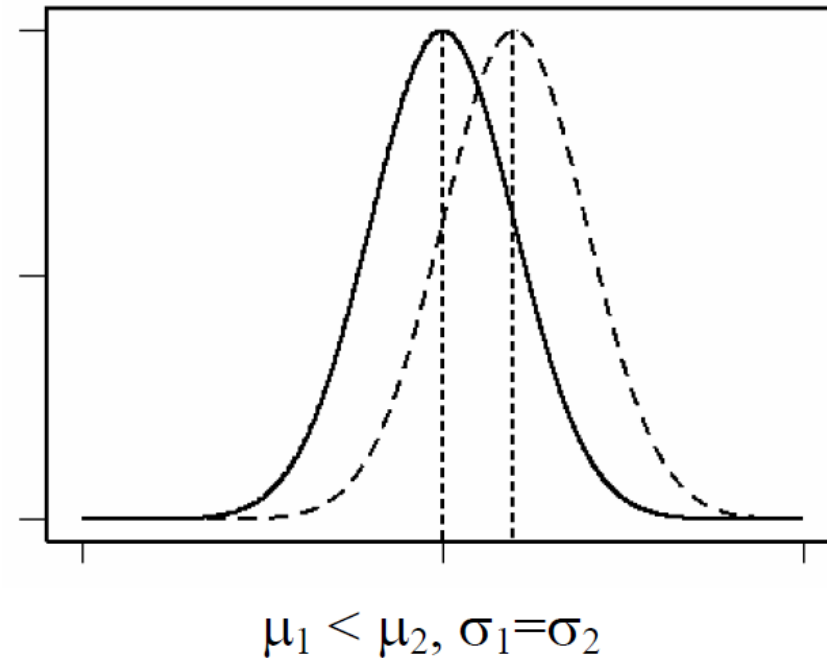


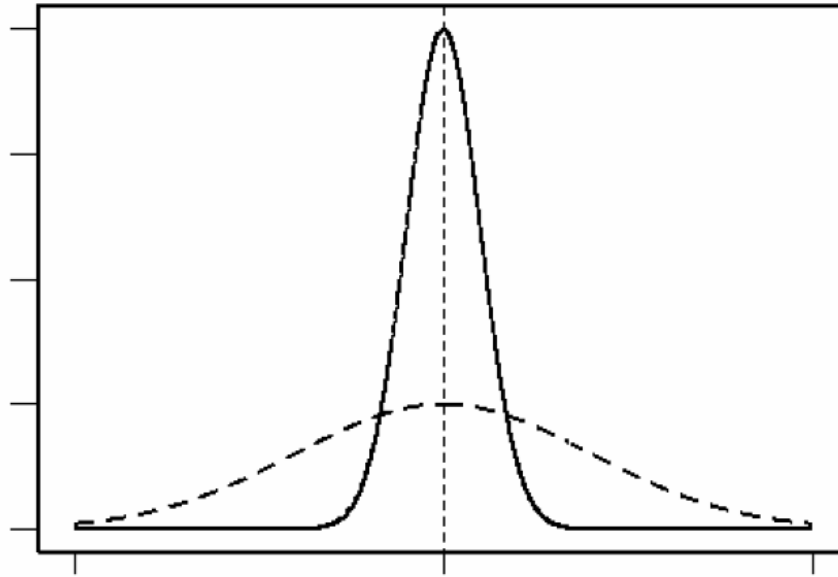
- The pdf of the normal distribution depends on two parameters: mean = $E(X) = \mu$ and variance = $\text{Var}(X) = \sigma^2$.
- If the random variable X has a normal distribution with mean μ and variance σ^2 , we write:
- $X \sim \text{Normal}(\mu, \sigma)$ or $X \sim N(\mu, \sigma)$
- The pdf of $X \sim \text{Normal}(\mu, \sigma)$ is given by:

$$f(x) = n(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} ; \begin{cases} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma > 0 \end{cases}$$

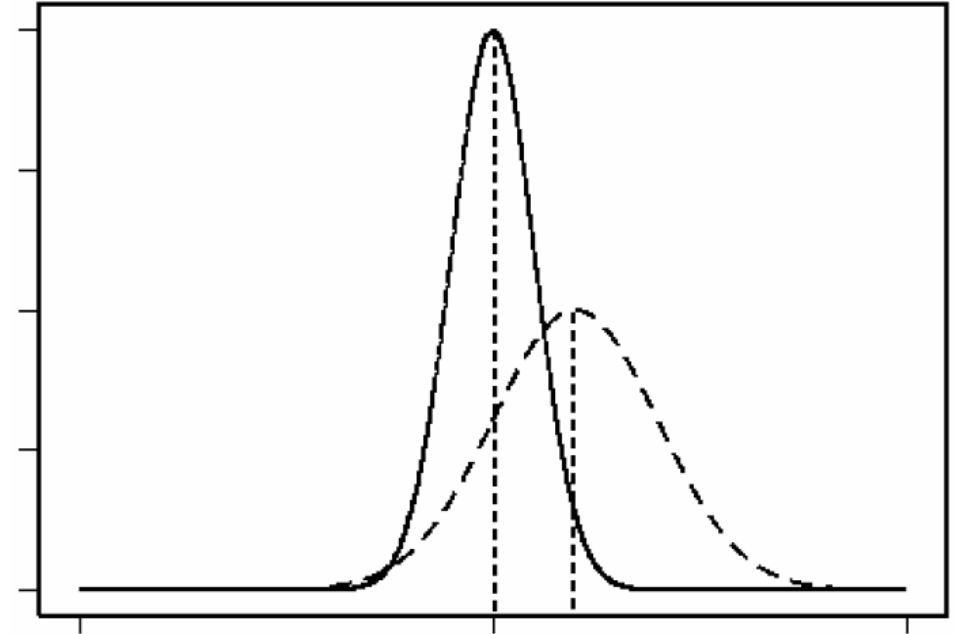
- The location of the normal distribution depends on μ and its shape depends on σ .
- Suppose we have two normal distributions:

_____ $N(\mu_1, \sigma_1)$
----- $N(\mu_2, \sigma_2)$





$$\mu_1 = \mu_2, \sigma_1 < \sigma_2$$



$$\mu_1 < \mu_2, \sigma_1 < \sigma_2$$

- Some properties of the normal curve $f(x)$ of $N(\mu, \sigma)$:
 1. $f(x)$ is symmetric about the mean μ .
 2. $f(x)$ has two points of inflection at $x = \mu \pm \sigma$.
 3. The total area under the curve of $f(x) = 1$.
 4. The highest point of the curve of $f(x)$ at the mean μ .

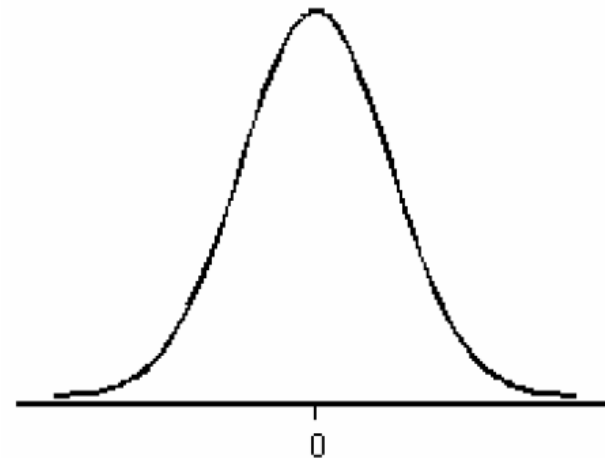
Areas Under the Normal Curve

Definition 6.1:

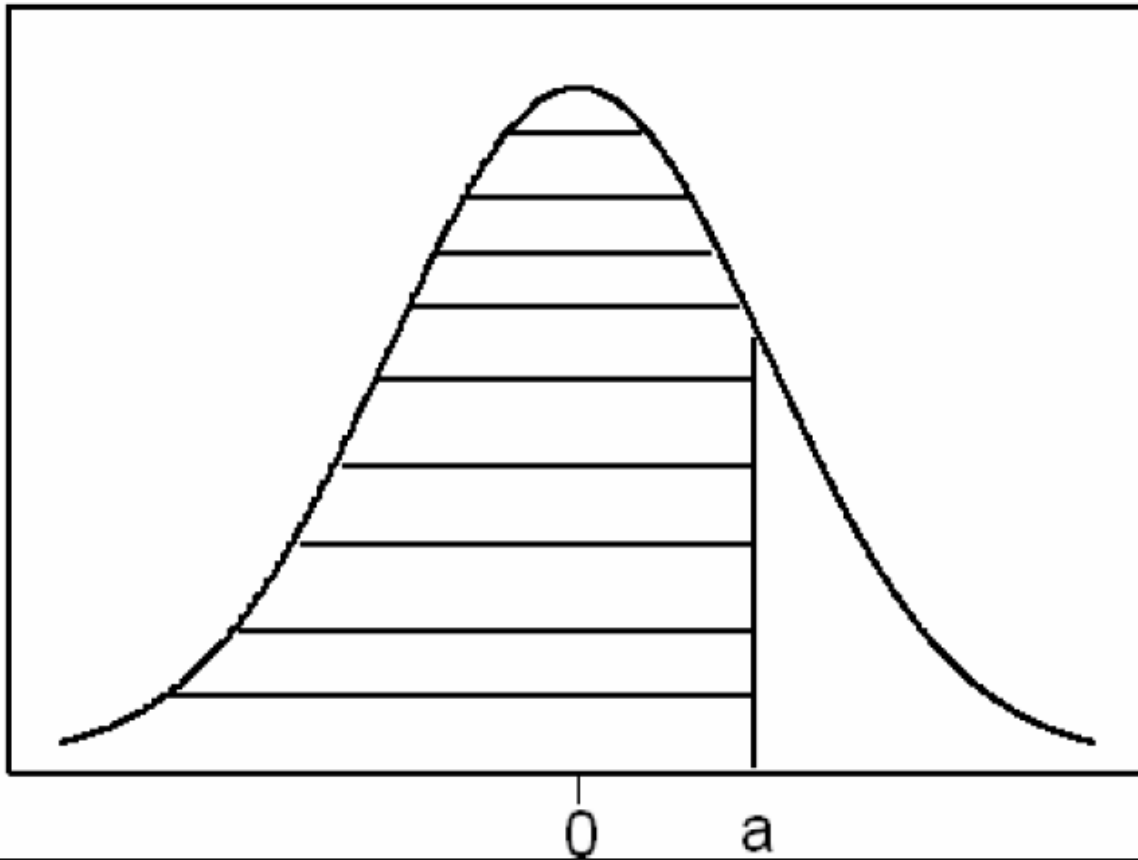
The Standard Normal Distribution:

- The normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$ is called the standard normal distribution and is denoted by Normal (0,1) or N(0,1). If the random variable Z has the standard normal distribution, we write $Z \sim \text{Normal}(0,1)$ or $Z \sim N(0,1)$.
- The pdf of $Z \sim N(0,1)$ is given by:

$$f(z) = n(z;0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$



- The standard normal distribution, $Z \sim N(0,1)$, is very important because probabilities of any normal distribution can be calculated from the probabilities of the standard normal distribution.
- Probabilities of the standard normal distribution $Z \sim N(0,1)$ of the form $P(Z \leq a)$ are tabulated (Table A.3).



$$\begin{aligned} P(Z \leq a) &= \int_{-\infty}^a f(z) dz \\ &= \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\ &= \text{from the table} \end{aligned}$$

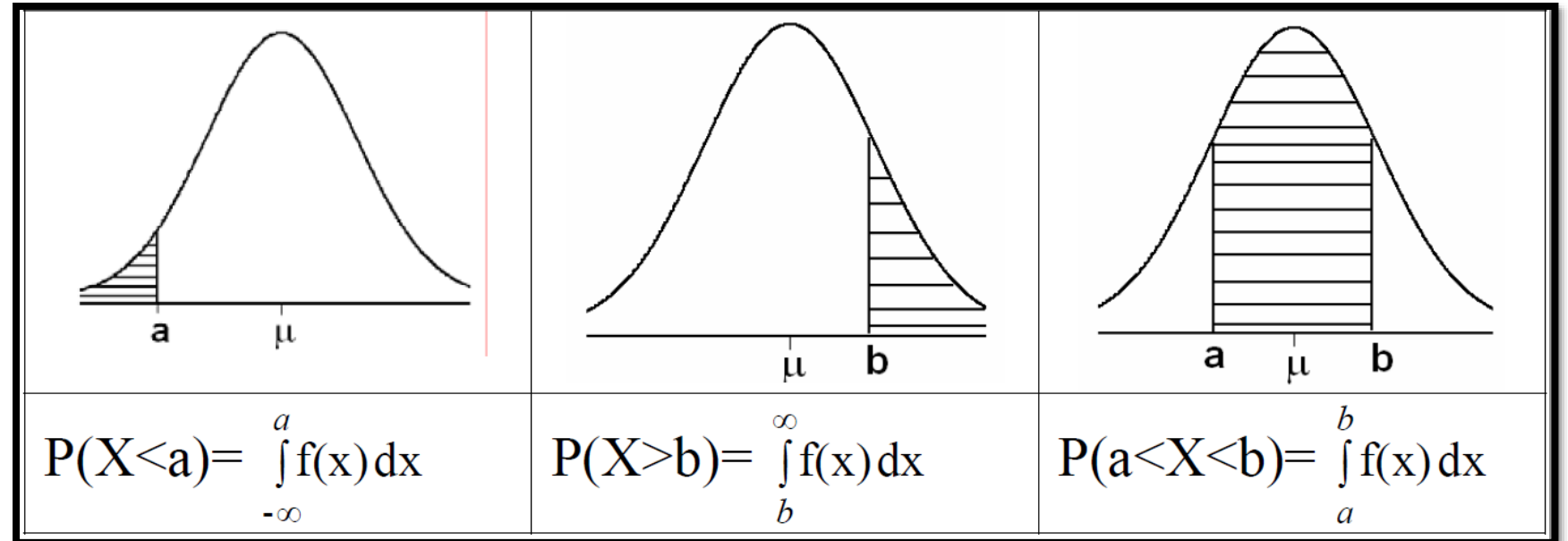
We can transfer any normal distribution $X \sim N(\mu, \sigma)$ to the standard normal distribution, $Z \sim N(0, 1)$ by using the following result.

- Result: If $X \sim N(\mu, \sigma)$, then

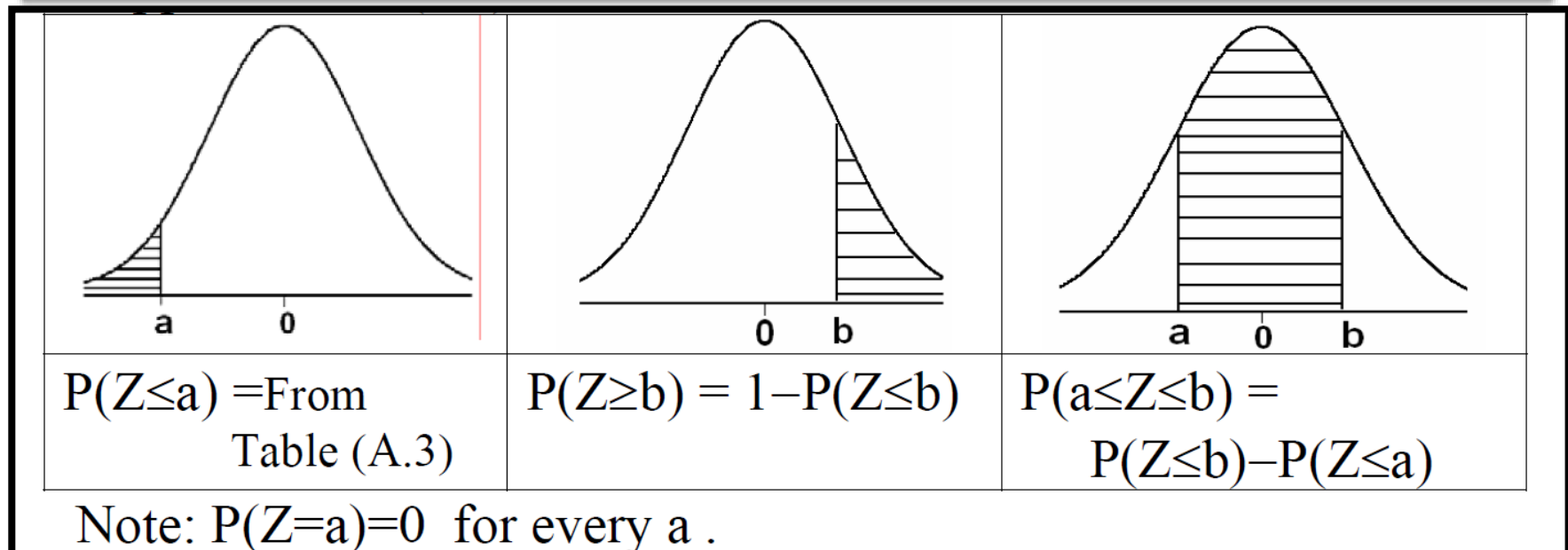
$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1).$$

Areas Under the Normal Curve of $X \sim N(\mu, \sigma)$

- The probabilities of the normal distribution $N(\mu, \sigma)$ depends on μ and σ .



Probabilities of $Z \sim N(0, 1)$:
Suppose $Z \sim N(0, 1)$.



Examples

Suppose $Z \sim N(0,1)$.

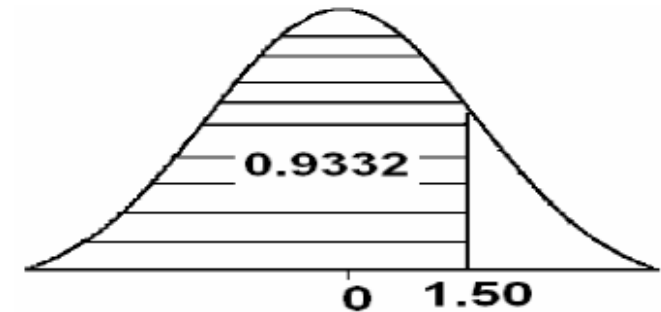
- $P(Z \leq 1.50) = 0.9332$

Z	0.00	0.01	...
:	↓		
1.5 \Rightarrow	0.9332		
:			

Table A.3 (continued) Areas under the Normal Curve

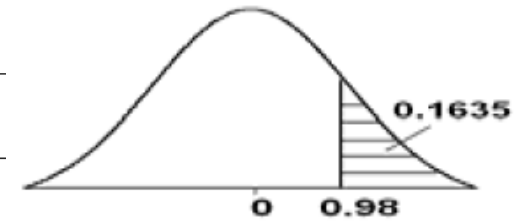
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9684	0.9691	0.9698	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

1.5 0.9332



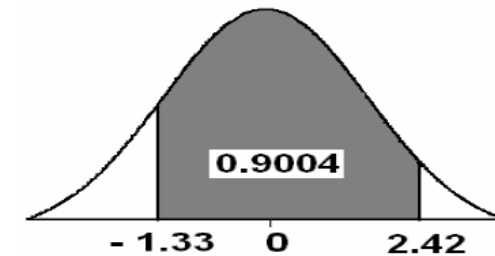
- $P(Z \geq 0.98) = 1 - P(Z \leq 0.98)$
 $= 1 - 0.8365$
 $= 0.1635$

Z	0.00	...	0.08
:	:	:	⇓
:	⇓
0.9⇒	⇒	⇒	0.8365



- $P(-1.33 \leq Z \leq 2.42)$
 $= P(Z \leq 2.42) - P(Z \leq -1.33)$
 $= 0.9922 - 0.0918$
 $= 0.9004$

Z	...	0.02	0.03
:	:	⇓	⇓
-1.3	⇒		0.0918
:		⇓	
2.4	⇒	0.9922	

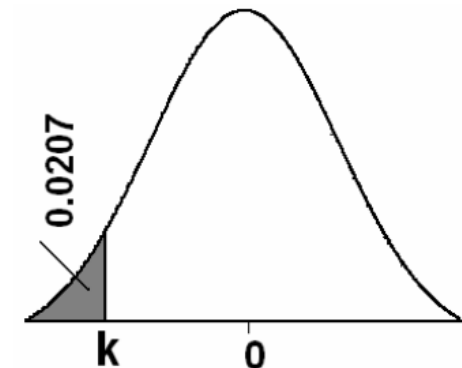


- $P(Z \leq 0) = P(Z \geq 0) = 0.5$

- Suppose $Z \sim N(0,1)$. Find the value of k such that $P(Z \leq k) = 0.0207$.

$.k = -2.04$

Z	...	0.04	
:	:	⇑	
		⇑	
-2.0	⇐⇐	0.0207	
:			



Probabilities of $X \sim N(\mu, \sigma)$:

■ Result: $X \sim N(\mu, \sigma) \Leftrightarrow Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$

■ $X \leq a \Leftrightarrow \frac{X - \mu}{\sigma} \leq \frac{a - \mu}{\sigma} \Leftrightarrow Z \leq \frac{a - \mu}{\sigma}$

1. $P(X \leq a) = P\left(Z \leq \frac{a - \mu}{\sigma}\right)$

2. $P(X \geq a) = 1 - P(X \leq a) = 1 - P\left(Z \leq \frac{a - \mu}{\sigma}\right)$

3. $P(a \leq X \leq b) = P(X \leq b) - P(X \leq a) = P\left(Z \leq \frac{b - \mu}{\sigma}\right) - P\left(Z \leq \frac{a - \mu}{\sigma}\right)$

4. $P(X = a) = 0$ for every a .

5. $P(X \leq \mu) = P(X \geq \mu) = 0.5$

Example:

Suppose that the hemoglobin level for healthy adults males has a normal distribution with mean $\mu=16$ and variance $\sigma^2 = 0.81$ (standard deviation $\sigma = 0.9$).

(a) Find the probability that a randomly chosen healthy adult male has hemoglobin level less than 14.

(b) What is the percentage of healthy adult males who have hemoglobin level less than 14?

- Let X = the hemoglobin level for a healthy adult male

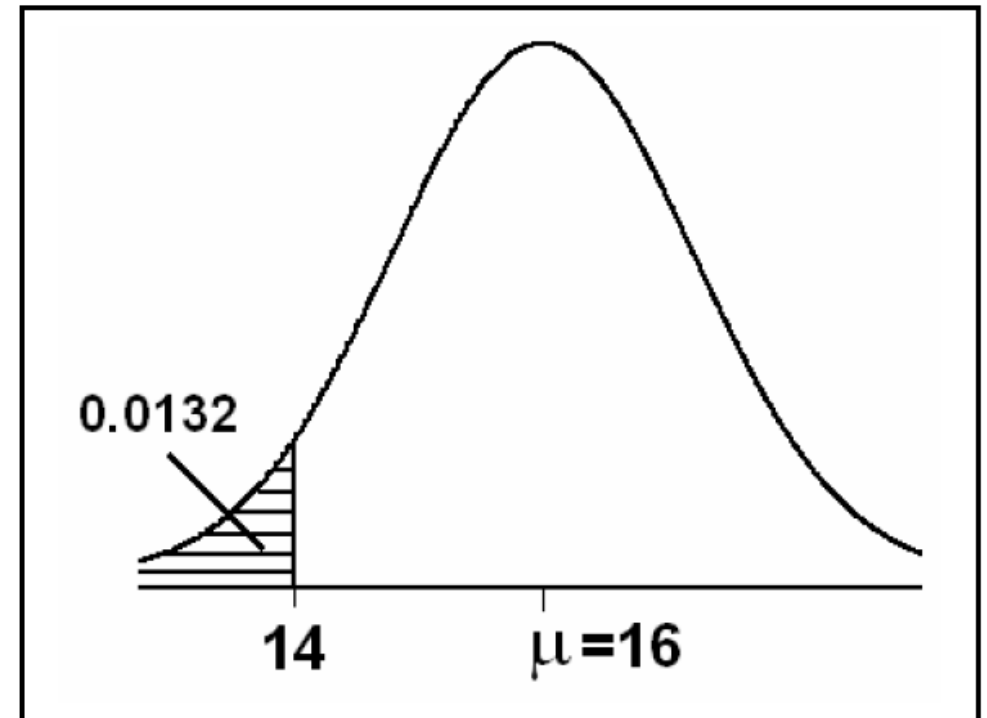
- $X \sim N(\mu, \sigma) = N(16, 0.9)$.

$$\begin{aligned} \text{(a) } P(X \leq 14) &= P\left(Z \leq \frac{14 - \mu}{\sigma}\right) = P\left(Z \leq \frac{14 - 16}{0.9}\right) \\ &= P(Z \leq -2.22) = 0.0132 \end{aligned}$$

(b) The percentage of healthy adult males who have hemoglobin level less than 14 is

$$\begin{aligned} P(X \leq 14) \times 100\% &= 0.01320 \times 100\% \\ &= 1.32\% \end{aligned}$$

Therefore, 1.32% of healthy adult males have hemoglobin level less than 14.



Example: Suppose that the birth weight of Saudi babies has a normal distribution with mean $\mu=3.4$ and standard deviation $\sigma=0.35$.

(a) Find the probability that a randomly chosen Saudi baby has a birth weight between 3.0 and 4.0 kg.

(b) What is the percentage of Saudi babies who have a birth weight between 3.0 and 4.0 kg?

- X = birth weight of a Saudi baby

$\mu = 3.4$, $\sigma = 0.35$

$X \sim N(3.4, 0.35)$

(a) $P(3.0 < X < 4.0) = P(X < 4.0) - P(X < 3.0)$

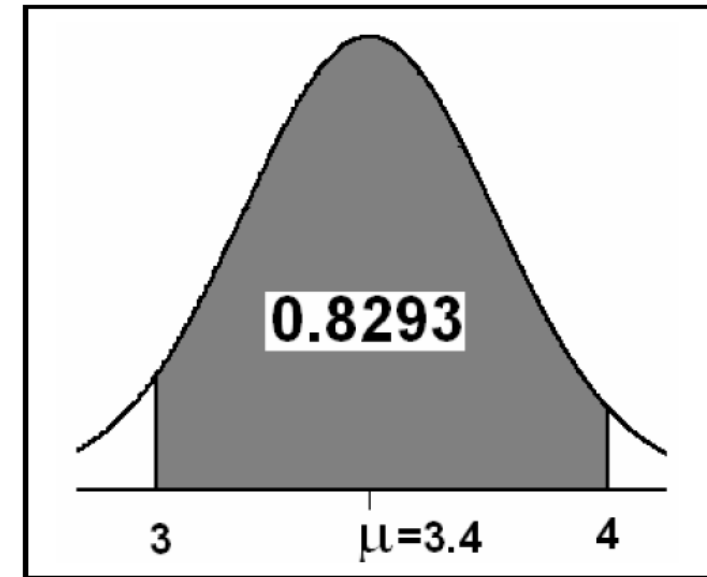
$$= P\left(Z \leq \frac{4.0 - \mu}{\sigma}\right) - P\left(Z \leq \frac{3.0 - \mu}{\sigma}\right)$$

$$= P\left(Z \leq \frac{4.0 - 3.4}{0.35}\right) - P\left(Z \leq \frac{3.0 - 3.4}{0.35}\right)$$

$$= P(Z \leq 1.71) - P(Z \leq -1.14)$$

$$= 0.9564 - 0.1271$$

$$= 0.8293$$



(b) The percentage of Saudi babies who have a birth weight between 3.0 and 4.0 kg is

$$P(3.0 < X < 4.0) \times 100\% = 0.8293 \times 100\% = 82.93\%$$

Notation:

- $P(Z \geq Z_A) = A$

Result:

- $Z_A = -Z_{1-A}$

- **Example:**

- $Z \sim N(0,1)$

- $P(Z \geq Z_{0.025}) = 0.025$

- $P(Z \geq Z_{0.95}) = 0.95$

- $P(Z \geq Z_{0.90}) = 0.90$

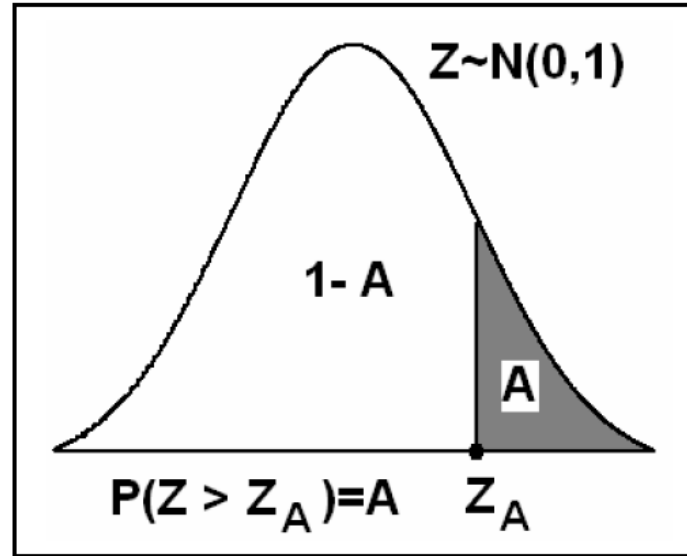
- **Example:**

- $Z \sim N(0,1)$

- $Z_{0.025} = 1.96$

- $Z_{0.95} = -1.645$

- $Z_{0.90} = -1.285$



Z	...	0.06
:	:	$\uparrow\uparrow$
1.9	$\Leftarrow\Leftarrow$	0.975
$P(Z \geq Z_{0.025}) = 0.025$		
$Z_{0.025} = 1.96$		

Application of the Normal Distribution

Example 6.9: In an industrial process, the diameter of a ball bearing is an important component part. The buyer sets specifications on the diameter to be 3.00 ± 0.01 cm. The implication is that no part falling outside these specifications will be accepted. It is known that, in the process, the diameter of a ball bearing has a normal distribution with mean 3.00 cm and standard deviation 0.005 cm. On the average, how many manufactured ball bearings will be scrapped?

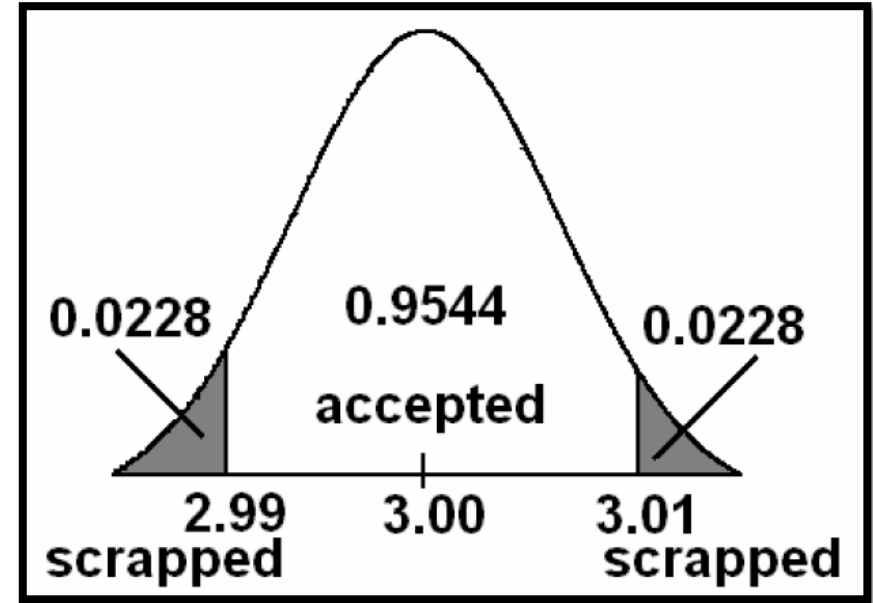
Sol.

- $\mu = 3.00$
- $\sigma = 0.005$
- $X = \text{diameter}$
- $X \sim N(3.00, 0.005)$
- The specification limits are:
- 3.00 ± 0.01
- $x_1 = \text{Lower limit} = 3.00 - 0.01 = 2.99$
- $x_2 = \text{Upper limit} = 3.00 + 0.01 = 3.01$
- $P(x_1 < X < x_2) = P(2.99 < X < 3.01) = P(X < 3.01) - P(X < 2.99)$



$$P(x_1 < X < x_2) = P(2.99 < X < 3.01) = P(X < 3.01) - P(X < 2.99)$$

$$\begin{aligned}
 &= P\left(Z \leq \frac{3.01 - \mu}{\sigma}\right) - P\left(Z \leq \frac{2.99 - \mu}{\sigma}\right) \\
 &= P\left(Z \leq \frac{3.01 - 3.00}{0.005}\right) - P\left(Z \leq \frac{2.99 - 3.00}{0.005}\right) \\
 &= P(Z \leq 2.00) - P(Z \leq -2.00) \\
 &= 0.9772 - 0.0228 \\
 &= 0.9544
 \end{aligned}$$



Therefore, on the average, 95.44% of manufactured ball bearings will be accepted and 4.56% will be scrapped.

Example 6.10:

Gauges are used to reject all components where a certain dimension is not within the specifications $1.50 \pm d$. It is known that this measurement is normally distributed with mean 1.50 and standard deviation 0.20. Determine the value d such that the specifications cover 95% of the measurements.

Solution:

$$\mu = 1.5$$

$$\sigma = 0.20$$

X = measurement

$$X \sim N(1.5, 0.20)$$

The specification limits are:

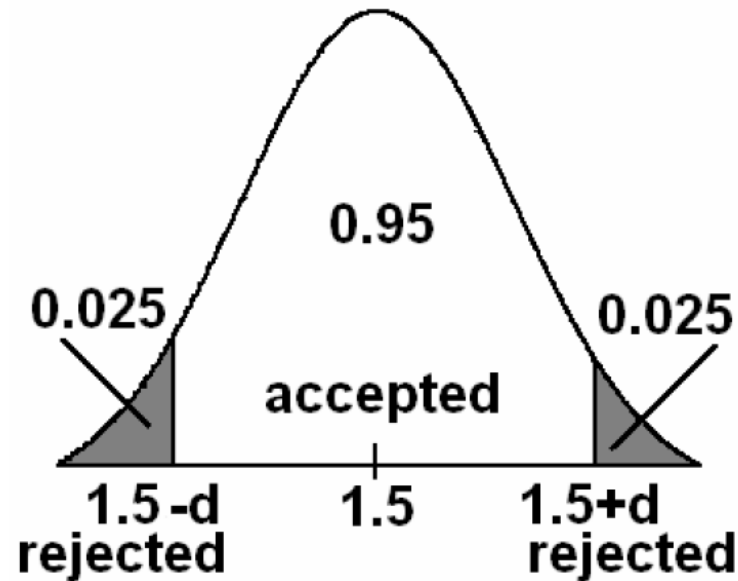
$$1.5 \pm d$$

$$x_1 = \text{Lower limit} = 1.5 - d$$

$$x_2 = \text{Upper limit} = 1.5 + d$$

$$P(X > 1.5 + d) = 0.025 \Leftrightarrow P(X < 1.5 + d) = 0.975$$

$$P(X < 1.5 - d) = 0.025$$



$$P\left(\frac{X - \mu}{\sigma} \leq \frac{(1.5 - d) - \mu}{\sigma}\right) = 0.025$$

$$P\left(Z \leq \frac{(1.5 - d) - \mu}{\sigma}\right) = 0.025$$

$$P\left(Z \leq \frac{(1.5 - d) - 1.5}{0.20}\right) = 0.025$$

$$P\left(Z \leq \frac{-d}{0.20}\right) = 0.025$$

$$\frac{-d}{0.20} = -1.96$$

$$-d = (0.20)(-1.96)$$

$$d = 0.392$$

Z	...	0.06	
:	:	↑↑	
		↑↑	
-1.9	⇐⇐	0.025	
$P(Z \leq \frac{-d}{0.20}) = 0.025$ $\frac{-d}{0.20} = -1.96$ Note: $\frac{-d}{0.20} = Z_{0.025}$			

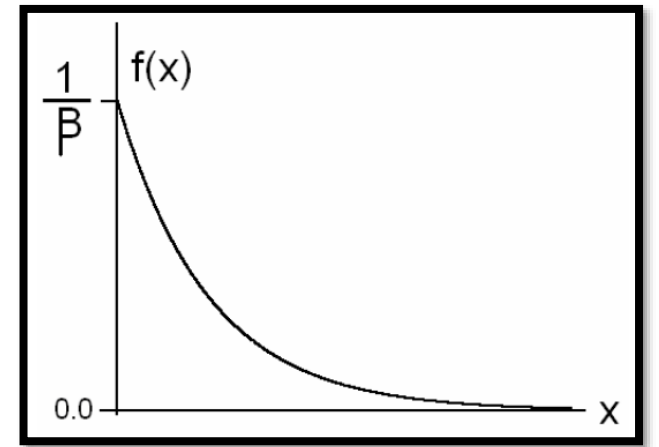
The specification limits are:

$$X_1 = \text{Lower limit} = 1.5 - d = 1.5 - 0.392 = 1.108$$

$$x_2 = \text{Upper limit} = 1.5 + d = 1.5 + 0.392 = 1.892$$

Therefore, 95% of the measurements fall within the specifications (1.108, 1.892).

Exponential Distribution



Definition:

- The continuous random variable X has an exponential distribution with parameter β , if its probability density function is given by:

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & ; x > 0 \\ 0 & ; elsewhere \end{cases}$$

- and we write $X \sim \text{Exp}(\beta)$

• Theorem:

- If the random variable X has an exponential distribution with parameter β , i.e., $X \sim \text{Exp}(\beta)$, then the mean and the variance of X are:

$$\begin{aligned} E(X) &= \mu = \beta \\ \text{Var}(X) &= \sigma^2 = \beta^2 \end{aligned}$$

Example 6.17:

Suppose that a system contains a certain type of component whose time in years to failure is given by T . The random variable T is modeled nicely by the exponential distribution with mean time to failure $\beta=5$. If 5 of these components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years?

Solution:

- $\beta=5$
- $T \sim \text{Exp}(5)$
- The pdf of T is
- The probability that a given component is still functioning after 8 years is given by:

$$f(t) = \begin{cases} \frac{1}{5} e^{-t/5} & ; t > 0 \\ 0 & ; \text{elsewhere} \end{cases}$$
$$P(T > 8) = \int_8^{\infty} f(t) dt = \int_8^{\infty} \frac{1}{5} e^{-t/5} dt = e^{-8/5} = 0.2$$

Now define the random variable:

X = number of components functioning after 8 years out of 5 components

$X \sim \text{Binomial}(5, 0.2)$ ($n=5, p= P(T>8)= 0.2$)

$$f(x) = P(X = x) = b(x; 5, 0.2) = \begin{cases} \binom{5}{x} 0.2^x 0.8^{5-x} ; & x = 0, 1, \dots, 5 \\ 0 ; & \text{otherwise} \end{cases}$$

The probability that at least 2 are still functioning at the end of 8 years is:

$$\begin{aligned} P(X \geq 2) &= 1 - P(X < 2) = 1 - [P(X=0) + P(X=1)] \\ &= 1 - \left[\binom{5}{0} 0.2^0 0.8^{5-0} + \binom{5}{1} 0.2^1 0.8^{5-1} \right] \\ &= 1 - [0.8^5 + 5 \times 0.2 \times 0.8^4] \\ &= 1 - 0.7373 \\ &= 0.2627 \end{aligned}$$