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Reviewed work(s):

Source: *Journal for Research in Mathematics Education*, Vol. 35, No. 4 (Jul., 2004), pp. 258-291

Published by: [National Council of Teachers of Mathematics](#)

Stable URL: <http://www.jstor.org/stable/30034810>

Accessed: 20/03/2012 17:29

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Learning Mathematics in a Classroom Community of Inquiry

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This article considers the question of what specific actions a teacher might take to create a culture of inquiry in a secondary school mathematics classroom. Sociocultural theories of learning provide the framework for examining teaching and learning practices in a single classroom over a two-year period. The notion of the zone of proximal development (ZPD) is invoked as a fundamental framework for explaining learning as increasing participation in a community of practice characterized by mathematical inquiry. The analysis draws on classroom observation and interviews with students and the teacher to show how the teacher established norms and practices that emphasized mathematical sense-making and justification of ideas and arguments and to illustrate the learning practices that students developed in response to these expectations.

Key words: Classroom interaction; Learning theories; Reform in mathematics education; Secondary mathematics; Teaching (role, style, methods); Vygotsky

In recent years, educational policymakers and researchers have called for significant changes to the way that mathematics is taught in schools. In the United States, for example, the series of influential curriculum documents produced by the National Council of Teachers of Mathematics (NCTM) has placed increased emphasis on the processes of problem solving, reasoning, and communication (NCTM, 1989, 2000). A similar shift in priorities has occurred in Australia, where the intent of the NCTM's *Principles and Standards* document is echoed in the *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991). Like the NCTM agenda, the reformist goals for school mathematics in Australia are concerned with developing students' communication skills and problem-solving capacities and allowing students to experience the actual processes through which mathematics develops (e.g., conjecture, generalization, proof, refutation).

This article is based in part on my doctoral dissertation completed at The University of Queensland under the supervision of Peter Galbraith and Peter Renshaw. The research reported here was supported by a Large Grant from the Australian Research Council. I am grateful to Vince Geiger and his students for welcoming me into their classroom for the duration of this study. I would also like to thank Edward Silver and the anonymous reviewers who gave helpful and encouraging feedback on earlier versions of the manuscript.

Forman (2003) argues that sociocultural perspectives on learning can provide a theoretical rationale for these moves for mathematics education reform. Sociocultural approaches are distinguished from other theoretical frameworks by their association with the Vygotskian school of thought, which claims that human thinking is inherently social in its origins (Sfard, Forman, & Kieran, 2001). According to Forman, sociocultural theory offers a way forward in understanding the fundamental link between instructional practices and learning outcomes, and also in showing how mathematics learning entails communication in social contexts. From this perspective, mathematics teaching and learning are viewed as social and communicative activities that require the formation of a classroom community of practice (Lave & Wenger, 1991) where students progressively appropriate and enact the epistemological values and communicative conventions of the wider mathematical community.

In a sense, *all* classrooms are communities of practice—but classroom communities differ in the kinds of learning practices that become codified and accepted as appropriate by teachers and students (Boaler, 1999). For example, in mathematics classrooms using a traditional, textbook-dominated approach, effective participation involves students in listening to and watching the teacher demonstrate mathematical procedures, and then practicing what was demonstrated by completing textbook exercises. Teaching methods that foster learning mathematics by memorization and reproduction of procedures can be contrasted with the more open approaches in reform-oriented mathematics classrooms, where quite different learning practices such as discussion and collaboration are valued in building a climate of intellectual challenge. Rather than rely on the teacher as an unquestioned authority, students in these classrooms are expected to propose and defend mathematical ideas and conjectures and to respond thoughtfully to the mathematical arguments of their peers. Thus, the practices and beliefs developed within reform classrooms frame learning as participation in a community of practice characterized by *inquiry mathematics*—where students learn to speak and act mathematically by participating in mathematical discussion and solving new or unfamiliar problems (Richards, 1991). Such classrooms could be described as *communities of mathematical inquiry*.

This article draws on the findings of a larger study, carried out in an Australian secondary school, that aimed to examine the teacher's role in creating a classroom culture of inquiry that promoted mathematical habits of mind and to investigate patterns of discourse that mediated mathematical reasoning when students worked collaboratively on challenging problems. Clearly, these aims resonate with the two sociocultural themes—regarding links between teaching practices and learning outcomes and the nature of mathematical communication—identified by Forman (2003) as being consistent with frameworks for mathematics education reform. Elsewhere I have reported on findings related to the second aim of this larger study, examining discourse and reasoning within small groups of students (Goos, 2002; Goos, Galbraith, & Renshaw, 2002). The focus of this article, then, is concerned with the larger study's first aim and addresses the question of how the teacher initiated students into mathematical practice.

The next section introduces key concepts from Vygotsky's original writings and outlines how deeper interpretations of these concepts have led to contemporary developments in sociocultural theory, particularly in mathematics education. The theoretical framework for the study reported in this article is then located within this broader landscape. This is followed by details of the research design and methods of data collection and analysis relevant to the findings reported here. First, I present an account of teaching and learning practices and participants' beliefs in a mature community of inquiry—a Grade 12 classroom that was observed over a full school year. Second, I analyze changing participation patterns in a Grade 11 class taught by the same teacher in the following year, to show how this teacher began to engage his students in mathematical inquiry.

OVERVIEW OF SOCIOCULTURAL THEORIES OF LEARNING

It is generally accepted that sociocultural theories trace their origins to the work of the Russian psychologist Vygotsky in the early 20th century (e.g., Forman, 2003; Sfard et al., 2001). Wertsch (1985), one of the first scholars to interpret this work to Western researchers, identified three general themes at the heart of Vygotsky's theoretical approach. The first of these is a reliance on a genetic or developmental method: in other words, to understand mental phenomena, we need to concentrate on the *process* of growth and change rather than the *product* of development. The second theme is concerned with the social origins of higher mental functions: voluntary attention, memory, concepts, and reasoning appear first between people on the social plane and then within an individual on the psychological plane. The third theme claims that mental processes are mediated by tools and signs such as language, writing, systems for counting, algebraic symbol systems, diagrams, and so on. In connection with these three themes, Vygotsky analyzed the related concepts of internalization and the zone of proximal development (ZPD). He viewed internalization as a process whereby social phenomena, performed initially on an external plane, are transformed into psychological phenomena, executed on an internal, mental plane. It is within the zone of proximal development that such a transformation can occur, since a child's interaction with an adult or more capable peer may awaken mental functions that have not yet matured and thus lie in the region between actual and potential developmental levels, between unassisted and assisted performance.

In the West, early attempts to apply Vygotsky's theory in educational research in the 1970s and 1980s led to studies of how children learned through collaborative interaction with adults. Here, the metaphor of *scaffolding* was introduced by Wood, Bruner, and Ross (1976) to elaborate on the role of tutoring in enabling novices to solve problems beyond their unassisted efforts. Although the work of Wood and colleagues did not explicitly refer to Vygotsky's notion of the zone of proximal development, it soon became common to use the term scaffolding to describe the interaction between adult and child within the ZPD (e.g., Bruner, 1986; Rogoff & Wertsch, 1984). However, this first generation of Vygotskian-inspired

educational research suffered from several limitations, arising from the narrow ways in which Vygotsky's ideas were initially interpreted rather than from any weaknesses in the ideas themselves. For example, Stone (1993) argued that the scaffolding metaphor, as it was applied in the research of that time, relied on literal notions of internalization of the interchange between child and adult, rather than more subtle semiotic mechanisms that might account for the transformation and appropriation of meanings during these interchanges. He also drew attention to the interpersonal dimensions of scaffolding, pointing out that interactions—whether between adult and child or within peer groups—are influenced by the motives, goals, and status of the participants (see Forman & Cazden, 1985; Rogoff, 1990). Consequently, through the 1980s and 1990s, more sophisticated interpretations began to emerge in a second generation of research that extended and enriched the emerging sociocultural framework, seen, for example, in the collection of theoretical and empirical work edited by Forman, Minick, and Stone (1993). The studies in this volume contributed to a new conception of sociocultural theory, which recognized the need to give attention to the institutional context of social interactions, the importance of interpersonal relationships in teaching and learning interactions, and the idea that modes of thinking are closely linked to forms of social practice.

In addition to these developments, contemporary sociocultural theory acknowledges that learning involves increasing participation in a community of practice composed of experts and novices (Lave & Wenger, 1991). The concept of a community of practice was neither originally focused on school classrooms nor on pedagogy. Also, Lave and Wenger developed their work on situated learning, in part, through criticizing the notion of learning as internalization, which they claimed could be too easily construed as a process of transmission or assimilation. However, their concern for “the whole person acting in the world” (p. 49), and their emphasis on sociocultural transformation, resonates with elements of the emergent framework referred to above.

Recent research on mathematics learning has employed sociocultural approaches in a variety of ways, and likewise there are many ways in which one could reasonably classify and organize these studies within the theoretical perspective sketched above. It is beyond the scope of this article to provide a complete review; however, the themes identified by Forman (2003) and mentioned at the start of this article will serve to map some features of this territory. Thus, there is a body of research that identifies with a *discursive* perspective, focusing on the dynamics of mathematical communication in teacher-orchestrated discussion within whole-class settings (e.g., O'Connor, 2001) or interactive student discussion within small-group problem solving (e.g., Kieran, 2001). Interest here centers on the role of semiotic mediation in mathematics learning. Researchers have also given their attention to relationships between instructional *practices* and learning outcomes (e.g., Boaler, 1999, 2000; Lampert, 1990b), often invoking the concept of learning mathematics in a community of practice. Whether the emphasis is on discourse or practices, in this research there is a clear shift away from viewing mathematics learning as *acquisition* toward understanding mathematics learning as *participation* in the discursive

sive and cultural practices of a community. (The acquisition and participation metaphors are further elaborated in Sfard, 1998.)

The shift from acquisition to participation is neatly captured by van Oers (2001) in his discussion of educational forms of initiating children into the culture of mathematical practice. How might children be assisted so as to improve their abilities for participation? Van Oers proposes that this process begins with the teacher's demonstration of a mathematical attitude, that is, a willingness to deal with mathematical concepts and to engage in mathematical reasoning according to the accepted values in the community, and consequently, from the teacher's mathematical expectations about the learners' activity. He suggests that learners appropriate this mathematical attitude through participation in shared practice structured by the teacher's actions and expectations. Lerman (2001a; 2001b) frames this increasing participation in mathematical speaking and thinking as pulling learners forward into their zones of proximal development. He notes that his use of the ZPD is less "internalist" than many interpretations of Vygotsky's original formulation. For Lerman, the ZPD is not a physical space, but a symbolic space created through the interaction of learners with more knowledgeable others and the culture that precedes them. It is in this sense that I use the zone of proximal development as a key theoretical construct in the research study presented in this article. The following section (drawing on Minick, 1987; van der Veer & Valsiner, 1994; Vygotsky, 1978) outlines how the ZPD provided a framework for the analysis of learning within a classroom where participants were pulled forward into mathematical inquiry.

THE ZONE OF PROXIMAL DEVELOPMENT AS A FRAMEWORK FOR ANALYZING LEARNING

Teacher-Student Interaction: The ZPD as Scaffolding

The first element of the framework builds on Vygotsky's original definition of the ZPD as the distance between a child's problem-solving capability when working alone and with the assistance of a more advanced partner, such as a teacher or peer tutor. As I described in the previous section, the term *scaffolding* became associated with interactions where the teacher structured tasks to allow students to participate in joint activities that would otherwise be beyond their reach—for example, by using predictable dialogue structures or negotiating a division of labor between teacher and learner. Central to this notion was the gradual withdrawal of teacher support as the learner came to understand the task and to perform more independently. Early research inspired by this scaffolding version of the ZPD tended to have a transmissive flavor that implied teaching and learning are simply processes of demonstration and imitation, resulting in the orderly transfer of information and skills from teacher to learner in some kind of predetermined sequence. However, critics of these transmissive interpretations point out that creating a ZPD always involves mutual appropriation by teacher and learner of each other's actions and

goals (Griffin & Cole, 1984; Newman, Griffin, & Cole, 1989; Renshaw, 1998), and that this requires learners to identify with the teacher and the values of the knowledge community he or she represents (Litowitz, 1993). Thus, the active contribution of the learner is essential in negotiating the co-construction of the ZPD, and sequencing is seen in the resulting reorganization of social interactions rather than the preplanned segmentation and presentation of tasks. Concepts of identification and resistance are crucial to understanding what motivates learners to participate in these interactions.

Student-Student Interaction: The ZPD as Collaboration

Vygotsky also analyzed the notion of the ZPD in terms of more equal status partnerships, noting that when children played together they were able to regulate their own and their partners' behavior according to more general social scripts and take the perspective of others. From an educational perspective, there is learning potential in peer groups where students have incomplete but relatively equal expertise, each partner possessing some knowledge and skill but requiring the others' contribution in order to make progress. In mathematics education, this approach has informed research on small-group problem solving to explain how interaction between students of comparable expertise can create a collaborative ZPD (e.g., Forman & McPhail, 1993; see also Goos, Galbraith, & Renshaw, 2002, for an analysis involving students who took part in the study reported here).

Working in collaborative peer groups, students have an opportunity to own the ideas they are constructing and to experience themselves and their partners as active participants in creating personal mathematical insights. Nevertheless, it is important to recognize that not all student constructions are equally valid, although incomplete or unacceptable constructions can form the basis of classroom activities and discussion of different interpretations. Here, the teacher as a more experienced knower in the discipline plays a crucial role in selecting student ideas that are fruitful to pursue.

Everyday and Scientific Concepts: The ZPD as Interweaving

A third aspect of the ZPD is derived from Vygotsky's theorizing in relation to schools and the access that formal schooling provides to more organized and abstract forms of knowledge. He distinguished between two types of concepts: (1) everyday or spontaneous concepts arising from the experiences available in the child's immediate community and (2) scientific or theoretical concepts that have been elaborated and refined over time to form coherent systems of understanding. Thus, the ZPD is conceptualized here in terms of the distance between learners' intuitive notions and the formalized concepts, or cultural tools, of a particular academic community. Mature knowledge is achieved with the merging of everyday and scientific concepts—not by replacing the former with the latter as in a transmission model of teaching, but by interweaving the two conceptual forms. Again, as the representative of mathematical culture in the classroom, it is the teacher who

has the capacity to see in the students' ideas the links to the language and concepts of the wider community of mathematicians.

The features of the ZPD discussed above highlight the pivotal position of the teacher in structuring learning activities and social interactions to facilitate students' increasing participation in a culture of mathematical inquiry. Within mathematics education, accounts of reform-related research have attempted to identify conditions supporting the creation of learning cultures that challenge students to engage regularly in authentic mathematical activities (e.g., Boaler, 1998; Brown, Stein, & Forman, 1995; Forman, 1996; Lampert, 1990a; Renshaw & Brown, 1997; Stein, Grover, & Henningsen, 1996). Recently, however, there have been calls for more attention to be given to the *detailed* practices of teaching and learning through which reform approaches—often described in research reports only in general terms—are enacted in classroom communities (Boaler, 2002; McClain & Cobb, 2001). The purpose of this article is to provide such a level of detail by analyzing some of the teaching and learning practices used by one teacher in helping students appropriate the ways of knowing, speaking, and acting characteristic of a community of mathematical inquiry.

RESEARCH DESIGN AND METHODS

Participants and Setting

Highfields School¹ is an independent secondary school located in a large city in the Australian State of Queensland. The school, which opened in 1987, aims to provide its students with a balance between academic development, personal growth, and physical challenge (e.g., all students participate in an outdoor education program). Teachers, students, and parents contribute to the running of the school through a range of democratic structures such as the School Council. Highfields is a medium-sized school by Australian standards, with around 600 students in Grades 8 to 12, and approximately equal numbers of boys and girls. Enrolment is restricted to 600 in order to promote students' sense of belonging and reinforce the school's espoused commitment to an ethic of care in student-teacher relationships. The student population is fairly homogenous with respect to cultural and socioeconomic background, with most students coming from white, Anglo-Australian middle class families. Like all independent schools in Australia, Highfields charges tuition fee for students to attend. These fees are considered to be moderate by comparison with other larger, longer established independent schools in the same city.

The Head of the school's mathematics department was invited to participate in this study because of his interest in developing his students' mathematical thinking and problem solving abilities through an inquiry mathematics approach. (He had

¹ The school name and all student names are pseudonyms.

recently completed a small research project on this topic as part of the requirements for a master's degree in education.) This teacher was recognized locally as a leader in the mathematics teaching profession in Queensland and known nationally through his involvement with curriculum development groups and professional associations. At the time of the study he had been teaching for about 12 years, most of which had been spent at Highfields School.

In this study I focused on the senior secondary school (Grades 11 and 12) because of recent changes to the Queensland senior mathematics syllabuses. Three new subjects—Mathematics A, B, and C—had just been accredited by the Queensland Board of Senior Secondary School Studies for inclusion on students' Senior Certificate, the record of their academic achievement in Grades 11 and 12. Mathematics A concentrates on applications for daily living offered by such topics as financial mathematics and applied geometry and is described in the syllabus as the mathematics required for intelligent citizenship. Mathematics B is a prerequisite for entry to university science and business courses and provides an introduction to calculus and statistics. Mathematics C is a more advanced subject that can only be taken in conjunction with Mathematics B and prepares students for further study of mathematics at the university level. For each of these syllabuses, the three general objectives of Mathematical Techniques, Mathematical Applications, and Communication reflect recent moves to encourage a more open, problem-solving approach to school mathematics in Australia. Thus, students are assessed on their ability not only to recall and use learned procedures (Techniques), but also to solve unfamiliar, life-related problems (Applications) and justify their methods of solution (Communication).

Teachers of senior secondary mathematics are sometimes reluctant to participate in any long-term research project that threatens to disrupt regular classroom arrangements. This reluctance is related to time constraints imposed by the need to cover prescribed content and pressures associated with high-stakes assessment in the final years of school. The teacher who took part in the present study took these factors into account when he offered his Mathematics C classroom as a research site. As the only teacher in the school responsible for this subject (one class in Grade 11 and one in Grade 12), he enjoyed a degree of autonomy that allowed him to implement and further develop his preferred, inquiry-oriented teaching approaches. This instructional approach appeared to be compatible with the broad theoretical interests that shaped my role as a participant-observer. In addition to the formal data collection methods detailed in the next section, I had many informal conversations with the teacher where we compared our respective interpretations of lesson events. This sometimes led to the teacher modifying his plans for the following lesson; for example, by finding ways of making a task more challenging so as to steer students toward deeper exploration of mathematical concepts. Thus, a research partnership evolved in which key theoretical ideas were elaborated in the context of classroom practice, and possibilities for changing classroom practices were created by emerging theoretical insights.

Data Collection

As the emphasis was on interpreting learning in complex social settings rather than experimental manipulation and control of variables, the multiple research methods used in the study were consistent with a naturalistic inquiry approach (Lincoln & Guba, 1985). These methods are described next.

Classroom observation. Weekly lesson observations, supplemented by video and audiotaped recording of teacher-student and student-student interactions, took place throughout the 2 years of the study (a Grade 12 class in the first year and a Grade 11 class in the second). The video camera was positioned on a tripod near the front, and to one side, of the classroom to allow panning to film the teacher or students seated at their desks or tables. Students' conversations during small-group work were captured by an external microphone and/or audiotape recorder placed on their desks. Field notes were also made of each lesson observed to record teacher actions, student actions, board work, and the nature of any materials used by the students such as textbooks, handouts, calculators, or computers. These records were later annotated with additional observations made while viewing the lesson videotapes.

Interviews with teacher and students. Stimulated recall interviews (Leder, 1990) were conducted with the teacher and groups of students on several occasions to seek their interpretations of videotape excerpts. Videotapes for teacher interviews were selected and previewed in order to identify a few key moments that warranted elaboration by the teacher, with the specific focus being on how students developed understanding of the mathematics. The videotape of the whole lesson was played without interruption, with the teacher invited to use the pause button whenever he wanted to comment on any incident. If a key moment passed without comment, I paused the tape and requested comment through questions such as "What was happening here?" These interviews lasted around 60 minutes. All were recorded on audiotape, with portions being transcribed and inserted into the field notes of the lesson. Stimulated recall interviews were carried out with groups of students whose classroom interactions had previously been videotaped. A small number of segments was chosen for replay, and students were invited to interpret their own and their partners' talk. These group interviews generally lasted 15 to 20 minutes and were audiotaped for later transcription to supplement the lesson field notes.

Students' views about learning mathematics were also investigated via semi-structured individual and whole-class interviews. Questions focused on how students thought their teacher expected them to work in class, what was the best way they had found to learn and understand mathematics, what they did to learn mathematics in the classroom, how their teacher and their classmates helped them to learn, and strategies they used for tackling mathematical problems. Information from all of these data sources is integrated into the account that follows.

Data Analysis

Sociocultural research requires a unit of analysis that unites the individual and the social setting and takes into account motives, goals, norms, beliefs, and values in relationships between people. Analysis of the data, therefore, involved situated practices—the mathematical activities of the teacher and his students in the Mathematics C classroom. I carried out data collection and analysis simultaneously in order to generate categories and develop theoretical insights. The process of bringing structure and meaning to the data corpus involved three iterations (cf Anfara, Brown, & Mangione, 2002): category creation, category refinement, and theory development.

The first iteration of analysis used field notes, video and audio recordings from the first four weeks of lesson observations to create initial categories for giving meaning to teacher actions. This resulted in nine categories, expressed as the following action statements:

1. The teacher models mathematical thinking.
2. The teacher asks students to clarify, elaborate, and justify their responses and strategies.
3. The teacher emphasizes sense-making.
4. The teacher makes explicit reference to mathematical conventions and symbolism.
5. The teacher encourages reflection, self-monitoring, and self-checking.
6. The teacher uses the students' ideas as starting points for discussion.
7. The teacher structures students' thinking.
8. The teacher encourages exploratory discussion.
9. The teacher structures students' social interactions.

In the second iteration, which continued through the remainder of the 1st year of the study, additional data from lesson observations were compared within and between categories to refine their meaning. For example, the second category was modified to read, "The teacher asks students to clarify, elaborate, and justify their responses and strategies, *both to the teacher and to each other during whole class discussions.*" Evidence of teacher actions supporting this category included the following observations (each of which was recorded across many lessons):

- 2.1 The teacher uses questioning to elicit these processes.
- 2.2 The teacher encourages/permits other students to comment on the contribution of previous student speakers.
- 2.3 The teacher encourages argumentation between students, not mediated by the teacher.
- 2.4 The teacher insists that students take responsibility for validating solutions.

Early in the 2nd year of the study, a third iteration advanced the analysis toward developing theoretical insights about the teacher's role in initiating students into

a culture of mathematical inquiry. In this process, I attempted to identify the mathematical attitudes and pedagogical expectations underlying the teacher's actions (cf van Oers, 2001) and to understand how these were appropriated by students. Drawing in part on the ZPD framework elaborated earlier in this article, I regrouped the existing categories of teacher actions to formulate a set of five statements that appeared to reflect the teacher's assumptions (i.e., attitudes and expectations) about mathematics teaching and learning. These are listed in the left column of Table 1. The first two assumptions identify in a general way the practices of mathematical inquiry in which the teacher wished his students to participate, whereas the next three assumptions correspond to the interpretations of learning in the ZPD I have described as scaffolding, collaboration, and interweaving. I checked the plausibility of these statements via comparison with data already collected via teacher and student interviews and lesson observations and continued this comparative analysis throughout the 2nd year of the study as I conducted further observations and interviews. I also searched the data corpus for evidence of students' actions that suggested that they were (or were not) being pulled forward into the practices demonstrated and valued by the teacher. Thus, this third iteration featured a constant interplay between data and theory, from which emerged the overall findings shown in Table 1. The teacher actions and student actions listed here represent a synthesis of evidence from the study as a whole.

The overview in Table 1 gives only a broad indication as to how the teacher created a culture of mathematical inquiry. In the remainder of the article, I use examples from the data collected over the 2 years of the study to illustrate these findings in more detail, and in particular to capture the changes in the teacher's and students' participation over time, as emphasized in Vygotsky's genetic method.

THE MATURE COMMUNITY OF INQUIRY

The teacher offered his Grade 12 Mathematics C classroom as a research site in the 1st year of the study. There were fourteen students in this class, six boys and eight girls, aged 16–17 years. The teacher had also taught this class in Grade 11. From initial lesson observations, it soon became clear that this was a classroom where a culture of mathematical inquiry appeared to have taken hold. For example, explanation and justification of ideas featured strongly in classroom social interactions (see Assumption 1 in Table 1), self-directed thinking and personal sense-making were emphasized (Assumption 2), and there was a high incidence of mathematical discussion among students (Assumption 4). The teacher articulated his commitment to these principles during a stimulated recall interview as he watched a video recording of a lesson in which he had introduced the concept of simple harmonic motion to this class.

He began the lesson by asking the students to line up facing him in single file in order of height, with the smallest student at the front of the line and the tallest at the back. This allowed all students to watch as the teacher slowly spun a large wheel, to the rim of which he had attached a marking pen. As students observed the spin-

Table 1

Assumptions About Teaching and Learning Underlying Teacher and Student Mathematical Activity in the Mathematics C Classroom

Assumptions	Teacher actions	Student actions
1. Mathematical thinking is an act of sense-making and rests on the processes of specializing, generalizing, conjecturing, and convincing.	The teacher models mathematical thinking using a dialogic format to invite students to participate.	Students begin to offer conjectures and justifications without the teacher's prompting.
	The teacher invites students to take ownership of the lesson content by providing intermediate or final steps in solutions or arguments initiated by the teacher.	During whole-class discussion, students initiate argumentation between themselves, without teacher mediation.
	The teacher withholds judgment on students' suggestions while inviting comment or critique from other students.	
2. The processes of mathematical inquiry are accompanied by habits of individual reflection and self-monitoring.	The teacher asks questions that encourage students to question their assumptions and locate their errors.	Students begin to point out and correct their own and each other's errors and those made by the teacher.
	The teacher allows time in class for students to read textbook explanations and interrogate worked examples.	Students spontaneously consult textbooks and examples to clarify their understanding.
3. Mathematical thinking develops through teacher scaffolding of the processes of inquiry.	The teacher calls on students to clarify, elaborate, critique, and justify their assertions.	Students spontaneously provide clarification, elaboration, critiques, and justifications.
	The teacher structures students' thinking by leading them through strategic steps or linking ideas to previously or concurrently developed knowledge.	Students take increasing responsibility for suggesting strategic steps and making links to prior knowledge.
4. Mathematical thinking can be generated and tested by students through participation in equal-status peer partnerships.	The teacher structures social interactions between students by asking them to explain and justify ideas and strategies to each other.	Students form informal groups to monitor their progress, seek feedback on ideas, and explain ideas to one another.
5. Interweaving of familiar and formal knowledge helps students to adopt the conventions of mathematical communication.	The teacher makes explicit reference to mathematical language, conventions, and symbolism, labeling conventions as traditions that permit communication.	Students begin to debate the appropriateness and relative advantages of different symbol conventions.
	The teacher links technical terms to commonsense meanings and uses multiple representations of new terms and concepts.	

ning wheel from side on, the teacher asked them to focus on the pen as it moved in the vertical plane and think about how they would describe its motion. Students noticed that the pen seemed to be moving “up and down,” “along the diameter.” When the teacher pressed them to be more explicit, asking “What do you notice about *how* it’s going up and down?” they replied, “Its velocity is changing” and “It’s accelerating in the middle.” After the students returned to their seats, the teacher initiated a discussion that brought into play their prior knowledge of circular motion (“What related stuff have we been studying recently?”) and asked them to apply this knowledge to explain the apparent motion of the pen (“How might this be useful in explaining the phenomenon observed earlier in the lesson?”). At all times during this discussion, the teacher expected students to clarify their ideas (“Be more specific!”) and monitor their thinking (“Are you sure?” and “Tell me if I’m wrong.”). At several points, he interrupted the whole-class discussion to allow students to take some responsibility for constructing the mathematical description of simple harmonic motion (e.g., by asking students to work together to “now find out something about its acceleration” after having developed the formula for velocity).

The teacher’s commentary on the lesson videotape, elicited during a stimulated recall interview, revealed some of the pedagogical expectations he held for students and the mathematical attitudes from which these derived. Three themes that emerged from this interview concerned student engagement and ownership in making sense of mathematics, teacher modeling of the processes of inquiry, and the importance of communicating and explaining one’s thinking. When the teacher viewed part of the videotape where he stepped back from the discussion and invited students to develop the acceleration equation for themselves, he commented that:

I want to try as much as possible to get them to work it out for themselves. . . . The other important thing about it as well, by doing it this way you’ve got a degree of ownership involved . . . the kids are engaged, and I really think that’s because they’re owning what’s going on, it’s not just sitting there, listen to this and away you go.

Similarly, when he asked students to reconstruct a mathematical argument developed in a previous lesson on circular motion, he pointed out that:

You’re getting them to try to build some sense into it, by getting them to reconstruct it themselves they have to be able to make some sense out of it even if it’s only internal consistency with the mathematics.

Yet the teacher did not abandon students to work alone: on the contrary, he saw one of his responsibilities as modeling and scaffolding mathematical thinking. For example, when students were unsure whether the acceleration equation should contain $\dot{\theta}$ or $(\dot{\theta})^2$, he urged them to “think about the result you’re getting to, what are you trying to get out of this thing?” When viewing the videotape, he described this as:

. . . a deliberate attempt to get them to focus on an endpoint, to think about where they’re getting to, to see if that would help them along. There’s an element of attempting to model the problem solving process in this as well.

The teacher expected students to explain and justify the ideas they contributed to whole-class discussions, as well as to critique the contributions of other students. He had at least two reasons for insisting that students did so, during the interview referring to the potential for peer interaction to promote understanding and also to connect everyday and mathematical language:

I do think it's important that they're able to communicate with other people and their peers. They will learn at least as much from each other as they will with me. To be able to do that they have to talk to each other. It's also a part, one of the reasons I often force them to say things because they need to be able to use the language because the language itself carries very specific meanings; and unless they have the language they probably don't have the meanings properly either. They need the language to be able to, obviously communicate, but I think it also has something to do with their understanding as well.

In contrast with traditional classrooms where public talk in whole-class discussion must be channeled through the teacher, students in this classroom frequently directed their comments to each other without the teacher's mediation, thus sparking the kind of argumentation that would otherwise have to be orchestrated through teacher intervention. One such instance occurred during a lesson soon after the one that was the subject of the teacher's stimulated recall interview and involved students interrogating a worked example.

The teacher regularly allowed time in class for students to study worked examples so that they would learn to find their way independently through mathematical texts. The examples, which also introduced students to the formal reasoning involved in applying new concepts, were then required to be fully explicated by students during whole-class discussion. Although students initially read in silence, after a short time they invariably turned to their neighbors either to seek clarification or to confirm their individual interpretations of the examples. This particular example demonstrated how to describe the motion of a mass executing simple harmonic motion while suspended from a spring. During the ensuing whole-class discussion, some students questioned the change of notation from $x = r \cos \omega t$ (as used in the lesson described above that introduced simple harmonic motion) to $x = a \cos nt$ (a more general form that applies to all kinds of simple harmonic motion, not just that derived from a projection of uniform circular motion on a diameter of the circle). Rather than provide a rationale, the teacher withdrew from the discussion to allow students to resolve the issue for themselves:

Rob: Why did they suddenly skip to a ?

Belinda: Because x is equal to $a \cos nt$.

Ben: Why use a and n when we have the exact same formula with r and ω ? Does it refer to ω involving radians?

Rob: On this side [referring to the handout containing the example—also used in the lesson that introduced simple harmonic motion] they said $x = r \cos \omega t$, on the other side $x = a \cos nt$.

Belinda: Excuse me, I have a point to make here! You can't always use r because—[to teacher] Oh, sorry! [Teacher indicates she should continue.] I don't know if

anyone will agree with me— because you're not always using a circle, it's not always going to be the radius.

Rob: Radius, yeah.

Belinda: So the amplitude's not always the radius.

By ceding control of this debate, the teacher provided an opportunity for students to seek and receive explanations from each other concerning the nature and appropriateness of different notations until they were satisfied that they understood.

The Grade 12 students' responses to interview questions during the 1st year of the study showed that they were remarkably well attuned to their teacher's goals and were aware that their classroom operated differently from others they had experienced. From these interviews emerged three themes that corresponded closely to the beliefs expressed by the teacher in justifying his approach, indicating that students were appropriating his values and learning goals.

First, students described how engagement in worthwhile tasks engendered a sense of personal ownership and developed their understanding. When the topic of working with peers arose during a whole-class interview, students commented:

Duncan: You can work it out yourself, amongst the group, without having to be told.

Belinda: With groups, if you've actually worked it through yourself and not just learned it off by heart then you're more likely to—

Rob: —understand it.

The second theme emerging from students' reflections on the classroom culture concerned the status of knowledge claims as conjectures that had to be validated by mathematical argument. This is illustrated by comments from a stimulated recall interview conducted with a group of Grade 12 students and based on a videotaped lesson segment that captured their lively discussion about a simple harmonic motion problem (MG is the author/interviewer, Rob a student):

MG: One of the interesting things is that you don't just accept what each other says.

Rob: We always assume everyone else is wrong about it!

MG: But it's not just saying "No it isn't," "Yes it is."

Rob: Yeah, we've got to be proven [sic] beyond all doubt!

Finally, students voiced beliefs about the importance of explaining their thinking as a means of both evaluating and strengthening their understanding, echoing the teacher's commitment to encouraging students to explain their reasoning. For example, in an interview with the Grade 12 class near the end of the 1st year of the study, students reflected on the cognitive benefits of explaining to peers:

Duncan: So many times I find myself trying to explain something to other people, and you find something you've kind of missed yourself. . . . Even if they don't really know what they're doing, explaining it to them imprints it to *your* mind.

Belinda: Yeah, and if you can explain it to someone else it means you *know*.

Interestingly, these students insisted that mathematics was different from (and more enjoyable than) other subjects they studied at school because it was a more open field of inquiry involving learning through discussion rather than memorization:

Ben: In other subjects like Biology the teacher doesn't give you much time to talk to other students. Most of the time, she's [i.e., the teacher] talking. When I talk to Duncan about something, we get in trouble for talking.

Duncan: It's more like learning parrot fashion.

Belinda: It's mostly pure learning, so what do you discuss? It's already all proven . . .

Together, these beliefs about teaching and learning and the practices through which they were enacted provide insights into the culture of mathematical inquiry that had become established in this classroom. After confirming that this culture had already taken hold by Grade 12, I wished to investigate the particular teaching practices the teacher employed with a new group of students he had not previously taught. For this reason, the study continued in the following year with a Grade 11 Mathematics C class, comprising nine boys and one girl aged 15–16 years.

CREATING A COMMUNITY OF INQUIRY

To understand how the community of inquiry developed over a period of time, it is necessary to examine both the specific teaching practices used by the teacher to create a variety of ZPDs and also the changing nature of students' participation over time. This relationship between teacher actions and students' changing participation is outlined in general terms in Table 1 and illustrated in more detail via the classroom vignettes presented below in chronological order as they occurred throughout the 1996 school year, comprising 41 teaching weeks.

Week 11 (16 and 17 April)

Early in the school year, the teacher placed explicit emphasis on modeling the processes of mathematical inquiry, structuring students' thinking and social interactions, and connecting students' developing ideas to mathematical language and symbolism. An example from two lessons on matrices about 2 months into the school year illustrates his actions and the students' responses. The annotated field notes from these lessons, which focus particularly on the teacher's actions, appear in the Appendix.

The aim of the lessons was to have the students discover for themselves the algorithm for finding the inverse of a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The teacher first chose a matrix A with a determinant of 1 and asked the students to find the inverse A^{-1} by using their existing knowledge of simultaneous equations to solve the matrix equation $AA^{-1} = I$. He then elicited students' conjectures about the general form of the inverse matrix, based on the specific case they had examined. Since the nature of the example ensured that students would offer

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

as the inverse, the teacher was able to provide a realistic context for students to test this initial conjecture. A counterexample, whose inverse was found to have the form

$$n \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

allowed the students to find a formula for n , which only then was labeled by the teacher as the determinant.

In this example, the teacher modeled mathematical thinking by presenting a specific problem for students to work on and eliciting a series of conjectures that had to be tested by students in order to arrive at a valid generalization (thus illustrating the first assumption underlying a culture of inquiry; see Table 1). He prompted reflection and self-monitoring (Assumption 2, also concerning the nature of mathematical inquiry) by asking questions such as “Can you check via matrix multiplication that you do get the identity matrix?” and “How could we verify this?” He further scaffolded the processes of inquiry (Assumption 3, ZPD as scaffolding) by moving students’ thinking forward (“How is this [the form of the inverse for the counterexample] related to Luke’s conjecture?”), reminding them of the task goals (“What was the reason we wanted to find matrix inverses in the first place?”), and drawing on ideas developed during the lesson (“What did you divide by in the previous example?”). Students were asked to explain their solutions to each other, thus signaling that the teacher valued social interaction with peers as a means of generating and testing mathematical thinking (Assumption 4, ZPD as collaboration). He also avoided using technical terms until students had developed an understanding of the underlying mathematical ideas (Assumption 5, ZPD as interweaving), saying “This thing is called the determinant,” and then “Let’s formalize what you’ve found.”

Week 14 (10 May)

By about one third of the way through the school year, there was evidence that the Grade 11 students were beginning to appropriate forms of reasoning and patterns of social interaction consistent with the notion of inquiry mathematics that were valued by the teacher. The first example comes from a lesson on vectors, 3 weeks after the matrix lessons summarized above.

In this lesson, the teacher wished to clarify the distinction between position and displacement vectors, and to have the students discover and prove the relationship between them: if \mathbf{a} and \mathbf{b} are the position vectors of the points A and B, respectively, then the displacement vector \mathbf{AB} is given by $\mathbf{b} - \mathbf{a}$. Instead of modeling his own thinking and inviting students to contribute specific segments of the solution as he had in earlier lessons, the teacher issued a very general instruction to “prove that to me geometrically, by drawing something.” He then allowed sufficient time

for students to work on the task and circulated around the room posing questions and offering comments to individual students as needed to guide their thinking, such as “What’s the really basic definition of vectors—what basic operation did we do first?” (past-oriented structuring); “You don’t need numbers to prove something”; “If we’re talking about position vectors they’ve got to go somewhere”; and “What is it we’re trying to prove?” (all future-oriented structuring).

It was noticeable that most students were now working together and sharing their ideas without the teacher’s prompting. More significant was the observation that some were also beginning to offer their own ideas and questions during whole-class discussion. After the teacher elicited the proof most students had constructed, shown in Figure 1(a), one student (Adam) asked if he could “draw something up there” and proceeded to take charge at the whiteboard, modifying the diagram by constructing the vector sum of \mathbf{b} and $-\mathbf{a}$ as shown in Figure 1(b). Impressed, the teacher validated Adam’s alternative proof by carefully paraphrasing his reasoning to check his understanding:

Teacher: OK that’s very good, excellent! Stay there so I get it right. You’ve said that this is negative \mathbf{a} and this goes this way, that’s just \mathbf{b} . So you’ve said \mathbf{b} plus negative \mathbf{a} is just \mathbf{AB} .

Immediately another student asked whether subtracting \mathbf{a} was different in this case from adding negative \mathbf{a} , demonstrating that students were beginning to take responsibility for clarifying their understanding. Similarly, later in the lesson a group of students quizzed the teacher on the difference between the symbols used to denote

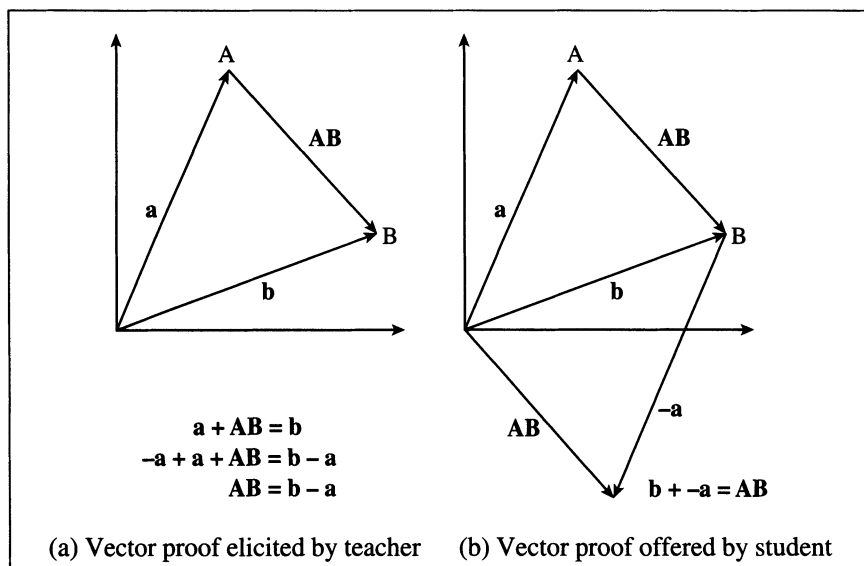


Figure 1. Vector proofs

the magnitude of a vector (e.g., $|\mathbf{a}|$) and the absolute value of a number (e.g., $|-5|$), and asked him to explain the meaning of the “squiggle” used when hand writing the vector \mathbf{a} as \underline{a} . Distinguishing between similar symbols with different meanings, and different symbols with the same meaning, was an aspect of understanding mathematical language often emphasized by the teacher, and the students’ actions suggest they were becoming aware of the importance of such conventions.

Week 17 (31 May)

In a lesson that took place 3 weeks later, the teacher asked students to develop a method for finding the angle between two vectors

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 5 \\ 1 \end{pmatrix},$$

based on their knowledge of the formula for the dot product, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \vartheta$, encountered for the first time only the previous day. Now the students were expected to advance their thinking without the teacher’s scaffolding, and most spontaneously formed small groups and pairs to work on the task without his assistance for over 10 minutes. Nevertheless, one student (Dennis) remained reluctant to share his thinking with peers and had to be deliberately encouraged by the teacher to do so:

Teacher: Don’t look at me in horror, Dennis, you can do it! But *talk* to people about it—talk to Rebecca, talk to Dean.

(See Goos, Renshaw, & Galbraith, 1998, for a discussion of resistance to participation in this and other classrooms.)

Other students were still coming to terms with the teacher’s insistence that they explain the reasoning that led to their answers. For example, when Adam calculated that the angle between the vectors was 22.4° and announced that he had the answer, his friend Dean pointed out that the teacher “doesn’t want the answer, he wants *how* you work out the angle.” This point was reinforced by the teacher himself when he reconvened the class and nominated a student (Alex) to come to the whiteboard to present his solution. As Alex began to calculate the value of

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \end{pmatrix},$$

the teacher reminded him that he wanted a general equation first before any numerical substitution. Other students then began offering Alex suggestions and hints as to how to proceed:

Adam: Rearrange it, Alex.

Aaron: Yeah, rearrange it.

Alex: Using . . . ?

Aaron: Using, like, symbols.

Adam: Look up on the board! [i.e., at the formula $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \vartheta$]. Just write down the equation.

Alex: So you work out \mathbf{a} dot \mathbf{b} using this method—[starts to substitute numbers again]

Teacher: I don't want to see anything to do with those numbers at all!

Aaron: Alex, rearrange that equation so you get theta by itself. [Alex begins to do so.]

Teacher: How's he going? Is he right? [Chorus from class, "Yes." Alex finishes rearranging formula to give

$$\vartheta = \cos^{-1} \left[\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \right].$$

Alex, that's great, that's spot on!

Here Adam and Aaron appear to have appropriated teacher-like strategies in moving a peer's thinking forward, and they bypassed the teacher completely in directing their comments to the student at the whiteboard. Note that the purpose of their actions was not to publicly display their own knowledge for the teacher's evaluation and approval—often the only reason why students are permitted to speak in traditional classrooms—but to assist in the construction of a jointly owned solution.

Toward the end of the lesson, the teacher again pulled the students toward personal sense-making by asking how they could use the dot product formula to find out if two vectors were parallel or perpendicular. After allowing a few minutes for students to discuss this question with peers, the teacher called on Dylan to "tell us what he's found":

Dylan: If $\mathbf{a} \cdot \mathbf{b}$ is zero they're perpendicular, because cosine ϑ is $\cos 90^\circ$ which is zero, so $|\mathbf{a}|$ times $|\mathbf{b}|$ times zero is zero.

Teacher: Did everyone get that?

Sean: No, I didn't!

[Teacher paraphrases Dylan's answer.]

Done! [indicating he now understands]

Teacher: And how do you know if two vectors are parallel?

Alex: They're in the same proportion, ratio.

Teacher: What does that mean? How can we say that?

Rhys: One is a scalar multiple of another.

Teacher: [Tone of admiration] Very good, Rhys!

Dennis: If one's negative, aren't they in different directions?

Teacher: One must be two times the other, or a half, and if it's negative two times they're in different directions, right, but they're still parallel.

This short exchange illustrates a growing maturity in students' participation—seen, for example, in Sean's willingness to publicly admit his confusion and ask for further explanation and in the ease with which Rhys responded to the teacher's demand for precision in formulating a mathematical definition of the properties of parallel vectors. Noteworthy, too, is Dennis's action in *asking* a question to clarify his understanding of the conditions for parallelism, since this student had previously avoided contributing to whole-class discussions unless specifically called on to *answer* the teacher's questions.

Week 29 (4 and 6 September)

Well before the end of the school year, mathematical practices had been estab-

lished that were similar to those observed in the Grade 12 group of the previous year. One strategy the teacher used to actively engage students was to structure entire lessons around a complex problem so that the students would learn new concepts as they developed solution strategies. Pitching these problems at the edge of students' existing competence seemed to be highly effective in promoting collaborative peer interactions that brought together students with incomplete knowledge but comparable expertise. One memorable comment from a Grade 11 student (Dean) illustrates this point and highlights the synergy inherent in complementary competence. When Dean was shown part of a videotaped lesson in which he and his friend Adam were deeply engaged together in exploring a novel problem, he commented:

Dean: Adam helps me [. . .] see things in different ways. Because, like, if you have two people who think differently and you both work on the same problem you both see different areas of it, and so it helps a lot more. More than having twice the brain, it's like having *ten* times the brain, having two people working on a problem.

Peer tutoring interactions were also observed in which students guided the learning of less expert peers. One striking example of this type of assistance occurred during a lesson when students were investigating the iterative processes underlying fractals and chaos theory. The class had considered the example of the Middle Thirds Cantor Set, a fractal constructed by starting with a line of length 1, removing the middle third, then removing the middle third of the remaining segments and repeating this process infinitely many times. The point of the example was to prove that the sum of all lengths removed is equal to the length of the original line, a surprising and counterintuitive result. Students were then asked in a subsequent activity to find how much space is removed from the Middle *Fifths* Cantor Set. A common error made by many students was simply to substitute $1/5$ for $1/3$ in the proof provided in the worked example. The worked example for Middle Thirds Cantor Set that was available to the students is shown in Table 2. The following edited transcript reveals how one student (Adam) helped his friend (Luke) to recognize this error.

Luke: It's going to be a fifth instead of a third [pointing to example, no response from Adam]. Adam, it's going to be a fifth [points to example].

Adam: Just think . . . start, work through it from the beginning.

Luke: I am. [Writes]

Adam: [Glances at Luke's work.] Work through it! [Emphatically]

Luke: [Not looking up.] I am. [Looks up, puzzled.] What am I doing? [Checks example.] The size remaining's right, isn't it? [Adam looks at Luke's work and chuckles.] That's right!

Adam: OK, you just do it.

Luke: OK, fine, whatever.

Adam: [Opens his own book and checks his working, grins at Luke.] Wrong!

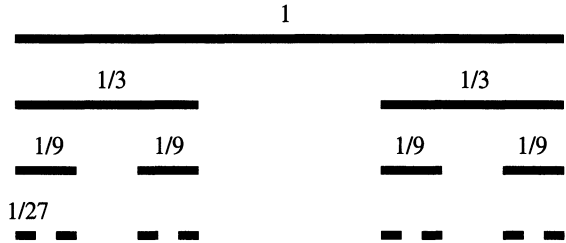
Luke: [Expression of disbelief on his face, looks at his working.] How and where? I cannot see where it could possibly be wrong!

Adam: [Pauses, raises eyebrows, makes the decision to rescue Luke.] OK, explain this to me. Explain . . . explain this to me [pointing to Luke's working].

Table 2
Middle Thirds Cantor Set and Worked Example

The Cantor Set

The Cantor Set is constructed by starting with an interval of length 1 and removing the middle third, leaving the two remaining end intervals. This is the beginning of an iterative process, so the next step is to remove the middle third of the two remaining intervals. This can be repeated infinitely many times.



Let us find out the total amount of space removed as it would appear that there might be nothing left after the final iteration.

After the first iteration, $1/3$ of the interval has been removed; after the second iteration, two sections of length $(1/3)^2$ have been removed; and after the third interval, four sections of length $(1/3)^3$ are deleted. This gives us the following pattern:

Level	Length of Section Removed
0	0
1	$\frac{1}{3}$
2	$2 \times \left(\frac{1}{3}\right)^2$
3	$4 \times \left(\frac{1}{3}\right)^3$
n	$2^{n-1} \times \left(\frac{1}{3}\right)^n$

Finding the sum of these terms:

$$\begin{aligned} S_{\infty} &= \frac{1}{3} + 2\left(\frac{1}{3}\right)^2 + 4\left(\frac{1}{3}\right)^3 + 8\left(\frac{1}{3}\right)^4 + \dots \\ &= \frac{1}{3} + 2^1\left(\frac{1}{3}\right)^2 + 2^2\left(\frac{1}{3}\right)^3 + 2^3\left(\frac{1}{3}\right)^4 + \dots + 2^{n-1}\left(\frac{1}{3}\right)^n \\ &= \frac{1}{3} \left[1 + 2^1\left(\frac{1}{3}\right)^1 + 2^2\left(\frac{1}{3}\right)^2 + 2^3\left(\frac{1}{3}\right)^3 + \dots + 2^{n-1}\left(\frac{1}{3}\right)^{n-1} \right] \\ &= \frac{1}{3} \left[1 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots + \left(\frac{2}{3}\right)^{n-1} \right] \end{aligned}$$

The section in the large set of brackets represents a converging GP with $a = 1$ and $r = 2/3$.

Table 2 (Continued)

Using the formula for the sum to infinity of a converging GP:

$$\begin{aligned}
 S_{\infty} &= \frac{a}{1-r} \\
 &= \frac{1}{1-\frac{2}{3}} \\
 &= \frac{1}{\frac{1}{3}} \\
 &= 3
 \end{aligned}$$

Thus, the sum of all the lengths removed is $1/3 \times 3 = 1$, which is the length of the original section.

- Luke:* What do you mean? This bit? OK, we start off with this thing. Length removed is nothing. Then we get here, length that's been removed is one fifth.
- Adam:* Uh huh.
- Luke:* Now we get, here. What we've lost is the original fifth, and . . . [turns to Adam] two fifths [sounding hesitant].
- Adam:* No, you're doing each section, you haven't added them up yet. So you haven't lost the original fifth yet, we're ignoring that.
- Luke:* OK. Out of each portion we—
- Adam:* [Immediately] OK, how big's the section? [Smiles] Mmm?
- Luke:* A fifth of . . . two fifths! ["Aha" expression on face, turns to Adam and grins. Adam grins back and raises eyebrows. Both laugh uproariously.] Now it gets tricky! [Luke circles the incorrect working and crosses it out.] Phew! [to Adam] You've given me food for thought!

In this interaction, Adam refrained from telling Luke where the error was or showing how to fix it; instead, he acted as his teacher typically did in pressing for explanations and posing questions that would help his partner to find his error. By directing Luke's attention to the key aspect of the task and questioning him in this way, Adam adopted the teacher's role in the teacher-student dialogue pattern that had become so familiar in this classroom.

A culture of collaboration such as that established by the teacher "primed" students to seek assistance from peers, and some of the Grade 11 students were very aware of the teacher's motives in this regard. For example, Alex chided Dylan for raising his hand to ask the teacher a question, pointing out that "He'll probably tell you to ask me, or ask other people!" Dylan later echoed this comment in declining the teacher's help:

- Dylan:* [To teacher] Actually, you probably don't have to worry because Alex is trying to explain it to me. If he can't explain it I'll come to you.
- Alex:* [To teacher] Yeah, you normally say to ask someone else first.

One of the most convincing signs that the Grade 11 group had appropriated the teacher's view of mathematics as sense-making came 2 days after the Cantor Set lesson, when the class had moved on to investigating another example of geometric

iteration, the Koch snowflake. Dylan, a student who had previously displayed a highly instrumental approach to understanding mathematics (i.e., he was satisfied with knowing rules without reasons), was struggling with a task which asked students to prove that there is a limit to the area of a Koch snowflake curve. The following dialogue occurred after Dylan had spent several minutes with his hand raised hoping to seek the teacher's assistance.

Dylan: [Plaintively] I can't keep going! I want to know why!

Alex: [Looks up, both laugh] Have you got my disease?

Dylan: Yeah!

Alex: Dylan, wanting to know why!

Dylan: Me wanting to know why is a first, but I just want to. It's a proof—you need to understand it. (Alex resumes work, Dylan still has his hand raised.)

Dylan's newfound insistence on knowing why, rather than glossing over elements of a proof he did not understand, indicates that he was moving toward fuller participation in the practices of mathematical inquiry—where understanding involves making personal sense of the concepts and reasoning conventions accepted by the wider community of mathematicians.

DISCUSSION AND CONCLUSION

The notion of learning mathematics in a community of practice has struck a chord with researchers interested in creating classroom environments that foster mindful, strategic learning by engaging students in collaborative forms of inquiry. This article has attempted to shed some light on how such communities can be created in upper-secondary-school mathematics classrooms. In particular, the research reported here extends the existing body of literature in this area by focusing in some detail on the enactment of teaching and learning practices through which the goals of reform-oriented curricula—with their emphasis on communication, reasoning, and problem solving—might be achieved. This research was conducted over an extended period of time, allowing the formation of the classroom community to be documented over the 2 years of senior secondary schooling. Although two different classes were observed during this period—a Grade 12 class in the 1st year, and a new Grade 11 class in the 2nd year—the emergent design of the study allowed attention to be given to how the teacher worked with the Grade 11 students to establish the culture of inquiry that appeared to be taken for granted by the Grade 12 group. The involvement of classes in the senior years of secondary schooling is especially significant, since accountability pressures associated with syllabus coverage and high-stakes assessment might seem to favor more traditional approaches to teaching at this level of schooling.

Sociocultural perspectives on learning offer mathematics education researchers a useful theoretical framework for analyzing learning as initiation into social practices and meanings. However, conceptualizing learning as increasing participation in a community of practice raises two important questions: first, in *what kinds of practices* do we wish students to participate; and second, *what specific actions* should

a teacher take to improve students' participation? The first question is related to the nature of mathematical inquiry, which, for the teacher in this study, involved mathematical attitudes he demonstrated through a commitment to personal sense-making and by his willingness to deal with more abstract ideas concerning conjecture, justification, and proof. Typically, he modeled this process of inquiry by presenting students with a significant problem designed to engage them with a new mathematical concept, eliciting their initial conjectures about the concept, withholding his own judgment to maintain an authentic state of uncertainty regarding the validity of these conjectures, and orchestrating discussion or presenting further problems that would assist students to test their conjectures and justify their thinking to others.

This example provides a generalized template for what a teacher might actually do to facilitate students' participation in a community of mathematical inquiry. Specifically, the zone of proximal development was invoked as an explanatory framework for learning that can explicitly inform teaching practice. This study outlined three ways in which a teacher and students can set up ZPDs—through scaffolding, peer collaboration, and the interweaving of spontaneous and theoretical concepts—each of which highlights the teacher's central position in assisting students to appropriate mathematics as cultural knowledge.

In the early stages of Grade 11, the teacher scaffolded the students' thinking by providing a predictable structure for inquiry through which he enacted his expectations regarding sense-making, ownership, self-monitoring, and justification. As the school year progressed, the teacher gradually withdrew his support to pull students forward into more independent engagement with mathematical ideas. For their part, the students responded by completing tasks with decreasing teacher assistance but also by proposing and evaluating alternative solutions, often bypassing the teacher in whole-class discussions to answer questions posed by fellow students. Some students additionally adopted teacher-like scaffolding strategies to assist less capable peers, for example, by asking questions that led their partner to locate an error or reconsider a solution plan. This reorganization of classroom social interactions is crucial to understanding that the creation of the ZPD is a process of negotiating personal meanings and comparing these with conventional interpretations from the community of mathematicians.

Peer interactions can also create a collaborative ZPD, and it was clear from students' interview responses that they believed this kind of participation framework provided opportunities to test their understanding and validate their conjectures through mathematical argument with peers. Again, the teacher played a vital role in initially orchestrating these interactions, inviting students to explain their thinking to neighbors or ask a friend for help before consulting him as a last resort.

ZPDs created by interweaving spontaneous and theoretical concepts challenge students to integrate their existing language and experiences with the more abstract concepts and precise terminology of mathematics. Making connections between everyday and scientific concepts was accomplished in a variety of settings in the classroom. For example, during initial class discussion of a task, the teacher paraphrased or reinterpreted students' language to introduce appropriate mathematical

terms for the ideas they expressed. Typically, students then formed pairs or small groups to tackle the task and were free to use their own informal and imprecise language. When the teacher subsequently reconvened the class and invited students to explain their solutions, he insisted that they use conventional mathematical language and explicate their reasoning in full.

These positive examples of learning as initiation into mathematical practice probably present too idealized an image of the classroom as a community of inquiry, when it is more correct to say that there were noticeable differences in the nature and extent of students' participation. A small number of students resisted the teacher's efforts to move them toward more independent and critical engagement with mathematical tasks, for example, by waiting for the teacher to hand over the required knowledge or by avoiding constructive interaction with peers. Learning in the ZPD is a process of coming to *be* as well as one of coming to *know* (Litowitz, 1993), and one should not naively assume that participation implies inclusion or that all students willingly identify with the teacher's mathematical attitude and expectations.

In addressing the research question of how the teacher initiated his students into a culture of inquiry, analysis of the data presented here illuminated some issues but inevitably left others in shadow. In keeping with contemporary sociocultural theory, a more complete analysis would take into account the students' previous histories of mathematics learning and the knowledge, values, and experiences they brought to school from their home and family contexts, and investigate how the teacher's interpretation of these histories structured his interactions with students. The influence of the institutional context on learning in this classroom also deserves more careful analysis by expanding the concept of "community" to include the school and those who supported its goals and activities. For example, one question that might arise from this study concerns the teacher's relationships with other teachers within and outside the mathematics department, with the school's leadership team (e.g., principal and deputy principal), and with parents of the students he taught, and how he situated his pedagogical practices and goals within this network of relationships. Further research is also needed to widen the scope of this study beyond the Mathematics C classrooms and school considered here to provide a richer picture of constraints and opportunities in implementing inquiry mathematics approaches in different settings. A sociocultural theoretical approach has the potential to inform this research and improve understanding of how learners can be pulled forward into mature participation in communities of mathematical practice.

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APPENDIX

Annotated Field Notes for Grade 11 Lessons on Matrices

Annotations in the first column link observations to the categories outlined in the data analysis methods section of the article. S and Ss refer to student(s); T refers to teacher.

Grade 11 Mathematics C lesson #1: Finding the inverse of a 2×2 matrix

Annotation	Interaction	Whiteboard
<i>Structures S's thinking (past)</i>	T reminds Ss of procedure for finding inverse of a 2×2 matrix using simultaneous equations. Asks Ss to solve the resulting equations. Ss provide equations and solution. T: So the inverse of $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ is $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$.	$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $3a + c = 1$ $5a + 2c = 0$ $3b + d = 0$ $5b + 2d = 1$ $a = 2, b = -1, c = -5, d = 3$
<i>Encourages self-checking</i>	T: Can you check via matrix multiplication that you do get the identity matrix? Ss confirm this is so.	
<i>Models mathematical thinking</i> <i>Withholds judgment</i> <i>Invites ownership</i>	T: Is it inefficient to do this every time? Ss concur. T: Could we find a shortcut? Luke suggests reversing the position of a and d , and placing minus signs in front of b and c . T elicits symbolic representation and writes on whiteboard.	Inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is } \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}?$
<i>Models mathematical thinking</i>	T: How could we verify this? Ss suggest doing another one. T provides another example; asks students to use "Luke's conjecture" to write down the hypothetical inverse and check via matrix multiplication. Ss do so—they are convinced the method works.	$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{\text{inverse}} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

Annotation	Interaction	Whiteboard
<i>Structures S's thinking (future)</i>	T gives another example for Ss to try. Gradual increase in S talk as they realize Luke's conjecture doesn't work for this one (matrix multiplication does not yield the identity matrix).	$\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \xrightarrow{\text{inverse}} \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} ?$ $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$
<i>Structures S's thinking (past)</i>	T reminds Ss they can still find the inverse by solving simultaneous equations. Ss do so and verify via matrix multiplication.	Inverse is $\begin{pmatrix} \frac{2}{5} & \frac{-1}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{pmatrix}$
<i>Structures S's thinking (future)</i>	T: How is this related to Luke's conjecture? (which is half right). Ss reply that the first attempt is too big by a factor of 5, so they need to divide by 5.	
<i>Structures S's thinking (present)</i>	T: What did you divide by in the previous example? Ss realize they could divide by 1.	$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{\text{inverse}} \begin{pmatrix} \frac{1}{1} & \frac{-1}{1} \\ \frac{-1}{1} & \frac{2}{1} \end{pmatrix}$
<i>Models mathematical thinking</i>	T: So the new method (dividing by something) works. But how do you know what to divide by?	
<i>Sense-making; invites ownership</i>	Homework: Find a rule that works for these two cases. Test it on another matrix of your choice.	

Grade 11 Mathematics C lesson #2: Inverse and determinant of a 2×2 matrix

Annotation	Interaction	Whiteboard
	<p>T asks Ss to remind him of the matrix worked on last lesson (homework). The first try gave</p> $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix};$ <p>you had to adjust by dividing by five. (Ss were to find a rule for the divisor.) <i>T</i>: What was the divisor? <i>Dean</i>: $ad - bc$.</p>	$\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ \downarrow $\begin{pmatrix} \frac{2}{5} & \frac{-1}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{pmatrix}$
<i>Invites ownership</i>	<p><i>T</i>: Did you invent your own matrix and test it? <i>Ss</i>: Yes, it worked.</p>	
<i>Mathematical conventions and symbolism</i>	<p><i>T</i> names “this thing” ($ad - bc$) as the <i>determinant</i>. <i>T</i>: Let’s formalize what you’ve found. What would I write as the inverse of</p> $\begin{pmatrix} a & b \\ c & d \end{pmatrix}?$ <p>Alex volunteers the formula, which <i>T</i> writes on whiteboard. <i>Luke</i>: Would the inverse of a 3×3 matrix be similar? <i>T</i>: Yes, but it’s messy—you can use your graphics calculator to do it. You need to be able to find the inverse of a 2×2 matrix longhand. <i>Rhys</i>: What part of that is the determinant? <i>T</i> labels $ad - bc$ and writes the symbol and name “del” on whiteboard.</p>	$\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ $\nabla = ad - bc$ <i>del</i>

Annotation	Interaction	Whiteboard
<i>Models mathematical thinking (test conjecture with another example)</i>	T puts another example on whiteboard and asks Ss to find the inverse. After working for a short time Ss begin to murmur “zero.” They find that $ad - bc$, the determinant of the matrix, is zero; therefore, the inverse cannot be calculated.	Find $\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}^{-1}$
<i>Demonstrates ownership of idea</i>	<i>Rhys</i> : Is our method <i>still</i> wrong? <i>T</i> : No. Remember, some elements of the real number system have no inverse. So what is the test to find if a matrix is noninvertible? <i>Luke</i> : The determinant is zero.	
<i>Mathematical language</i>	<i>T</i> : A noninvertible matrix is called a <i>singular</i> matrix. What happens if you try to invert this matrix using your graphics calculator? Ss try it: see “error” message.	
<i>Structures S’s thinking (present)</i>	<i>T</i> : We can think about this another way. Remember how to use simultaneous equation method to find the inverse. . . . What happens if the matrix is singular? First find the inverse of this matrix, using simultaneous equations. Ss work on solving the simultaneous equations. T tours the room. Asks Adam, “Have you done it?” <i>Adam</i> : No.	$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $2a + c = 1$ $2b + d = 0$ $a + c = 0$ $b + d = 1$ $a = 1, c = -1, b = -1, d = 2$
<i>Structures S’s social interaction</i>	<i>T</i> : Then ask Aaron (beside Adam) to explain it. Ss finish finding solutions.	

Annotation	Interaction	Whiteboard
<i>Structures S's thinking (past)</i>	<p>T: What is this related to, from Junior maths?</p> <p>Ss: Finding the intersection of two lines.</p> <p>T: These are all linear equations so we could solve them by graphing.</p> <p>Ss use graphics calculators to find graphical solutions.</p> <p>T: So one way to find the inverse is to set up simultaneous equations and solve (algebraically or graphically). Now try to find the inverse of</p> $\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}$ <p>(which we just found is singular) by solving simultaneous equations graphically.</p> <p>Ss find parallel lines—no solution.</p>	
<i>Structures S's thinking (future)</i>	<p>T: Another interesting thing . . . you know how to turn a matrix equation into simultaneous equations . . . (Ss do the conversion and solve the equations).</p> <p>T: Can we do the reverse? What if I gave you the simultaneous equations—how would you make a matrix equation?</p> <p>Aaron explains how the numbers and the letters are arranged in matrix formation.</p> <p>T: What was the reason we wanted to find matrix inverses in the first place?</p> <p>Rhys: We couldn't divide by a matrix!</p>	$\begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \end{pmatrix}$ $4a + 2b = 10$ $a + b = 3$ $a = 2, b = 1$

Annotation	Interaction	Whiteboard
<i>Structures S's thinking (past)</i>	<p>T reminds Ss where they left off previous work on solving a problem that required division of one matrix by another (like the equation on the whiteboard).</p> <p>T: Recall the parallel with the real number system . . . to solve this algebraic equation you'd multiply both sides by the multiplicative inverse of 3.</p>	$\begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \end{pmatrix}$ $3 \quad x = 6$
<i>Sense-making; invites ownership</i>	Homework: Solve the matrix equation (by "inventing" matrix algebra).	