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Hermitian matrix polynomials with real eigenvalues of definite type. Part I: Classification ☆

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ABSTRACT

The spectral properties of Hermitian matrix polynomials with real eigenvalues have been extensively studied, through classes such as the definite or definitizable pencils, definite, hyperbolic, or quasi-hyperbolic matrix polynomials, and overdamped or gyroscopically stabilized quadratics. We give a unified treatment of these and related classes that uses the eigenvalue type (or sign characteristic) as a common thread. Equivalent conditions are given for each class in a consistent format. We show that these classes form a hierarchy, all of which are contained in the new class of quasidefinite matrix polynomials. As well as collecting and unifying existing results, we make several new contributions. We propose a new characterization of hyperbolicity in terms of the distribution of the eigenvalue types on the real line. By analyzing their effect on eigenvalue type, we show that homogeneous rotations allow results for matrix polynomials with nonsingular or definite leading coefficient to be translated into results with no such requirement on the leading coefficient, which is important for treating definite and quasidefinite polynomials. We also give a sufficient and necessary condition for a quasihyperbolic matrix polynomial to be strictly isospectral to a real diagonal quasihyperbolic matrix polynomial of the same degree, and show that this condition is always satisfied in the quadratic case and for any hyperbolic matrix polynomial, thereby identifying an important new class of diagonalizable matrix polynomials.

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1. Introduction

Eigenvalue problems $Ax = \lambda x$ with Hermitian A have many desirable properties which lead to a variety of special algorithms. Here we consider what can be regarded as the closest analogs of this class of problems for the generalized eigenvalue problem $L(\lambda)x = 0$, with $L(\lambda) = \lambda A - B, A = A^*, B = B^*$, and for the polynomial eigenvalue problem $P(\lambda)x = 0$, with

$$P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j, \quad A_j = A_j^*, \quad j = 0{:}\, m, \tag{1.1}$$

namely, the classes of definite, definitizable, hyperbolic, quasihyperbolic, overdamped and gyroscopically stabilized eigenproblems [7,16,19,24,28,33]. A property common to all these problems is that the eigenvalues are all real and of definite type, that is, $x^*P'(\lambda_0)x \neq 0$ for all nonzero $x \in \ker P(\lambda_0)$ and for all eigenvalues λ_0. Here $P'(\lambda)$ denotes the first derivative of P with respect to λ. We assume throughout that the matrix polynomial P is *regular*, that is, $\det P(\lambda) \not\equiv 0$.

The interest in matrix polynomials with real eigenvalues of definite type comes from systems of differential equations with constant coefficients of the form

$$\sum_{j=0}^{m} i^j A_j \frac{d^j u}{dt^j} = 0, \quad t \in \mathbb{R}, \tag{1.2}$$

where $i = \sqrt{-1}$, $A_j = A_j^* \in \mathbb{C}^{n\times n}$, $j = 0{:}\, m$, and A_m nonsingular. It is known [9, Theorem 13.1.1] that the general solution of (1.2) is given by

$$u(t) = \begin{bmatrix} 0 & \cdots & 0 & I_n \end{bmatrix} e^{-itC} u_0, \tag{1.3}$$

where

$$C = \begin{bmatrix} -A_m^{-1} & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} \begin{bmatrix} A_{m-1} & A_{m-2} & \dots & A_0 \\ I_n & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & I_n & 0 \end{bmatrix} \tag{1.4}$$

is the companion form of $A_m^{-1}P(\lambda)$ and $u_0 \in \mathbb{C}^{nm}$ is arbitrary. The solutions (1.3) are bounded on the half line $[0, \infty)$ if and only if C, or equivalently $P(\lambda)$, has all its eigenvalues real and semisimple,[1] and these solutions remain bounded under small perturbations of the matrix coefficients A_j of $P(\lambda)$ if and only if the eigenvalues of P are real and of definite type [9, Theorem 13.2.1].

The results presented in this paper are useful in the solution of the inverse problem of constructing quasidefinite Hermitian matrix polynomials and their subclasses from given spectral data, as will be shown in part II of this paper [1].

In this work we investigate the many subclasses of Hermitian matrix polynomials having real eigenvalues, giving a unified treatment that provides a consistent set of properties of each class and shows the relations between the classes. A key unifying tool is the eigenvalue type (or sign characteristic). We show that definite pencils and hyperbolic and definite matrix polynomials can all be characterized in terms of the eigenvalue types—something that does not seem well known for definite pencils.

We also extend the notion of quasihyperbolic matrix polynomials, which includes definitizable pencils as a special case, in a way that relaxes the requirement of nonsingularity of the leading coefficient matrix, yielding what we call quasidefinite matrix polynomials. A key idea here is the use of homogeneous rotations and consideration of its effect on eigenvalue types. The quasidefinite matrix polynomials include all the other classes considered here as special cases. Fig. 1 depicts all these classes and the connections between them and thereby provides a diagrammatic summary of most of the results of this paper.

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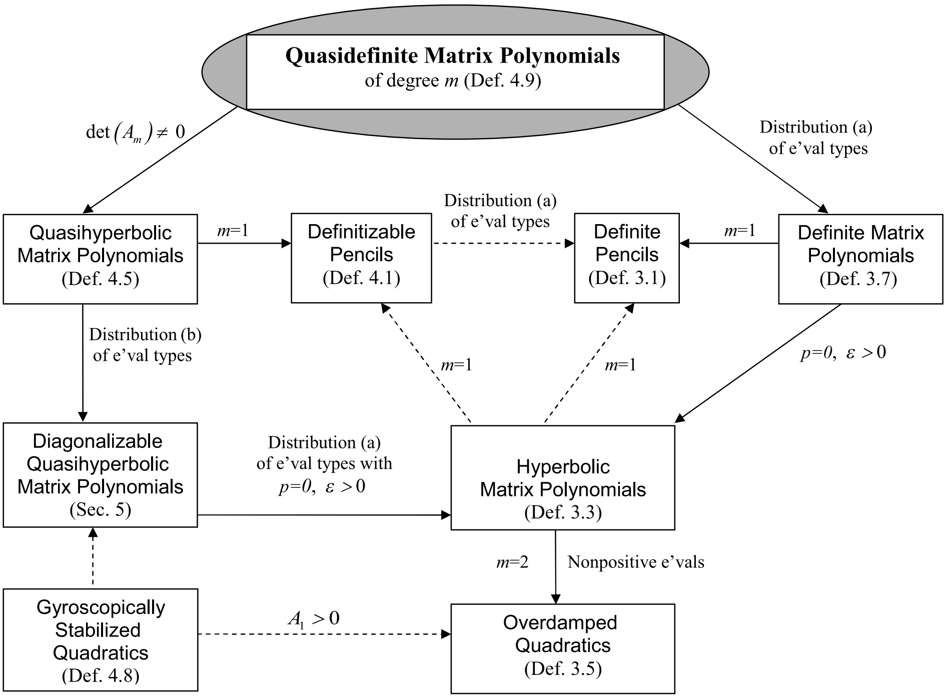
[1] An eigenvalue of a matrix polynomial $P(\lambda) = \sum_{k=0}^{m} \lambda^k A_k$ is semisimple if it appears only in 1×1 Jordan blocks in a Jordan form for P [10].



Distribution (a) :

$$\underbrace{\lambda_{mn} \le \cdots \le \lambda_{(m-1)n+p+1}}_{\substack{n-p \text{ eigenvalues} \\ \text{of } (-1)^{m-1}\varepsilon \text{ type}}} < \cdots < \underbrace{\lambda_{jn+p} \le \cdots \le \lambda_{(j-1)n+p+1}}_{\substack{n \text{ eigenvalues of} \\ (-1)^{j-1}\varepsilon \text{ type}, 1 \le j \le m-1}} < \cdots < \underbrace{\lambda_p \le \cdots \le \lambda_1}_{\substack{p \text{ eigenvalues} \\ \text{of } -\varepsilon \text{ type}}},$$

with $0 \le p < n$ and where "$\alpha\varepsilon$ type" denotes positive type when $\alpha\varepsilon > 0$ and negative type otherwise, λ_{p+1} being of ε type.

Distribution (b) : there is a grouping of the eigenvalues into n subsets $\{\lambda_{i_j} : j = 1:m\}_{i=1}^{n}$ that can be ordered so that $\underbrace{\lambda_{i_m} < \cdots < \lambda_{i_2} < \lambda_{i_1}}_{\text{alternating types}}$, $i=1:n$.

Fig. 1. Quasidefinite $n \times n$ matrix polynomials $P(\lambda) = \sum_{i=0}^{m} \lambda^i A_i$ and their subclasses. A subclass $\mathcal{A}$ pointing to a subclass $\mathcal{B}$ with a solid line (dotted line) and property "C" means that the subclass $\mathcal{A}$ with the property "C" is exactly (is contained in) the subclass $\mathcal{B}$.

Matrix polynomials cannot in general be simultaneously diagonalized by a strict equivalence transformation. However, Lancaster and Zaballa [25] have recently characterized a class of quadratic matrix polynomials that can be diagonalized by applying a strict equivalence transformation or congruences to a linearization of the quadratic while preserving the structure of the linearization. Along the same line, we identify amongst all quasidefinite matrix polynomials of arbitrary degree those that can be diagonalized by a congruence transformation applied to a Hermitian linearization L of the matrix polynomial P while maintaining the block structure of the linearization L. In particular, we show that all hyperbolic matrix polynomials can be transformed to diagonal form in this way.

The paper is organized as follows. We recall in Section 2 the notions of sign characteristic and eigenvalue type. We also study the effects of homogeneous rotation and linearization on the eigenvalue types. Definite matrix polynomials and their subclasses are investigated in Section 3, while Section 4 deals with quasidefinite matrix polynomials and their subclasses. Finally quasidefinite matrix polynomials that can be transformed to diagonal form are identified in Section 5.

2. Preliminaries

An $n \times n$ Hermitian matrix polynomial P of degree m as in (1.1) has mn eigenvalues, which are all finite when A_m is nonsingular. Infinite eigenvalues occur when A_m is singular and zero eigenvalues are present when A_0 is singular. Because P is Hermitian, $\Lambda(P)$ is symmetric with respect to the real axis, where $\Lambda(P)$ denotes the spectrum of P.

We write $A > 0$ to denote that the Hermitian matrix A is positive definite. A Hermitian matrix A is definite if either $A > 0$ or $-A > 0$. Two definite matrices have opposite parity if one is positive definite and the other is negative definite. A sequence $A_0, A_1, A_2, \ldots$of definite matrices has alternating parity if A_j and A_{j+1} have opposite parity for all j.

We concentrate here on Hermitian matrix polynomials whose eigenvalues are all real and of definite type. For this reason we begin this section with a brief review of the eigenvalue types and sign characteristic (detailed discussions can be found in [9,10]).

2.1. Eigenvalue types and sign characteristic

We begin by defining the concept of eigenvalue type.

Definition 2.1 (*positive type/negative type*)**.** Let $P(\lambda)$ be a Hermitian matrix polynomial. A finite real eigenvalue λ_0 of P is of positive type (negative type) if $x^*P'(\lambda_0)x > 0$ ($x^*P'(\lambda_0)x < 0$) for all nonzero $x \in \ker P(\lambda_0)$, respectively.

Thus for an eigenvalue λ_0 of positive type (negative type), the graph of the scalar polynomial $x^*P(\lambda)x$ for any nonzero $x \in \ker P(\lambda_0)$ crosses the x-axis at λ_0 with a positive slope (negative slope). Note that simple eigenvalues are either of positive type or of negative type since for any nonzero $x \in \ker P(\lambda_0)$, $x^*P'(\lambda_0)x \neq 0$ [2, Theorem 3.2]. This does not necessarily hold for semisimple eigenvalues: for example the pencil

$$L(\lambda) = \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -a & 0 \\ 0 & a \end{bmatrix}$$

has a semisimple eigenvalue $\lambda_0 = a$ with corresponding eigenvectors $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and it is easily seen that $e_1^*L'(a)e_1 = 1$ and $e_2^*L'(a)e_2 = -1$. The eigenvalue $\lambda_0 = a$ is of mixed type. Note that $x^*L'(\lambda_0)x = 0$ for $x = e_1 + e_2$.

Definition 2.2 (*definite type/mixed type*)**.** A finite real eigenvalue of a Hermitian matrix polynomial is of definite type if it is either of positive type or of negative type. It is of mixed type otherwise.

If λ is a real eigenvalue of P of mixed type then there exist $x, y \in \ker P(\lambda)$ such that $x^*P'(\lambda)x > 0$ and $y^*P'(\lambda)y < 0$. But $x + \alpha y \in \ker P(\lambda)$, $\alpha \in \mathbb{C}$ and clearly $(x + \alpha y)^*P'(\lambda)(x + \alpha y) = 0$ for some nonzero α (see the previous example).

Lemma 2.3. *A finite real eigenvalue λ_0 of a Hermitian matrix polynomial is of definite type if and only if $x^*P'(\lambda_0)x \neq 0$ for all nonzero $x \in \ker P(\lambda_0)$.*

As shown in [3, Lemma 2.1], eigenvalues of definite type are necessarily semisimple. Indeed, if λ_0 is not semisimple then there is an eigenvector x and a generalized eigenvector y such that $P(\lambda_0)y + P'(\lambda_0)x = 0$. Multiplying on the left by x^* yields

$$x^* \left(P(\lambda_0)y + P'(\lambda_0)x \right) = x^* P'(\lambda_0)x = 0.$$

Hence λ_0 is of mixed type.

We will need a notion of eigenvalue type at infinity. To this end use the reversal of P denoted by $\mathrm{rev}\, P$, which is obtained by reversing the order of the coefficient matrices of P:

$$\mathrm{rev}\, P(\lambda) = \lambda^m P(1/\lambda) = \lambda^m A_0 + \lambda^{m-1} A_1 + \cdots + \lambda A_{m-1} + A_m. \tag{2.1}$$

Note that λ_0 is an eigenvalue of $P(\lambda)$ if and only if $1/\lambda_0$ is an eigenvalue of $\mathrm{rev}\, P(\lambda)$ with 0 and ∞ regarded as reciprocals. Easy manipulations show that when $\lambda_0 \neq 0$ the equation $(m/\lambda_0)x^* P(\lambda_0)x = 0$ can be rewritten as

$$x^* P'(\lambda_0)x = -\lambda_0^{m-2} x^* (\mathrm{rev}\, P)'(1/\lambda_0)x. \tag{2.2}$$

This suggests the following definition.

Definition 2.4 (*type at* ∞)**.** The type of $\lambda_0 = \infty$ as an eigenvalue of a Hermitian matrix polynomial P is given by the type of $1/\lambda_0 = 0$ as an eigenvalue of $-\mathrm{rev}\, P$. In other words, $\lambda_0 = \infty$ is of positive type if $x^* A_{m-1} x < 0$ for every nonzero $x \in \ker \mathrm{rev}\, P(0)$ and of negative type if $x^* A_{m-1} x > 0$ for every nonzero $x \in \ker \mathrm{rev}\, P(0)$. The eigenvalue $\lambda_0 = \infty$ is of definite type whenever it is of positive type or negative type.

The notion of eigenvalue type is connected with the more general notion of sign characteristic of a Hermitian matrix polynomial with nonsingular leading term [9,10]. When all the eigenvalues λ_j of P are real, finite and of definite type, the sign characteristic, for a given ordering $\lambda_1, \ldots, \lambda_{mn}$, is a set of signs $\{\varepsilon_j\}_{j=1}^{mn}$ with $\varepsilon_j = \mathrm{sign}(x_j^* P'(\lambda_j)x_j)$, where x_j is an eigenvector corresponding to λ_j.

We will show in Sections 3 and 4 that the sign characteristic of definite pencils, overdamped and gyroscopically stabilized quadratics, and hyperbolic and definite polynomials has a particular distribution over the extended real line. Indeed the eigenvalues of these matrix polynomials belong to disjoint intervals, each interval containing eigenvalues of a single type. We say that an interval $\mathcal{I}$ of $\mathbb{R}$ is of positive (negative) type for a matrix polynomial P if every $\lambda \in \Lambda(P) \cap \mathcal{I}$ is of positive (negative) type. The interval $\mathcal{I}$ is of definite type if every $\lambda \in \Lambda(P) \cap \mathcal{I}$ is of definite type. We also use the wording "ε type" to denote positive type for $\varepsilon > 0$ and negative type for $\varepsilon < 0$.

2.2. Homogeneous rotations

We will use the homogenous forms of the matrix polynomial $P(\lambda)$ in (1.1) and the pencil $L(\lambda) = \lambda A - B$, which are given by

$$P(\alpha, \beta) = \sum_{j=0}^{m} \alpha^j \beta^{m-j} A_j, \quad L(\alpha, \beta) = \alpha A - \beta B.$$

This form is particularly useful when A_m or A is singular or indefinite. An eigenvalue λ is identified with any pair $(\alpha, \beta) \neq (0, 0)$ for which $\lambda = \alpha/\beta$. Note that $P(0, \beta) = \beta^m A_0$ so that $\lambda = 0$ represented by $(0, \beta)$ is an eigenvalue of P if and only if A_0 is singular. Similarly, $\lambda = \infty$ represented by $(\alpha, 0)$ is an eigenvalue of P if and only if A_m is singular. Without loss of generality we can take $\alpha^2 + \beta^2 = 1$. We then have a direct correspondence between eigenvalues on the extended real line $\mathbb{R} \cup \{\infty\}$ and the unit circle (see Fig. 2). Note the two copies of $\mathbb{R} \cup \{\infty\}$, represented by the upper semicircle and the lower semicircle.

The matrix polynomial $\widetilde{P}(\widetilde{\alpha}, \widetilde{\beta})$ is obtained from $P(\alpha, \beta)$ by homogenous rotation if

$$G \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} =: \begin{bmatrix} \widetilde{\alpha} \\ \widetilde{\beta} \end{bmatrix}, \quad c, s \in \mathbb{R}, \quad c^2 + s^2 = 1 \tag{2.3}$$

and

$$P(\alpha, \beta) = \sum_{j=0}^{m} (c\widetilde{\alpha} - s\widetilde{\beta})^j (s\widetilde{\alpha} + c\widetilde{\beta})^{m-j} A_j =: \sum_{j=0}^{m} \widetilde{\alpha}^j \widetilde{\beta}^{m-j} \widetilde{A}_j =: \widetilde{P}(\widetilde{\alpha}, \widetilde{\beta}). \tag{2.4}$$

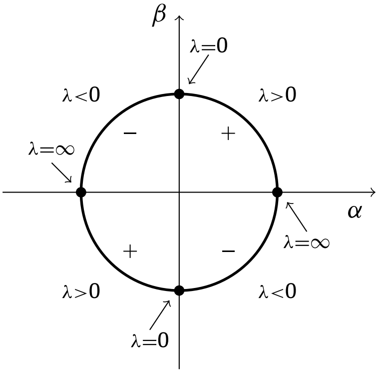


Fig. 2. Correspondence between λ and (α, β).

Homogeneous rotations can be seen as an analog of translations of λ in the nonhomogeneous case: one common feature is that they both preserve the eigenvectors. Indeed P and $\widetilde{P}$ have the same eigenvectors but the corresponding eigenvalues are rotated. On using $P(\alpha, \beta) = \widetilde{P}(\widetilde{\alpha}, \widetilde{\beta})$, the binomial expansion theorem leads to an expression for each $\widetilde{A}_j$. In particular we find that

$$\begin{aligned}
\widetilde{A}_m &= P(c, s),\\
\widetilde{A}_{m-1} &= \sum_{j=0}^{m} \left(-jc^{j-1}s^{m+1-j} + (m-j)c^{j+1}s^{m-j-1}\right) A_j,\\
\widetilde{A}_0 &= P(-s, c).
\end{aligned} \tag{2.5}$$

We will use homogeneous rotations to transform a polynomial P with singular or indefinite leading coefficient A_m to a polynomial $\widetilde{P}$ with nonsingular or positive definite leading coefficient $\widetilde{A}_m = P(c, s)$, which we can do provided that a pair (c, s) on the unit circle is known such that $\det(P(c, s)) \neq 0$ or $P(c, s) > 0$, respectively (see Example 2.5).

Example 2.5. The pencil

$$L(\lambda) = \lambda \mathrm{diag}(1, 1, -1) - \mathrm{diag}(2, 3, -5) =: \lambda A - B$$

has indefinite leading coefficient matrix A. Note that for $\mu = 4$, $L(\mu) = \mathrm{diag}(2, 1, 1) > 0$. We homogeneously rotate L into $\widetilde{L}$ so that μ corresponds to ∞. This is achieved by taking $c = \mu/\sqrt{\mu^2+1}$ and $s = 1/\sqrt{\mu^2+1}$ in (2.3). Then G rotates $L(\lambda)$ into $\widetilde{L}(\widetilde{\lambda}) =: \widetilde{\lambda}\widetilde{A} - \widetilde{B}$, where $\widetilde{A} = L(\mu)/\sqrt{\mu^2+1} > 0$. Note that L has eigenvalues 2 and 3 of positive type and eigenvalue 5 of negative type. These eigenvalues are rotated to 4.5, 13 and -21, respectively, all of positive type since $\widetilde{A}$ is positive definite.

Example 2.5 shows that homogeneous rotation does not preserve the eigenvalue types, but as the next lemma shows it always preserves definite type. To avoid ambiguity, P'_λ denotes the first derivative of P with respect to the variable λ.

Lemma 2.6. *Let $\widetilde{P}$ of degree m be obtained from P by homogeneous rotation (2.3). Let the real numbers $\lambda_0 = \frac{\alpha_0}{\beta_0}$ and $\widetilde{\lambda}_0 = \frac{\widetilde{\alpha}_0}{\widetilde{\beta}_0}$ with $\begin{bmatrix}\widetilde{\alpha}_0\\ \widetilde{\beta}_0\end{bmatrix} = G\begin{bmatrix}\alpha_0\\ \beta_0\end{bmatrix}$ be eigenvalues of P and $\widetilde{P}$, respectively, with corresponding eigenvector x.*

(*i*) *If λ_0 and $\widetilde{\lambda}_0$ are both real and finite then $c - \lambda_0 s \neq 0$ and*

$$x^* P'_\lambda(\lambda_0) x = (c - \lambda_0 s)^{m-2} x^* \widetilde{P}'_{\widetilde{\lambda}}(\widetilde{\lambda}_0) x.$$

(*ii*) *If* λ_0 *is real and finite and* $\widetilde{\lambda}_0 = \infty$ *then* $s \neq 0$ *and*

$$x^* P'_\lambda(\lambda_0)x = s^{2-m}\, x^* \left(-(\operatorname{rev}\widetilde{P})'_{\widetilde{\lambda}}(0)\right) x.$$

(*iii*) *If* $\lambda_0 = \infty$ *and* $\widetilde{\lambda}_0$ *is real and finite then* $s \neq 0$ *and*

$$x^* \left(-(\operatorname{rev} P)'_\lambda(0)\right) x = (-s)^{m-2}\, x^* \widetilde{P}'_{\widetilde{\lambda}}(\widetilde{\lambda}_0)x.$$

Proof. (i) Let $\lambda = \alpha/\beta$ and $\widetilde{\lambda} = \widetilde{\alpha}/\widetilde{\beta}$, where $\begin{bmatrix}\widetilde{\alpha}\\ \widetilde{\beta}\end{bmatrix} = G\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$. When λ and $\widetilde{\lambda}$ are finite, $\beta \neq 0$ and $\widetilde{\beta} = \beta c - \alpha s \neq 0$ hence $c - \lambda s \neq 0$. It follows from (2.4) that $\beta^m P(\lambda) = \widetilde{\beta}^m \widetilde{P}(\widetilde{\lambda}) = (\beta c - \alpha s)^m \widetilde{P}(\widetilde{\lambda})$ so that

$$P(\lambda) = (c - \lambda s)^m \widetilde{P}(g(\lambda)), \quad g(\lambda) = \frac{\lambda c + s}{c - \lambda s} = \widetilde{\lambda}. \tag{2.6}$$

Differentiating with respect to λ yields

$$P'_\lambda(\lambda) = -ms(c - \lambda s)^{m-1}\widetilde{P}(g(\lambda)) + (c - \lambda s)^{m-2}\widetilde{P}'_{\widetilde{\lambda}}(g(\lambda)). \tag{2.7}$$

Multiplying (2.7) on the left by x^* and on the right by x, evaluating at λ_0 and using $\widetilde{\lambda}_0 = g(\lambda_0)$ and $\widetilde{P}(\widetilde{\lambda}_0)x = 0$ yield the desired result.

(ii) When λ_0 is finite, $\beta_0 \neq 0$ and $\widetilde{\lambda}_0$ infinite implies that $\widetilde{\beta}_0 = \beta_0 c - \alpha_0 s = 0$, that is, $c = s\lambda_0$ with $s \neq 0$ since $c^2 + s^2 = 1$. Using (2.5) we obtain

$$s^{-m}\widetilde{A}_{m-1} = \sum_{j=0}^{m}\left(-j\lambda_0^{j-1} + (m-j)\lambda_0^{j+1}\right)A_j. \tag{2.8}$$

Multiplying (2.8) on the left by x^* and on the right by x we find that

- if $\lambda_0 \neq 0$ then

$$\begin{aligned} s^{-m}x^*\widetilde{A}_{m-1}x &= -x^*P'_\lambda(\lambda_0)x + \lambda_0^m x^*\left(A_{m-1} + \cdots + \frac{m-1}{\lambda_0^{m-2}}A_1 + \frac{m}{\lambda_0^{m-1}}A_0\right)x, \\ &= -x^*P'_\lambda(\lambda_0)x + \lambda_0^m x^*(\operatorname{rev} P)'_\lambda(1/\lambda_0)x, \\ &= -x^*P'_\lambda(\lambda_0)x - \lambda_0^2 x^*P'_\lambda(\lambda_0)x, \end{aligned}$$

  where we used (2.2) for the last equality. The relation in (ii) follows on noting that $1 + \lambda_0^2 = s^{-2}$ and $x^*\left((\operatorname{rev}\widetilde{P})'_{\widetilde{\lambda}}(0)\right)x = x^*\widetilde{A}_{m-1}x$.
- if $\lambda_0 = 0$ then $c = 0$, $s = \pm 1$ and

  $$s^{-m}x^*\widetilde{A}_{m-1}x = -x^*A_1x$$

  which is the relation in (ii) since $s = \pm 1$.

(iii) Suppose that G rotates $\lambda_0 = \infty$ to a finite eigenvalue $\widetilde{\lambda}_0$ then G^{-1} rotates $\widetilde{\lambda}_0$ to $\lambda_0 = \infty$ and we can apply (ii) to obtain the desired result. □

2.3. Hermitian linearizations

A standard way of treating the polynomial eigenvalue problem $P(\lambda)x = 0$, both theoretically and numerically, is to convert $P(\lambda)$ into an equivalent linear matrix pencil $L(\lambda) = \lambda A - B \in \mathbb{C}^{mn\times mn}$ by the process known as linearization. To be more specific, L is a linearization of P if it satisfies

$$E(\lambda)L(\lambda)F(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{(m-1)n}\end{bmatrix}$$

for some matrix polynomials $E(\lambda)$ and $F(\lambda)$ with constant nonzero determinants. It is clear that the eigenvalues of L and P coincide. As an example the pencil $\lambda I - C$ with C as in (1.4) is a linearization of $A_m^{-1}P(\lambda)$. Note that this linearization is not Hermitian when P is Hermitian.

In recent work [27] a vector space of pencils has been studied, namely,

$$\mathbb{L}_1(P) = \left\{ L(\lambda) : L(\lambda)(\Lambda \otimes I_n) = v \otimes P(\lambda),\ v \in \mathbb{C}^m \right\},$$

where $\Lambda = [\lambda^{m-1}, \lambda^{m-2}, \dots, 1]^T \in \mathbb{C}^m$, that generalizes the first companion form C in (1.4) (indeed $\lambda I - C \in \mathbb{L}_1(A_m^{-1}P)$ with $v = e_1$) and provides a rich source of interesting linearizations. In particular it is shown in [14] that for Hermitian P,

$$\begin{aligned} \mathbb{H}(P) &:= \left\{ L(\lambda) = \lambda A - B \in \mathbb{L}_1(P) :\ A^* = A,\ B^* = B \right\}, \\ &= \left\{ \sum_{j=1}^{m} v_j L_j(\lambda),\ v \in \mathbb{R}^m \right\}, \end{aligned} \tag{2.9}$$

where $L_j(\lambda) = \lambda B_j - B_{j-1}$ and the matrices B_j are a direct sum of block Hankel matrices (see [14, (3.6)–(3.8)]). The pencil $\lambda B_j - B_{j-1} \in \mathbb{H}(P)$ with vector $v = e_j$, where e_j denotes the jth column of the identity matrix, is referred to as the jth basis pencil. When A_m is nonsingular,

$$\lambda B_m - B_{m-1} := \lambda \begin{bmatrix} & & & & A_m \\ & & & \iddots & A_{m-1} \\ & & \iddots & \iddots & \vdots \\ & \iddots & \iddots & & A_2 \\ A_m & A_{m-1} & \dots & A_2 & A_1 \end{bmatrix} - \begin{bmatrix} & & & A_m & \\ & & \iddots & A_{m-1} & \\ & \iddots & \iddots & \vdots & \\ A_m & A_{m-1} & \dots & A_2 & \\ & & & & -A_0 \end{bmatrix} \tag{2.10}$$

is always a linearization of P by the following theorem. Note that $\mathbb{DL}(P)$, which can be defined as in (2.9) with conjugate transpose replaced by block transpose, is a vector space of pencils that contains $\mathbb{H}(P)$ [14].

Theorem 2.7 (eigenvalue exclusion theorem [27, Theorem 6.7])**.** *Suppose that $P(\lambda)$ is a regular matrix polynomial of degree m and $L(\lambda) \in \mathbb{DL}(P)$ with vector $v \in \mathbb{C}^m$. Then $L(\lambda)$ is a linearization for $P(\lambda)$ if and only if no root of*

$$\mathsf{p}(x; v) = v_1 x^{m-1} + v_2 x^{m-2} + \dots + v_{m-1} x + v_m \tag{2.11}$$

is an eigenvalue of $P(\lambda)$, where, by convention, $\mathsf{p}(x; v)$ has a root at ∞ whenever $v_1 = 0$.

The scalar polynomial in (2.11) is called the v-polynomial.

Any linearization $L(\lambda) \in \mathbb{H}(P)$ with vector v has the property that x is a right eigenvector of P associated to the eigenvalue λ if and only if $\Lambda \otimes x$ (if $\lambda$is finite) or $e_1 \otimes x$ (if $\lambda = \infty$) is a right eigenvector for L with eigenvalue λ.

The following result relates the type of a real eigenvalue λ of P to its type as an eigenvalue of a linearization $L \in \mathbb{H}(P)$ and shows that linearizations in $\mathbb{H}(P)$ preserve definite type.

Lemma 2.8. *Let (λ_0, x) be an eigenpair of P with λ_0 finite and let $L(\lambda) \in \mathbb{H}(P)$ with vector v be a linearization of P. Then,*

$$z^* L'(\lambda_0) z = \mathsf{p}(\lambda_0; v) \cdot x^* P'(\lambda_0) x, \tag{2.12}$$

where $z = \Lambda_0 \otimes x$. Hence a real eigenvalue λ_0 of L is of definite type if and only if λ_0 as an eigenvalue of P is of definite type. Moreover, if $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ with A_m nonsingular then $\lambda B_m - B_{m-1}$ in (2.10) is a linearization of P that preserves the type of the finite real eigenvalues.

Proof. How to obtain (2.12) can be found in [15, Section 3]. Now if $L \in \mathbb{H}(P)$ is a linearization of P then by Theorem 2.7, $\mathrm{p}(\lambda_0; v) \neq 0$. Hence $z^*L'(\lambda_0)z \neq 0$ if and only if $x^*P'(\lambda_0)x \neq 0$. The pencil in (2.10) is in $\mathbb{H}(P)$ with vector $v = e_m$ so $\mathrm{p}(\lambda_0; e_m) = 1$. It is a linearization of P when A_m is nonsingular. $\square$

3. Definite matrix polynomials

The class of definite matrix polynomials (defined in Section 3.3) has recently been introduced and investigated by Higham et al. [16]. It includes definite pencils, hyperbolic matrix polynomials, and overdamped quadratics. We review these subclasses in the following way: for each subclass we provide a list of equivalent properties, named consistently according to

(P1) concerning the distribution of the eigenvalue type on the real line,
(P2) in terms of certain definiteness properties,
(P3) concerning the roots of the scalar equations $x^*P(\lambda)x = 0$ (or $x^*L(\lambda)x = 0$ for pencils).

Each subclass has extra equivalent properties listed, either because the property is commonly used to define the subclass or because the property is relevant to the rest of the paper. We do not claim to provide a full list of characterizations.

3.1. Definite pencils

We start with definite pencils, whose occurrence is frequent in applications in science and engineering (see [5, Chapter 9,7] for examples).

Definition 3.1 (*definite pencils*)**.** An $n \times n$ Hermitian pencil is definite if it satisfies any one (and hence all) of the equivalent properties of Theorem 3.2.

Theorem 3.2. *For an $n \times n$ Hermitian pencil $L(\lambda) = \lambda A - B$ the following are equivalent*:

(*P1*) *$\Lambda(L) \subset \mathbb{R} \cup \{\infty\}$ with all eigenvalues of definite type and where the eigenvalues of positive type are separated from the eigenvalues of negative type (see Fig. 3).*
(*P2*) *$L(\mu)$ is a definite matrix for some $\mu \in \mathbb{R} \cup \{\infty\}$, or equivalently $L(\alpha, \beta) > 0$ for some (α, β) on the unit circle.*
(*P3*) *For every nonzero $x \in \mathbb{C}^n$, the scalar equation $x^*L(\lambda)x = 0$ has exactly one zero in $\mathbb{R} \cup \{\infty\}$.*
(*P4*) *$(x^*Ax, x^*Bx) \neq 0$ for all nonzero $x \in \mathbb{C}^n$.*
(*D*) *There exists a nonsingular $X \in \mathbb{C}^{n\times n}$ such that*

$$X^*L(\lambda)X = \begin{bmatrix} L_+(\lambda) & 0 \\ 0 & L_-(\lambda) \end{bmatrix},$$

where $L_+(\lambda) = \lambda D_+ - J_+$ and $L_-(\lambda) = \lambda D_- - J_-$ are real diagonal pencils, such that $[\lambda_{\min}(L_+), \lambda_{\max}(L_+)] \cap [\lambda_{\min}(L_-), \lambda_{\max}(L_-)] = \emptyset$, D_+ has nonnegative entries, D_- has nonpositive entries and if $(D_+)_{ii} = 0$ then $(J_+)_{ii} > 0$ or if $(D_-)_{ii} = 0$ then $(J_-)_{ii} < 0$.

Proof. The proof of (P2) $\Leftrightarrow$ (P4) can be found in [31, Theorem 6.1.18] and the equivalence (P3) $\Leftrightarrow$ (P4) is immediate. We show that (P2) $\Rightarrow$ (P1) $\Rightarrow$ (D) $\Rightarrow$ (P2).

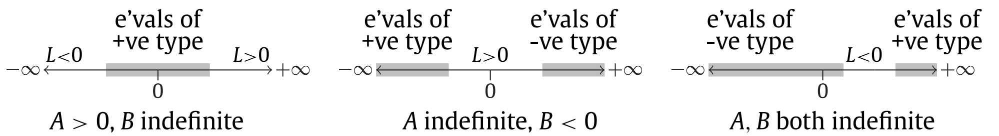


Fig. 3. Distribution of the eigenvalue types for three types of definite pencils $L(\lambda) = \lambda A - B$. On the shaded intervals L is indefinite.

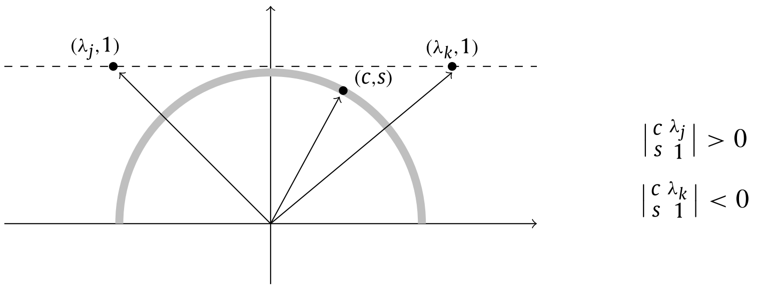


Fig. 4. Pictorial representation of (3.2).

(P2) $\Rightarrow$ (P1): Suppose $L(c,s) > 0$ for some $c, s \in \mathbb{R}$, $c^2 + s^2 = 1$. If $s = 0$ then A is definite so that all the eigenvalues belong to one interval of either positive type if $A > 0$ or negative type if $A < 0$ since for all eigenpairs (λ, x), $x^*L'(\lambda)x = x^*Ax$ is either positive or negative. Assume without loss of generality that $s > 0$ and homogeneously rotate L into $\widetilde{L}(\widetilde{\lambda}) = \widetilde{\lambda}\widetilde{A} - \widetilde{B}$ as in Example 2.5 so that $\widetilde{A} > 0$. Hence all the eigenvalues of $\widetilde{L}$ are real and of positive type. Let λ_j be an eigenvalue of L rotated to $\widetilde{\lambda}_j$. By Lemma 2.6, their types are related by

$$x^*\widetilde{L}'_{\widetilde{\lambda}}(\widetilde{\lambda}_j)x = \begin{cases} (c - \lambda_j s)x^*L'_{\lambda}(\lambda_j)x & \text{if } \lambda_j \text{ is finite,} \\ -sx^*\left(-(\operatorname{rev} L)'_{\lambda}(0)\right)x & \text{if } \lambda_j = \infty. \end{cases} \tag{3.1}$$

Note that

$$c - \lambda_j s = \det \begin{bmatrix} c & \lambda_j \\ s & 1 \end{bmatrix} \tag{3.2}$$

and the sign of these determinants is positive for any $\lambda_j = (\lambda_j, 1)$ that lies counterclockwise from (c,s), and negative for any that lies clockwise from (c,s); see Fig. 4. Hence it follows from (3.1) that eigenvalues of L lying clockwise from (c,s) (including $+\infty$) are of negative type and eigenvalues of L lying counterclockwise from (c,s) are of positive type. Also, there is a gap between the two types because c/s is not an eigenvalue of L.

(P1) $\Rightarrow$ (D): Recall that a Hermitian pencil is diagonalizable by congruence if and only if its eigenvalues belong to $\mathbb{R} \cup \{\infty\}$ and are semisimple [23]. Since eigenvalues of definite type are semisimple, there exists X nonsingular such that $X^*L(\lambda)X = \lambda D - J$, with D and J both real and diagonal. Their diagonal entries can be reordered so that $D = D_+ \oplus D_-$ and $J = J_+ \oplus J_-$, where the eigenvalues of $\lambda D_+ - J_+ = L_+(\lambda)$ are of positive type and that of $\lambda D_- - J_- = L_-(\lambda)$ are of negative type. If A is singular then one of D_+ or D_- (but not both otherwise $\lambda = \infty$ would be of mixed type) must be singular. Hence D_+ and $-D_-$ have nonnegative entries. Each zero entry on D corresponds to an infinite eigenvalue. By Definition 2.4 when $(D_+)_{ii} = 0$ we must have $(J_+)_{ii} > 0$ for $\lambda = \infty$ to be of positive type and when $(D_-)_{ii} = 0$ then $(J_-)_{ii} < 0$. Finally because the eigenvalues of positive type are separated from the eigenvalues of negative type, the intersection between $[\lambda_{\min}(L_+), \lambda_{\max}(L_+)]$ and $[\lambda_{\min}(L_-), \lambda_{\max}(L_-)]$ must be empty.

(D) $\Rightarrow$ (P2): It follows from (D) that

$$\begin{aligned} &L_+(\mu) < 0 \text{ for } \mu < \lambda_{\min}(L_+), \quad L_+(\mu) > 0 \text{ for } \mu > \lambda_{\max}(L_+) \text{ if } D_+ \text{ is nonsingular,} \\ &L_-(\mu) > 0 \text{ for } \mu < \lambda_{\min}(L_-), \quad L_-(\mu) < 0 \text{ for } \mu > \lambda_{\max}(L_-) \text{ if } D_- \text{ is nonsingular.} \end{aligned}$$

Hence

(i) if L_- is void then $L(\mu) < 0$ for $\mu < \lambda_{\min}(L_+)$,
(ii) if L_+ is void then $L(\mu) > 0$ for $\mu < \lambda_{\min}(L_-)$,
(iii) if $\lambda_{\max}(L_+) < \lambda_{\min}(L_-)$ then $L(\mu) > 0$ for $\lambda_{\max}(L_+) < \mu < \lambda_{\min}(L_-)$,
(iv) if $\lambda_{\max}(L_-) < \lambda_{\min}(L_+)$ then $L(\mu) < 0$ for $\lambda_{\max}(L_-) < \mu < \lambda_{\min}(L_+)$. $\square$

Characterizations (P2) and (P4) in Theorem 3.2 are commonly used as definitions of definite pencils. In (P2), $\mu = \infty$ is allowed and $L(\infty)$ definite means that A is definite. Note that (P4) is equivalent to saying that 0 is not in the field of values of $A + iB$ or that the Crawford number

$$\gamma(A, B) = \min_{\substack{z\in\mathbb{C}^n \\ z^*z=1}} \sqrt{(z^*Az)^2 + (z^*Bz)^2}$$

is strictly positive. Finally we remark that in property (D) all the eigenvalues of L_+ are of positive type and those of L_- are of negative type.

Pencils $L(\lambda) = \lambda A - B$ with $A > 0$ have computational advantages: the eigenvalues can be computed by methods that exploit the definiteness of A [6]. When A and B are both indefinite, characterization (P1) offers an easy way to check definiteness, but it is computationally unattractive since it requires all the eigenpairs. As an alternative, the recently improved arc algorithm of Crawford and Moon [4,12] efficiently detects whether $\lambda A - B$ is definite and determines μ such that $L(\mu) > 0$ at the cost of just a few Cholesky factorizations. The pencil can then be rotated to a pencil with positive definite leading term as in Example 2.5.

3.2. Hyperbolic matrix polynomials

Hyperbolic matrix polynomials generalize definite pencils $\lambda A - B$ with $A > 0$.

Definition 3.3 (*hyperbolic matrix polynomial*)**.** A Hermitian matrix polynomial is hyperbolic if it satisfies any one (and hence all) of the equivalent properties of Theorem 3.4.

Theorem 3.4. *For an $n \times n$ Hermitian matrix polynomial $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ the following are equivalent*:

(*P1*) *All eigenvalues are real and finite, of definite type, and such that*

$$\underbrace{\lambda_{mn} \leqslant \cdots \leqslant \lambda_{(m-1)n+1}}_{(-1)^{m-1}\text{ type}} < \cdots < \underbrace{\lambda_{2n} \leqslant \cdots \leqslant \lambda_{n+1}}_{\text{negative type}} < \underbrace{\lambda_{n} \leqslant \cdots \leqslant \lambda_{1}}_{\text{positive type}},$$

where "$(-1)^{m-1}$ type" denotes positive type for odd m and negative type for even m.

(*P2*) *There exist $\mu_j \in \mathbb{R} \cup \{\infty\}$ such that*

$$(-1)^j P(\mu_j) > 0, \quad j = 0\colon m-1, \quad \infty = \mu_0 > \mu_1 > \mu_2 > \cdots > \mu_{m-1}.$$

(*P3*) *$A_m > 0$ and for every nonzero $x \in \mathbb{C}^n$, the scalar equation $x^*P(\lambda)x = 0$ has m distinct real and finite zeros.*

(*L*) *P has a definite linearization $L(\lambda) \in \mathbb{H}(P)$ with vector $v \in \mathbb{R}^m$, where $v_1 \neq 0$, such that $L(\infty) > 0$ if $v_1 > 0$ and $L(\infty) < 0$ if $v_1 < 0$.*

Proof. That (P2) $\Leftrightarrow$ (P3) and (P2) $\Rightarrow$ (P1) is due to Markus [28, Section 31]. We show that (P1) $\Rightarrow$ (L) and (L) $\Rightarrow$ (P3).

(P1) $\Rightarrow$ (L): If $m = 1$ and $L \in \mathbb{H}(P)$ then $L(\lambda) = vP(\lambda)$ and L is a linearization if $v \neq 0$. By property (P1) of Theorem 3.2, $P(\lambda)$ is a definite pencil. Since all the eigenvalues are of positive type, property (D) of Theorem 3.2 implies that the leading coefficient of P is positive definite, i.e., $P(\infty) > 0$. Hence property (L) holds.

Now assume that $m > 1$. Let $v \in \mathbb{R}^m$ be such that the roots μ_j of the v-polynomial in (2.11) satisfy $\lambda_{jn} > \mu_j > \lambda_{jn+1}$, $j = 1\colon m-1$. Then by Theorem 2.7, $L(\lambda) \in \mathbb{H}(P)$ with vector v is a linearization of P. By construction, $\mathsf{p}(x; v) = v_1 \prod_{j=1}^{m}(x - \mu_j)$ with $v_1 \neq 0$ since all roots of $\mathsf{p}(x; v)$ are finite and $\operatorname{sign}(\mathsf{p}(\lambda_k; v)) = (-1)^{j-1}\operatorname{sign}(v_1)$ for $(j-1)n + 1 \leqslant k \leqslant jn$, $j = 1\colon m-1$ (see Fig. 5). By Lemma 2.8 we have that for each eigenpair (λ_k, x_k) of P, $z_k^* L'(\lambda_k) z_k = \mathsf{p}(\lambda_k; v) \cdot x_k^* P'(\lambda_k) x_k$, where $z_k = [\lambda_k^{m-1}, \lambda_k^{m-2}, \ldots, 1]^T \otimes x_k$ is an eigenvector of L with eigenvalue λ_k. Hence all eigenvalues of L are of positive type when $v_1 > 0$ and of negative type when $v_1 < 0$. Now properties (P1) and (D) of Theorem 3.2 imply that L is a definite pencil with $L(\infty) > 0$ if $v_1 > 0$ and $L(\infty) < 0$ if $v_1 < 0$.

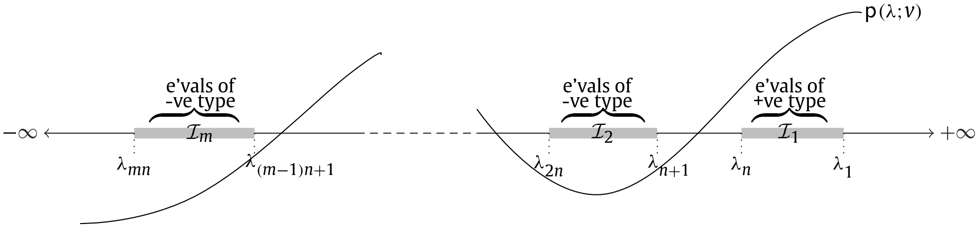


Fig. 5. Distribution of eigenvalue types of $n \times n$ hyperbolic polynomials of even degree m.

(L) $\Rightarrow$ (P3): If P has a definite linearization then by [16, Theorem 4.1], there exists $\mu \in \mathbb{R} \cup \{\infty\}$ such that $P(\mu)$ is definite and for every nonzero $x \in \mathbb{C}^n$ the scalar equation $x^*P(\lambda)x = 0$ has m distinct zeros in $\mathbb{R} \cup \{\infty\}$. By [16, Theorem 4.2], $L(\infty)$ definite implies $P(\infty)$ definite and [16, (4.7)] shows that $L(1, 0)$ is congruent to a block diagonal form whose (1, 1) block is $v_1P(1, 0)$. Now if $v_1 > 0$ and $L(1, 0) > 0$ or $v_1 < 0$ and $L(1, 0) < 0$ then $A_m = P(1, 0) > 0$. Hence $x^*A_mx \neq 0$ and $x^*P(\lambda)x = 0$ has m distinct real finite zeros. □

Property (P3) is used by Gohberg et al. [9, Section 13.4] as the definition of hyperbolicity. Characterization (P1) is stated for the quadratic case and without proof in [20, Section 9,24].

It is shown in [30, Corollary 2] that for a Hermitian triple (A, B, C), $(x^*Ax, x^*Bx, x^*Cx) \neq 0$ for all nonzero $x \in \mathbb{C}^n$ if and only if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha A + \beta B + \gamma C > 0$. So in view of (P2) in Theorem 3.4, the natural extension to quadratics of property (P4) for definite pencils in Theorem 3.2, i.e., $(x^*A_2x, x^*A_1x, x^*A_0x) \neq 0$ for all nonzero $x \in \mathbb{C}^n$, is a necessary condition for hyperbolicity (since $P(\mu) > 0$ for sufficiently large μ), but it is not sufficient: the quadratic $Q(\lambda) = \lambda^2 I_n + \lambda I_n + I_n$ is not hyperbolic since it has nonreal eigenvalues but $(x^*x, x^*x, x^*x) \neq 0$ for all nonzero $x \in \mathbb{C}^n$.

As for definite pencils, property (P1) is easy to check once all the eigenpairs of P are known. More efficient approaches to check whether a quadratic is hyperbolic or not include Guo, Higham and Tisseur's quadratically convergent matrix iteration based on cyclic reduction [11] or Guo, Higham and Tisseur's adaptation of the arc algorithm [12, Section 4.1]. For matrix polynomials of arbitrary degree, Niendorf and Voss [29] propose an algorithm that checks for hyperbolicity and which is based on a minmax and maxmin characterization of the eigenvalues.

3.2.1. Overdamped quadratics

Overdamped quadratics form a subclass of hyperbolic quadratics. They arise in overdamped systems in structural mechanics [7,18, Section 7.6] and are defined as follows.

Definition 3.5 (*overdamped quadratic*)**.** A quadratic matrix polynomial is overdamped if it satisfies any one (and hence all) of the equivalent properties of Theorem 3.6.

Theorem 3.6. *For a Hermitian quadratic matrix polynomial $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ the following are equivalent*:

(*P1*) *All eigenvalues are real, finite, nonpositive and distributed in two disjoint closed intervals, the left-most containing n eigenvalues of negative type and the right-most containing n eigenvalues of positive type*.

(*P2*) $A_2 > 0, A_1 > 0, A_0 \geqslant 0$, *and* $Q(\mu) < 0$ *for some* $\mu < 0$.

(*P3*) $A_2 > 0$ *and for every nonzero* $x \in \mathbb{C}^n$, *the scalar equation* $x^*Q(\lambda)x = 0$ *has 2 distinct real and finite nonpositive zeros*.

(*O*) $A_2 > 0, A_1 > 0, A_0 \geqslant 0$ *and*

$$(x^*A_1x)^2 > 4(x^*A_2x)(x^*A_0x) \quad \text{for all nonzero } x \in \mathbb{C}^n. \tag{3.3}$$

Proof. The equivalent characterizations of overdamping (P2), (P3) and (O) can be found in [11, Theorem 2.5]. Note that (P1) $\Leftrightarrow$ (P3) follows from (P1) $\Leftrightarrow$ (P3) in Theorem 3.4 to which is added the extra constraint that all the eigenvalues be nonpositive. □

Note that property (O) is usually taken as the definition of overdamped quadratics. If equality is allowed in (3.3) for some nonzero x then the quadratic is said to be weakly overdamped. Its $2n$ eigenvalues are real and when ordered, $\lambda_n = \lambda_{n+1}$ with partial multiplicities[2] at most 2. Hence λ_n is either of mixed type or if it is not then the property that the eigenvalues are distributed in two disjoint intervals, each interval containing exactly n eigenvalues of one type, is lost.

3.3. Definite matrix polynomials

Hyperbolic pencils $L(\lambda) = \lambda A - B$ are definite since their coefficient matrices are Hermitian with $A > 0$. However definite pairs are not necessarily hyperbolic since A and B can both be indefinite. By relaxing the requirement of definiteness of the leading coefficient, Higham et al. [16] introduced a new class of Hermitian matrix polynomials, the definite matrix polynomials, that extends the notion of hyperbolicity and is consistent with the definition of definite pencils. Definite matrix polynomials that are not hyperbolic arise in acoustic fluid–structure interaction problems [16].

Definition 3.7 (*definite matrix polynomial*)**.** A Hermitian matrix polynomial is definite if it satisfies any one (and hence all) of the equivalent properties of Theorem 3.8.

Theorem 3.8. *For an $n \times n$ Hermitian matrix polynomial $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ of degree m the following are equivalent*:

(*P1*) *All eigenvalues are real, of definite type and such that*

$$\underbrace{\lambda_{mn} \leqslant \cdots \leqslant \lambda_{(m-1)n+p+1}}_{\substack{n-p \text{ eigenvalues} \\ \text{of } (-1)^{m-1}\varepsilon \text{ type}}} < \cdots < \underbrace{\lambda_{jn+p} \leqslant \cdots \leqslant \lambda_{(j-1)n+p+1}}_{\substack{n \text{ eigenvalues of} \\ (-1)^{j-1}\varepsilon \text{ type, } 1 \leqslant j \leqslant m-1}} < \cdots < \underbrace{\lambda_p \leqslant \cdots \leqslant \lambda_1}_{\substack{p \text{ eigenvalues} \\ \text{of } -\varepsilon \text{ type}}} \leqslant \infty,$$

with $0 \leqslant p < n$, where "$\alpha\varepsilon$ type" denotes positive type when $\alpha\varepsilon > 0$ and negative type otherwise.

(*P2*) *There exist $\mu_j \in \mathbb{R} \cup \{\infty\}$ with $\mu_0 > \mu_1 > \mu_2 > \cdots > \mu_{m-1}$ ($\mu_0 = \infty$ being possible) such that $P(\mu_0), P(\mu_1), \ldots, P(\mu_{m-1})$ are definite matrices with alternating parity*.

(*P3*) *There exists $\mu \in \mathbb{R} \cup \{\infty\}$ such that the matrix $P(\mu)$ is definite and for every nonzero $x \in \mathbb{C}^n$ the scalar equation $x^*P(\lambda)x = 0$ has m distinct zeros in $\mathbb{R} \cup \{\infty\}$.*

(*L*) *P has a definite linearization $L(\lambda) \in \mathbb{H}(P)$.*

Note that when P is a definite matrix polynomial with A_m definite one of P or $-P$ is hyperbolic and by Theorem 3.4, $p = 0$ in property (P1) of Theorem 3.8.

Proof. The characterizations (P2), (P3) and (L) and their equivalence can be found in [16, Theorems 2.6 and 4.1].

(P1) $\Rightarrow$ (P3): Suppose P is not hyperbolic, i.e. $p \neq 0$. Let μ be such that $\lambda_{p+1} < \mu < \lambda_p$. Then homogeneously rotate P into $\widetilde{P}$ so that μ corresponds to $\widetilde{\mu} = \infty$. The rotation moves the p largest eigenvalues of P to the $n - p$ smallest ones to form a single group of n eigenvalues (see Figs. 2 and 4) which, by Lemma 2.6, are all of $(-1)^{m-1}\varepsilon$ type. The types of the remaining $m - 1$ groups of n eigenvalues remain unchanged. Hence by property (P1) of Theorem 3.4, $\widetilde{P}$ or $-\widetilde{P}$ is hyperbolic. By property (P3) of Theorem 3.4, $x^*\widetilde{P}(\widetilde{\lambda})x = 0$ has real distinct roots for all nonzero $x \in \mathbb{C}^n$ and therefore $x^*P(\lambda)x = 0$ has distinct roots in $\mathbb{R} \cup \{\infty\}$ ($x^*A_mx = 0$ is possible). Also by [16, Lemma 2.1], $P(\mu)$ is definite.

2 The partial multiplicities of an eigenvalue of Q are the sizes of the Jordan blocks in which it appears in a Jordan form for Q [10].

(P3) ⇒ (P1): Homogeneously rotate P into $\widetilde{P}$ so that μ corresponds to $\widetilde{\mu} = \infty$. Then $\widetilde{P}(\infty) = \widetilde{A}_m$ is definite, say $\widetilde{A}_m > 0$. Now if $x^*P(\lambda)x = 0$ has distinct roots in $\mathbb{R} \cup \{\infty\}$ then $x^*\widetilde{P}(\widetilde{\lambda})x = 0$ has real distinct roots (and no infinite root since $\widetilde{A}_m > 0$). Thus $\widetilde{P}(\widetilde{\lambda})$ is hyperbolic. Then property (P1) for P follows from property (P1) of Theorem 3.4, Lemma 2.6 and Fig. 4. □

The following result follows from (P2) in Theorem 3.8 and from counting sign changes in eigenvalues of the matrix $P(\mu)$ (see [16, Theorem 2.4 and its proof]).

Theorem 3.9. *For a definite matrix polynomial $P(\lambda)$ of degree m with eigenvalues as in property (P1) of Theorem 3.8, let*

$$\mathcal{I}_j = (\lambda_{jn+p+1}, \lambda_{jn+p}), \quad j = 1\colon m-1$$

and

$$\mathcal{I}_0 = \begin{cases} (\lambda_{p+1}, \lambda_p) & \text{if } p \neq 0, \\ (\lambda_1, +\infty) & \text{if } p = 0, \end{cases} \quad \mathcal{I}_m = \begin{cases} \emptyset & \text{if } p \neq 0, \\ (-\infty, \lambda_{mn}) & \text{if } p = 0. \end{cases}$$

Then $P(\mu)$ is definite for any $\mu \in \mathcal{I}_j$, $j = 0\colon m$ and if $\mu_j \in \mathcal{I}_j$, $\mu_{j+1} \in \mathcal{I}_{j+1}$ then $P(\mu_j)$ and $P(\mu_{j+1})$ have opposite parity.

Niendorf and Voss's algorithm [29] can be used to detect whether a Hermitian matrix polynomial is definite or not. For definite polynomials it also returns the μ_j of property (P2) so that a definite linearization can be built as shown in [16, Theorem 4.2].

4. Quasidefinite matrix polynomials

We have just seen that definite matrix polynomials are characterized by the fact that all their eigenvalues are real and of definite type and with a particular distribution of the eigenvalue types. We now consider a wider class of Hermitian matrix polynomials with real eigenvalues of definite type for which no assumption is made on the distribution of the eigenvalues types.

4.1. Definitizable pencils

Definite pencils form only a small subclass of Hermitian pencils with real and semisimple eigenvalues. We now consider a larger subclass of such pencils.

Definition 4.1 (*definitizable pencils*)**.** A Hermitian pencil $\lambda A - B$ is definitizable if it satisfies any one (and hence all) of the equivalent properties of Theorem 4.2.

Theorem 4.2. *For an $n \times n$ Hermitian pencil $L(\lambda) = \lambda A - B$ the following are equivalent:*

- (*P1*) *All the eigenvalues of L are real, finite, and of definite type.*
- (*P2*) *A is nonsingular and there exists a real polynomial q such that $Aq(A^{-1}B) > 0$.*
- (*P3*) *A is nonsingular and the scalar equation $x^*L(\lambda)x = 0$ has one zero in $\mathbb{R}$ for all eigenvectors $x \in \mathbb{C}^n$ of L.*
- (*P4*) *A is nonsingular and $(x^*Ax, x^*Bx, \ldots, x^*A(A^{-1}B)^{n-1}x) \neq 0$ for all nonzero $x \in \mathbb{C}^n$.*
- (*D*) *There exists a nonsingular $X \in \mathbb{C}^{n\times n}$ such that*

$$X^*AX = \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix}, \quad X^*BX = \begin{bmatrix} J_+ & 0 \\ 0 & -J_- \end{bmatrix}, \tag{4.1}$$

where $J_+ \in \mathbb{R}^{k\times k}$ and $J_- \in \mathbb{R}^{(n-k)\times(n-k)}$ are diagonal and $\Lambda(J_+) \cap \Lambda(J_-) = \emptyset$.

Proof. The equivalence of the characterizations (P1), (P2) and (P4) can be found in [24, Theorem 1.3]. (D) ⇒ (P3) is immediate. We show that (P3) ⇒ (P1) and (P1) ⇒ (D).

(P3) $\Rightarrow$ (P1) Suppose one eigenvalue is not real or is of mixed type then by [24, Lemma 2.2] there exists a corresponding eigenvector x such that $x^*Ax = 0$ and hence ($P3$) does not hold.

(P1) $\Rightarrow$ (D) A is nonsingular since all eigenvalues are finite and $\lambda A - B$ is simultaneously diagonalizable by congruence since all the eigenvalues are real and semisimple. Hence there exists X nonsingular such that $X^*(\lambda A - B)X = \lambda D - J$ is real diagonal. Since D is nonsingular, we can choose X such that $D = \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix}$ and partition J as $\begin{bmatrix} J_+ & 0 \\ 0 & -J_- \end{bmatrix}$ conformably with D. Then the property that all the eigenvalues are of definite type implies that $\Lambda(J_+) \cap \Lambda(J_-) = \emptyset$. $\square$

Lancaster and Ye [24] define definitizable pencils by property (P2) and add the adverb "*strongly*" to definitizable to emphasize the strict inequality in (P2). Note that the real polynomial q in (P2) is not unique and that J_+ in (4.1) contains the eigenvalues of positive type and J_- those of negative type. Now if $L(\lambda) = \lambda A - B$ with nonsingular A is definite then by property (P2) of Theorem 3.2, there exists $\mu \in \mathbb{R} \cup \{\infty\}$ such that the matrix $L(\mu)$ is definite. Then $Aq(A^{-1}B) > 0$ for $q(x) = -\varepsilon$ if $\mu = \infty$ and $q(x) = \varepsilon(x - \mu)$ otherwise, where $\varepsilon = 1$ if $L(\mu) < 0$ and $\varepsilon = -1$ if $L(\mu) > 0$. Hence definite pencils with nonsingular leading coefficient matrix are definitizable.

Though not necessarily computationally efficient, property (P1) provides an easy way to check whether a Hermitian pencil $\lambda A - B$ is definite or definitizable or none of these.

As a by-product of the proof of [24, Theorem 1.3], a real polynomial q of minimal degree such that $Aq(A^{-1}B) > 0$ can easily be constructed once all the eigenvalues of a definitizable pencil $\lambda A - B$ are known together with their types, as shown in the next theorem. The knowledge of q can be useful when constructing conjugate gradient iterations for solving saddle point problems [26].

Theorem 4.3. *For an $n \times n$ definitizable pencil $\lambda A - B$ with eigenvalues $\lambda_n \leqslant \cdots \leqslant \lambda_1$, let $k_j, j = 1{:}\,\ell - 1$ be the set of increasing integers such that*

$$\underbrace{\lambda_n \leqslant \cdots \leqslant \lambda_{k_{\ell-1}+1}}_{\substack{n-k_{\ell-1} \text{ eigenvalues} \\ \text{of } (-1)^{\ell-1}\varepsilon \text{ type}}} < \cdots < \underbrace{\lambda_{k_{j+1}} \leqslant \cdots \leqslant \lambda_{k_j+1}}_{\substack{k_{j+1}-k_j \text{ eigenvalues} \\ \text{of } (-1)^{j}\varepsilon \text{ type}}} < \cdots < \underbrace{\lambda_{k_1} \leqslant \cdots \leqslant \lambda_1}_{\substack{k_1 \text{ eigenvalues} \\ \text{of } \varepsilon \text{ type}}}.$$

Then $p(x) = \varepsilon \prod_{j=1}^{\ell-1}(x - \mu_j)$ with $\lambda_{k_j+1} < \mu_j < \lambda_{k_j}$ is a real polynomial of minimal degree $\ell - 1$ such that $Ap(A^{-1}B) > 0$.

Example 4.4. The pencils

$$L_1(\lambda) = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad L_2(\lambda) = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

have the same eigenvalues, i.e., $\Lambda(L_1) = \Lambda(L_2) = \{1, 2, 3\}$. Both pencils are definitizable because the eigenvalues are real, distinct and hence of definite type but only L_2 is definite since the eigenvalue 3, of negative type, is separated from the eigenvalues 1 and 2 of positive type as in the right-most depiction of Fig. 3. Furthermore, by Theorem 3.9, $L_2(\mu) > 0$ for all $\mu \in (2, 3)$. Also, by Theorem 4.3 any polynomial of the form $p(x) = (x - \mu_1)(x - \mu_2)$ with $\mu_1 \in (2, 3)$ and $\mu_2 \in (1, 2)$ satisfies $A_1 p(A_1^{-1}B_1) > 0$, where $L_1(\lambda) =: \lambda A_1 - B_1$.

4.2. Quasihyperbolic matrix polynomials

The notion of definitizable pencils extends to matrix polynomials of degree higher than one.

Definition 4.5 (*quasihyperbolic matrix polynomial*)**.** A Hermitian matrix polynomial is quasihyperbolic if it satisfies either (and hence both) of the equivalent properties of Theorem 4.6.

Theorem 4.6. *For an $n \times n$ Hermitian matrix polynomial $P(\lambda)$ the following are equivalent*:

(*P1*) *All the eigenvalues of P are real, finite and of definite type*.
(*L*) *Any linearization $L(\lambda) \in \mathbb{H}(P)$ is definitizable*.

Proof. We note that this result was proved in [21, Theorem 7.1] for a particular linearization in $\mathbb{H}(P)$.

A matrix polynomial has finite eigenvalues if and only if its leading coefficient matrix is nonsingular. Let $L \in \mathbb{H}(P)$ be a linearization of P. Then P has finite eigenvalues if and only if L has finite eigenvalues or equivalently L has nonsingular leading matrix coefficient. Moreover P has real eigenvalues if and only if L has real eigenvalues. By Lemma 2.8, the eigenvalues of P are of definite type if and only if those of L are of definite type. Hence by (P1) of Theorem 4.2, (P1) is equivalent to (L). $\square$

There is no obvious extension of properties (P2) and (P4) of Theorem 4.2 to quasihyperbolic matrix polynomials at the $n \times n$ level but by property (L) of Theorem 4.6 and property (P2) of Theorem 4.2, we have that P is quasihyperbolic if and only if there exists a real polynomial q such that $B_m q(B_m^{-1}B_{m-1}) > 0$, where $\lambda B_m - B_{m-1}$ is the $mn \times mn$ pencil (2.10). Property (P3) of Theorem 4.2 extends to quadratic matrix polynomials but not to higher degrees as shown by the next theorem and the following example.

Theorem 4.7. *Let P be a Hermitian matrix polynomial of degree m with nonsingular leading coefficient matrix*. *If the scalar equation $x^*P(\lambda)x = 0$ has m real distinct zeros for every eigenvector x of P then P is quasihyperbolic*. *The converse is also true when $m \leqslant 2$*.

Proof. Distinct real roots of $x^*P(\lambda)x = 0$ for all eigenvectors x of P implies that $\Lambda(P) \subset \mathbb{R}$ and $x^*P'(\lambda_0)x \neq 0$ for each eigenvalue $\lambda_0 \in \Lambda(P)$. Hence all eigenvalues are real, finite and of definite type, so P is quasihyperbolic by Theorem 4.6.

The converse is clearly true for linear P (see Theorem 4.2). Now for quadratic P, suppose the scalar quadratic $x^*P(\lambda)x = 0$, where x is an eigenvector, has a real double root. Then this double root is necessarily an eigenvalue of P, say λ_0 associated with x, and since it is a double root, $x^*P'(\lambda_0)x = 0$, so that λ_0 is of mixed type. Hence P is not quasihyperbolic. $\square$

Here is an example to show that the converse of Theorem 4.7 does not hold for polynomials of degree 3. The cubic polynomial

$$P(\lambda) = \lambda^3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \lambda^2 \begin{bmatrix} 9 & 0 \\ 0 & -6 \end{bmatrix} + \lambda \begin{bmatrix} -10 & 0 \\ 0 & 11 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -6 \end{bmatrix}$$

is quasihyperbolic. Any $x \in \mathbb{C}^2$ is an eigenvector associated with the eigenvalue $\lambda = 1$. It is easily checked that with $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the scalar polynomial $x^*P(\lambda)x$ has one real root and two complex conjugate roots.

4.3. Gyroscopically stabilized systems

Quadratic matrix polynomials associated with gyroscopic systems have the form

$$G(\lambda) = \lambda^2 M + \lambda C + K,$$

where M, K are Hermitian and C is skew-Hermitian [33]. As $G(\lambda)^* = G(-\bar{\lambda})$, the spectrum of $G(\lambda)$ is symmetric with respect to the imaginary axis. The quadratic $G(\lambda)$ is not Hermitian but

$$Q(\lambda) = -G(-i\lambda) = \lambda^2 M + \lambda(iC) - K =: \lambda^2 A_2 + \lambda A_1 + A_0$$

is. The gyroscopic system is said to be weakly stable if all the eigenvalues of G lie on the imaginary axis or equivalently, if the eigenvalues of Q are all real. The following definition appears in [3]. For a Hermitian B we write $|B| = (B^2)^{1/2}$, where the square root is the principal square root [13].

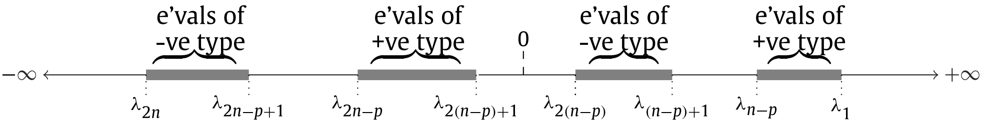


Fig. 6. Eigenvalue distribution of gyroscopically stabilized quadratics $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$, where p is the number of positive eigenvalues of A_1.

Definition 4.8. A Hermitian $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ is gyroscopically stabilized if $A_2 > 0$, $A_0 > 0$ and A_1 is indefinite and nonsingular with $|A_1| > \mu A_2 + \mu^{-1} A_0$ for some positive μ.

Barkwell et al. [3] prove that gyroscopically stabilized quadratics have real eigenvalues of definite type that belong to at most four distinct intervals with alternating types, with the number of eigenvalues in each interval depending on the number of positive eigenvalues p of A_1 (see Fig. 6). Hence gyroscopically stabilized quadratics are quasihyperbolic. They are overdamped when $A_1 > 0$.

4.4. Quasidefinite matrix polynomials

We remark that a definite pencil $\lambda A - B$ is not definitizable because A can be singular, and a definite matrix polynomial is not quasihyperbolic because its leading term can be singular. We therefore extend the definition of quasihyperbolic matrix polynomials to allow singular leading term and call such polynomials quasidefinite.

Definition 4.9. A Hermitian matrix polynomial P is quasidefinite if $\Lambda(P) \subset \mathbb{R} \cup \{\infty\}$ and each eigenvalue is of definite type.

Take a quasidefinite matrix polynomial P. Since P is regular, there exists $\mu \in \mathbb{R} \cup \{\infty\}$ such that $P(\mu)$ is nonsingular. Homogeneously rotate P into $\widetilde{P}$ so that μ corresponds to ∞ and $\widetilde{A}_m = \widetilde{P}(\mu)$ is nonsingular. Then by Lemma 2.6 the eigenvalues of $\widetilde{P}$ are all of definite type and $\widetilde{P}$ is quasihyperbolic. Hence any quasidefinite matrix polynomial is a "homogeneously rotated" quasihyperbolic one. Note that amongst the properties (P1), (P2) and (P3) we started with in Section 3, only a property of type (P1) remains for quasihyperbolic and quasidefinite matrix polynomials.

5. Diagonalizable quasidefinite matrix polynomials

Recall that a Hermitian pencil is diagonalizable by congruence if and only if its eigenvalues belong to $\mathbb{R} \cup \{\infty\}$ and are semisimple [23], a property shared by both definite and definitizable pencils. We now investigate how this property extends to (quasi)hyperbolic and definite matrix polynomials, thereby extending the simultaneous diagonalization property (D) in Theorems 3.2 and 4.2.

Two matrix polynomials are isospectral if they have the same eigenvalues with the same partial multiplicities. If furthermore they share the same sign characteristic then these two matrix polynomials are strictly isospectral [22]. For example any linearization $L(\lambda) \in \mathbb{H}(P)$ is isospectral to P but not necessarily strictly isospectral as shown by Lemma 2.8.

Now suppose that two $n \times n$ quasihyperbolic matrix polynomials P and $\widehat{P}$ of degree $m > 1$ are strictly isospectral. Let L_m and $\widehat{L}_m$ be the mth basis pencils of $\mathbb{H}(P)$ and $\mathbb{H}(\widehat{P})$, respectively (see (2.10)). By Lemma 2.8 L_m and $\widehat{L}_m$ are strictly isospectral and by Theorem 4.6 they are also definitizable. It follows from property (D) of Theorem 4.2, that there exist nonsingular matrices $X, \widehat{X} \in \mathbb{C}^{nm \times nm}$ such that

$$XL_m(\lambda)X^* = \lambda \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix} - \begin{bmatrix} J_+ & 0 \\ 0 & -J_- \end{bmatrix} = \widehat{X}\widehat{L}_m(\lambda)\widehat{X}^*.$$

The matrix $\widehat{X}^{-1}X$ defines a structure preserving congruence [22,25,32] since it preserves the block structure of $L_m(\lambda)$: $(\widehat{X}^{-1}X)L_m(\lambda)(\widehat{X}^{-1}X)^* = \widehat{L}_m(\lambda)$. Thus if there exists an $n \times n$ diagonal quasihyperbolic matrix polynomial $D(\lambda)$ of degree m strictly isospectral to $P(\lambda)$ then there exits a congruence transformation that preserves the block structure of the mth basis pencils of $\mathbb{H}(P)$ but also diagonalizes each block. If such structure preserving congruence exists then P is said to be diagonalizable.

In what follows $D(\lambda)$ has the form

$$D(\lambda) = \text{diag}\,(d_1(\lambda), \ldots, d_n(\lambda)), \tag{5.1a}$$

$$d_i(\lambda) = \delta_i(\lambda - \lambda_{i_m}) \cdots (\lambda - \lambda_{i_2})(\lambda - \lambda_{i_1}), \quad \delta_i \neq 0, \quad i = 1\colon n \tag{5.1b}$$

with $\bigcup_{i=1}^n \{i_1, \ldots, i_m\} = \{1, \ldots, mn\}$. The scalars $\lambda_{i_j}, j = 1\colon m, i = 1\colon n$ are the eigenvalues of $D(\lambda)$ and $P(\lambda)$ and are therefore real. We assume that they are ordered as

$$\lambda_{mn} \leqslant \cdots \leqslant \lambda_{j+1} \leqslant \lambda_j \leqslant \cdots \leqslant \lambda_1.$$

Theorem 5.1. *An $n \times n$ quasihyperbolic matrix polynomial of degree m with eigenvalues $\lambda_{mn} \leqslant \cdots \leqslant \lambda_1$ is strictly isospectral to an $n \times n$ diagonal matrix polynomial of degree m if and only if there is a grouping of its eigenvalues into n subsets of m distinct eigenvalues $\{\lambda_{i_j} : j = 1\colon m\}_{i=1}^n$ such that with the ordering $\lambda_{i_m} < \cdots < \lambda_{i_2} < \lambda_{i_1}, i = 1\colon n$, the eigenvalue λ_{i_j} is of $\delta_i(-1)^{j-1}$ type, where λ_{i_1} is of δ_i type.*

Proof. Let $P(\lambda)$ denote the $n \times n$ quasihyperbolic matrix polynomial of degree m.

($\Rightarrow$) Suppose $P(\lambda)$ is strictly isospectral to an $n \times n$ diagonal matrix $D(\lambda)$ of degree m as in (5.1). The scalar polynomials $d_i(\lambda)$ must have distinct roots since otherwise $0 = d_i'(\lambda_{i_j}) = e_i^* D'(\lambda_{i_j}) e_i$ for some eigenvalue λ_{i_j}, which implies that λ_{i_j} is not of definite type, a contradiction. Here e_i, the ith column of the $n \times n$ identity matrix, is a corresponding eigenvector. Consider the grouping $\{\lambda_{i_j}, j = 1\colon m\}_{i=1}^n$ of the eigenvalues. With the ordering $\lambda_{i_m} < \cdots < \lambda_{i_2} < \lambda_{i_1}, i = 1\colon n$, it is easily seen that this grouping must be such that, in each group, the eigenvalue λ_{i_j} is of $\delta_i(-1)^{j-1}$ type and the sign of δ_i is determined by the type of λ_{i_1}.

($\Leftarrow$) Let $\{\lambda_{i_j}, j = 1\colon m\}_{i=1}^n$ be a grouping of the eigenvalues of P into n subsets of m distinct eigenvalues, such that with the ordering $\lambda_{i_m} < \cdots < \lambda_{i_2} < \lambda_{i_1}, i = 1\colon n$, the eigenvalue λ_{i_j} is of $\delta_i(-1)^{j-1}$ type, where λ_{i_1} is of δ_i type. Let $D(\lambda)$ and $d_i(\lambda)$ be as in (5.1). Then by construction $D(\lambda)$ is quasihyperbolic and its eigenvalues and their types are the same as the eigenvalues of P and their types. Hence D is strictly isospectral to P. $\square$

Example 5.2. Let $P(\lambda)$ be a 2×2 cubic quasihyperbolic matrix polynomial with real eigenvalues $\lambda_1 > \lambda_2 > \lambda_3 = \lambda_4 > \lambda_5 > \lambda_6$ and associated types $\{+, -, +, +, +, -\}$, where $+$ means positive type and $-$ denotes negative type. This polynomial is not strictly isospectral to a diagonal cubic matrix polynomial because there is no sorting of the eigenvalues into two groups of three distinct eigenvalues, which when ordered have alternating types. Note that if the sign characteristic had been $\{+, -, +, +, -, +\}$ then P would have been strictly isospectral to a diagonal cubic matrix polynomial.

Quasihyperbolic matrix polynomials of degree m strictly isospectral to diagonal matrix polynomials of degree m form a new subclass of Hermitian polynomials with eigenvalues all real and of definite type. Note that $n \times n$ Hermitian quasihyperbolic quadratics have n eigenvalues of positive type and n eigenvalues of negative type [8, Theorem 1.3]. So there is always a sorting of the eigenvalues into n groups of two distinct eigenvalues with opposite types. By Theorem 3.4, the eigenvalues of an $n \times n$ hyperbolic matrix polynomial of degree m are distributed in m disjoint intervals each of which contains n eigenvalues and, the type of the intervals alternate. So we can always sort the eigenvalues in n subsets of m distinct eigenvalues, which when ordered have alternating types. Hence by Theorem 5.1, quasihyperbolic quadratics and hyperbolic matrix polynomials of arbitrary degree, say m, are strictly isospectral to diagonal matrix polynomials of degree m. This result also applies to quasidefinite quadratic matrix polynomials and definite matrix polynomials.

Corollary 5.3

(*a*) *A quasidefinite quadratic matrix polynomial is always strictly isospectral to a quasidefinite diagonal quadratic matrix polynomial.*

(*b*) *A definite matrix polynomial of degree m is always strictly isospectral to a definite diagonal matrix polynomial of degree m*.

Proof. The proofs of (a) and (b) are similar so we just provide that for (b).

A definite matrix polynomial P of degree m is a homogeneously rotated hyperbolic matrix polynomial $\widetilde{P}$ of degree m. From the comments preceding Corollary 5.3, $\widetilde{P}$ is strictly isospectral to a diagonal matrix polynomial $\widetilde{D}(\lambda)$ of degree m. Applying back the homogeneous rotation to $\widetilde{D}$ produces a diagonal matrix polynomial of degree m, which is strictly isospectral to P. □

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