PHYS 301 – PROJECT 1
Complex Numbers and Quantities in Physics – I

Solution of each problem contributes to 1 point bonus in the final course grade. HAND IN BY THE DAY OF 1ST MIDTERM EXAM.

1. A plane wave of light of angular frequency ω is represented by $e^{i\omega(t-nx/c)}$. In a certain substance the simple real index of refraction n is replaced by the complex quantity n-ik. What is the effect of k on the wave? What does k correspond physically?
(The generalization of a quantity from real to complex form occurs frequently in physics. Example range from the complex Young's modulus of viscoelastic materials to the complex potential of the "cloudy crystal ball" model of the atomic nucleus).

2. The electric field of a plane electromagnetic wave propagating along the positive direction of the z-axis is given by $\mathbf{E}_+(z,t) = E_0 \exp(-ikz - i\omega t)\mathbf{e}_+$, where $\mathbf{e}_+$ is the polarization vector of the filed. Assume another similar plane wave propagating along the **negative** direction with a different polarization vector $\mathbf{e}_-$. In this case the electric field is given by $\mathbf{E}_-(z,t) = E_0 \exp(ikz - i\omega t)\mathbf{e}_-$. Consider an electric field made up by the superposition of these two counter-propagating fields. Find the total electric field in the following two cases:

(a) The two beams have opposite circular polarizations σ^+ and σ^-. So $\mathbf{e}_+ = -\frac{1}{\sqrt{2}}(\mathbf{e}_X + i\mathbf{e}_Y)$ and $\mathbf{e}_- = \frac{1}{\sqrt{2}}(\mathbf{e}_X - i\mathbf{e}_Y)$.

(b) The two beams have linear orthogonal polarizations $\mathbf{e}_+ = \mathbf{e}_X$, $\mathbf{e}_- = \mathbf{e}_y$

The above physical situation is the basis of the so called polarization gradients which have played important role in the cooling and trapping of atoms

3. The two-dimensional irrotational fluid flow is conveniently described by a complex potential $f(z) = u(x,y) + iv(x,y)$. We label the real part $u(x,y)$, the **velocity potential**, and the imaginary part $v(x,y)$, **the stream function**. The fluid velocity **V** is given by $\mathbf{V} = \vec{\nabla} u$. If $f(z)$ is analytic,

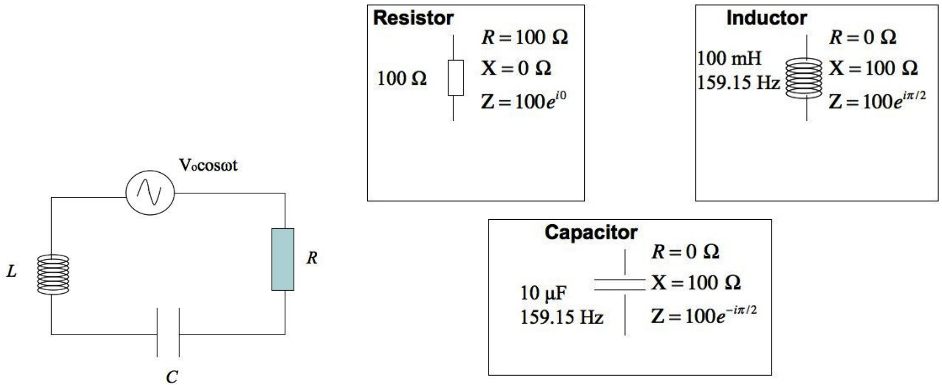
(a) Show that $df/dz = V_x - iV_y$,

(b) Show that $\vec{\nabla} \cdot \mathbf{V} = 0$ (i.e. no sources or sinks)

(c) Show that $\vec{\nabla} \times \mathbf{V} = 0$ (i.e. the flow is irrotational, nonturbulent flow)

4. Comlex numbers can be used to analyze AC electric circuits contanining Ohmic resistors, capacitors and inducances. All these circuit elements have complex resistances known as **impedances**. In the case of an ohmic resistor R the impedance is given as $Z_R = Re^{i0}$. In the case of a capacitor with capacity C the impedance is $Z_C = \frac{1}{2\pi fC} e^{-i\pi/2}$, while in the cace of the inductor L the impedance is given by $Z_L = 2\pi fLe^{i\pi/2}$. The following figures show examples

of how these impedances are defined in cases of ohmic resistors, capacitors and inductances respectively:



a) Consider that the three elements are connected in series to a AC generator which provides a voltage $V = V_0 \sin(2\pi ft)$ where $V_0 = 100V$ and $f = \left(\frac{100}{6}\right) Hz$. You are given also that $R = 100\Omega$, $C = 10\mu F$, $L = 50mH$. i) Find the total impedance of the circuit $Z = Z_R + Z_C + Z_L$, ii) Find the current in the circuit $I = V / Z$. Express your final results in exponential form.

b) Repeat the question (a) in the case where the three elements are connected in parallel to the generator. Recall that in this case $(1/Z) = (1/Z_R) + (1/Z_C) + (1/Z_L)$. Express your final results in exponential form.

c)

5. The use of complex numbers is very important in quantum mechanics. In the theory of spin we use the so called spin matrices.The spin matrix for the *y* spin component is,

$$s_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

(a) Find the matrix s_y^2.

(b) A random state of a particle with spin equal to $\hbar / 2$ is described by a column matric

$$\mathbf{X} = \begin{pmatrix} a \\ b \end{pmatrix}$$

where *a, b* are complex numbers. Calculate the average value of spin along direction the *y* direction which is given by:

$$\langle s_y \rangle = \begin{pmatrix} a^* & b^* \end{pmatrix} \cdot s_y \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$